

EXISTENCE THEORY FOR NONLINEAR STURM–LIOUVILLE PROBLEMS WITH NON-LOCAL BOUNDARY CONDITIONS

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Abstract. In this work we provide conditions for the existence of solutions to nonlinear Sturm-Liouville problems of the form

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(x(t))$$

subject to non-local boundary conditions

$$ax(0) + bx'(0) = \eta_1(x) \text{ and } cx(1) + dx'(1) = \eta_2(x).$$

Our approach will be topological, utilizing Schaefer's fixed point theorem and the Lyapunov-Schmidt procedure.

1. Introduction

In this paper we provide criteria for the solvability of nonlinear Sturm-Liouville problems of the form,

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(x(t)) \quad t \in [0, 1], \quad (1)$$

subject to non-local boundary conditions

$$ax(0) + bx'(0) = \eta_1(x) \text{ and } cx(1) + dx'(1) = \eta_2(x). \quad (2)$$

There are several standard ways in which one may define a solution to problem (1)–(2), and so to maintain completeness, we mention that in this paper we will be interested in proving the existence of classical solutions to (1)–(2). Formally, by a solution to (1)–(2) we mean a function $x : [0, 1] \rightarrow \mathbb{R}$ such that px' is continuously differentiable and satisfies (1)–(2).

Throughout our analysis, we will assume that $p, q : [0, 1] \rightarrow \mathbb{R}$ are continuous, $p(t) > 0$ for all $t \in [0, 1]$, $a^2 + b^2 > 0$ and $c^2 + d^2 > 0$, λ is an eigenvalue of the associated linear Sturm-Liouville problem, $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and for $i = 1, 2$, $\eta_i(x) = \int_{[0,1]} g_i(x) d\mu_i$, where $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ are continuous and μ_1 and μ_2 are finite Borel measures on $[0, 1]$.

The focus of this paper is the analysis of nonlinear Sturm-Liouville problems at resonance subject to non-local boundary conditions, where by resonance we mean that

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the linear homogeneous problem (7)–(8) has nontrivial solutions. Since the pioneering work of Landesman-Lazer, [12], much has been written about resonant nonlinear Sturm-Liouville boundary value problems with linear boundary conditions. Pertinent references from the point of view of this paper are [2, 4, 5, 6, 8, 11, 12, 14, 15, 18, 19]. Less has been said in regard to problems with nonlocal boundary conditions, even for the case of nonresonance; readers interested in results in this direction may consult [1, 7, 10, 13, 17, 20, 21, 22, 23].

The novelty of this work is due in large part to the generality of the nonlinear boundary conditions η_1 and η_2 . As an important special case we point out that, by taking μ_1 and μ_2 to be point-supported measures, our integral boundary conditions allow for nonlinear multipoint boundary conditions of the form

$$\eta_1(x) = \sum_{k=1}^n f_k(x(t_k)), \eta_2(x) = \sum_{j=1}^m h_j(x(t_j)),$$

where each f_k, h_j is a continuous function and each $t_k, t_j \in [0, 1]$.

Our main result, Theorem 3.1, provides conditions for the existence of solutions to (1)–(2) under a suitable interaction of the eigenspace of the linear Sturm-Liouville problem and the nonlinearities in both the differential equation and the boundary conditions. We would like to remark that the result we obtain in Theorem 3.1 constitutes a significant extension of the work found in [15] by allowing for much more generality in the boundary conditions, (2).

2. Preliminaries

The nonlinear boundary value problem (1)–(2) will be viewed as an operator equation. We let $C := C[0, 1]$ denote the space of real-valued continuous functions topologized by the supremum norm, $\|\cdot\|_C$. As usual, $L^2 := L^2[0, 1]$ will denote the space of real-valued square-integrable functions defined on $[0, 1]$. The topology on L^2 will be that induced by the standard L^2 -norm, $\|\cdot\|_{L^2}$. We use H^2 to denote the Sobelov space of functions with two weak derivatives in L^2 ; that is,

$$H^2 = \{x \in L^2 \mid x' \text{ is absolutely continuous and } x'' \in L^2\}.$$

Unless otherwise stated, the topology on H^2 will be the subspace topology inherited from L^2 . However, we will, on several occasions, topologize H^2 with the Sobelov norm,

$$\|x\|_{H^2} = \|x\|_{L^2} + \|x'\|_{L^2} + \|x''\|_{L^2}.$$

On occasion, we may also view H^2 as a subspace of C . We will use $|\cdot|$ to denote the Euclidean norm on \mathbb{R}^2 and $\langle \cdot, \cdot \rangle_2, \langle \cdot, \cdot \rangle_S$, and $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ will denote the inner products on L^2, H^2 , and \mathbb{R}^2 , respectively. Weak convergence in L^2 will be denoted by \xrightarrow{w} and weak convergence in the Sobelov space H^2 will be denoted by \xrightarrow{S} . We make $L^2 \times \mathbb{R}^2$

an inner product space with inner product

$$\left\langle \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} g \\ v_1 \\ v_2 \end{bmatrix} \right\rangle := m \left(\langle h, g \rangle_2 + \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle_{\mathbb{R}} \right), \tag{3}$$

where m is a positive constant which will be chosen later, and we will use $\|\cdot\|_{L^2 \times \mathbb{R}^2}$ to denote the norm generated by this inner product. Lastly, we give $C \times \mathbb{R}^2$ the product topology, and we will use $\|\cdot\|_{C \times \mathbb{R}^2}$ to denote the standard product norm on this space.

Linear boundary operators B_1 and B_2 will be defined as follows:

$B_1 : H^2 \rightarrow \mathbb{R}$ is given by

$$B_1x = ax(0) + bx'(0)$$

and $B_2 : H^2 \rightarrow \mathbb{R}$ is given by

$$B_2x = cx(1) + dx'(1).$$

We define $\mathcal{L} : H^2 \rightarrow L^2 \times \mathbb{R}^2$

$$\mathcal{L}x = \begin{bmatrix} \mathcal{A}x \\ B_1x \\ B_2x \end{bmatrix},$$

where $\mathcal{A} : H^2 \rightarrow L^2$ is defined by

$$\mathcal{A}x(t) = (p(t)x'(t))' + (q(t) + \lambda)x(t).$$

Similarly, we define a nonlinear operator $\mathcal{G} : H^2 \rightarrow L^2 \times \mathbb{R}^2$ by

$$\mathcal{G}(x) = \begin{bmatrix} \mathcal{F}(x) \\ \eta_1(x) \\ \eta_2(x) \end{bmatrix},$$

where $\mathcal{F}(x)(t) = f(x(t))$ and, as before, for $i = 1, 2$, $\eta_i(x) = \int_{[0,1]} g_i(x) d\mu_i$. Solving the nonlinear boundary value problem (1)–(2) is now equivalent to solving

$$\mathcal{L}x = \mathcal{G}(x). \tag{4}$$

The study of the nonlinear boundary value problem (1)–(2) will be intimately related to the linear nonhomogeneous boundary value problem

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = h(t), \quad t \in [0, 1] \tag{5}$$

$$ax(0) + bx'(0) = w_1 \quad \text{and} \quad cx(1) + dx'(1) = w_2, \tag{6}$$

where h is an element of L^2 and w_1 and w_2 are elements of \mathbb{R} . Using our notation from above, we have that solving (5)–(6) is equivalent to solving

$$\mathcal{L}x = \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}.$$

We begin our study of the nonlinear boundary value problem (1)–(2) by analyzing (5)–(6). To aid in this analysis, we first recall some well-known facts regarding the linear homogeneous Sturm-Liouville problem

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = 0 \tag{7}$$

$$ax(0) + bx'(0) = 0 \text{ and } cx(1) + dx'(1) = 0. \tag{8}$$

For those readers interested in a more detailed introduction to linear Sturm-Liouville problems, we suggest [9].

It is well known that λ is a simple eigenvalue; that is, $\text{Ker}(\mathcal{L})$ is one-dimensional. We may therefore choose a vector, ψ , which forms a basis for $\text{Ker}(\mathcal{L})$. Without loss of generality, we will assume $\|\psi\|_{L^2} = 1$. Since (7) is a second-order linear homogeneous differential equation, we may choose ϕ satisfying (7) so that $\{\psi, \phi\}$ forms a basis for the solution space of this linear homogeneous problem. We will assume $\langle \psi, \phi \rangle_2 = 0$.

For $u, v \in H^2$, let $wr(u, v)$ denote the Wronskian of u and v ; that is, $wr(u, v) = uv' - vu'$. It follows from standard ode theory that if u and v are linearly independent solutions to (7), then $p \cdot wr(u, v)$ is a nonzero constant. We will assume that ϕ has been chosen so that $p \cdot wr(\psi, \phi) = 1$ and define $\omega : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ by

$$\omega(t, s) = \begin{cases} \psi(t)\phi(s) & \text{if } 0 \leq t \leq s \leq 1 \\ \psi(s)\phi(t) & \text{if } 0 \leq s \leq t \leq 1 \end{cases}. \tag{9}$$

As a reminder to the reader, ω is often referred to as a fundamental solution of (7).

If we define $K : L^2 \rightarrow H^2$ by

$$Kh(t) = \int_0^1 \omega(t, s)h(s)ds, \tag{10}$$

then it is easy to verify that K is self-adjoint, compact, and satisfies $\mathcal{L}Kh = h$ for every $h \in L^2$. Differentiating under the integral symbol, one easily establishes that for every $h \in L^2$, $B_1Kh = \langle h, \phi \rangle_2 B_1\psi = 0$ and $B_2Kh = \langle h, \psi \rangle_2 B_2\phi$. Let

$$v_1 = B_1\phi \text{ and } v_2 = B_2\phi.$$

Since ϕ satisfies (7) and is linearly independent of ψ , we must have $B_1\phi \neq 0$ and $B_2\phi \neq 0$; this is a consequence of the uniqueness of solutions to initial value problems and that fact the linear Sturm-Liouville boundary conditions can be thought of as an orthogonality condition.

With the above ideas in hand, we are now in a position characterize the range of \mathcal{L} . We have the following result.

PROPOSITION 2.1. *Let $h \in L^2$ and $w_1, w_2 \in \mathbb{R}$. Then $\vec{h} := \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix} \in \text{Im}(\mathcal{L})$ if*

and only if $\langle \vec{h}, \vec{\psi} \rangle = 0$, where $\vec{\psi} := \begin{bmatrix} \psi \\ v_1^{-1} \\ -v_2^{-1} \end{bmatrix}$. That is, in $L^2 \times \mathbb{R}^2$, $\text{Im}(\mathcal{L}) = \{\vec{\psi}\}^\perp$.

Proof. $\mathcal{L}x = \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}$ if and only if $\mathcal{A}x = h$, $B_1x = w_1$, and $B_2x = w_2$. However,

$\mathcal{A}x = h$ if and only if $x = c_1\psi + c_2\phi + Kh$, for some real numbers c_1, c_2 . Applying the boundary map B_1 and recalling $B_1Kh = 0$, we get $B_1(c_1\psi + c_2\phi + Kh) = c_2v_1$. Similarly, using $B_2Kh = \langle h, \psi \rangle_2 B_2\phi$, we get $B_2(c_1\psi + c_2\phi + Kh) = (c_2 + \langle h, \psi \rangle_2)v_2$.

Now,

$$c_2v_1 = w_1 \quad \text{and} \quad (c_2 + \langle h, \psi \rangle_2)v_2 = w_2$$

if and only if

$$c_2 = \frac{w_1}{v_1} \quad \text{and} \quad \langle h, \psi \rangle_2 = \frac{w_2}{v_2} - \frac{w_1}{v_1} = \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} -v_1^{-1} \\ v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}},$$

which happens if and only if $\langle \vec{h}, \vec{\psi} \rangle = 0$. \square

With this characterization of the $Im(\mathcal{L})$ in hand, we make the following definitions which will play a crucial role in our ability to analyze the nonlinear Sturm-Liouville problem, (1)–(2), using a projection scheme.

DEFINITION 2.2. Define $P : L^2 \rightarrow L^2$ by $Px = \langle x, \psi \rangle_2 \psi$.

It is clear that P is the orthogonal projection onto $Ker(\mathcal{L})$.

Now, choose m , see (3), to be $\frac{1}{1 + \left\| \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\|^2}$. With this choice of m , $\vec{\psi}$ is a unit

vector in $L^2 \times \mathbb{R}^2$.

DEFINITION 2.3. Define $Q : L^2 \times \mathbb{R}^2 \rightarrow L^2 \times \mathbb{R}^2$ by

$$Q \left(\begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix} \right) = \left\langle \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}, \vec{\psi} \right\rangle \vec{\psi}.$$

From Proposition 2.1, we have that Q is the orthogonal projection of $L^2 \times \mathbb{R}^2$ on $Im(\mathcal{L})^\perp$. Thus, $I - Q$, is a projection onto the $Im(\mathcal{L})$.

In our analysis of the nonlinear Sturm-Liouville problem we will use a projection scheme often referred to as the Lyapunov-Schmidt procedure. The use of the Lyapunov-Schmidt reduction will allow us to write the operator equation (4) as an equivalent equation in which a fixed point argument may be applied to prove the existence of solutions. Interested readers may consult [3, 16] for a more detailed account of these ideas.

PROPOSITION 2.4. Solving $\mathcal{L}x = \mathcal{G}(x)$ is equivalent to solving the system

$$\left\{ \begin{array}{l} (I - P)x - M(I - Q)\mathcal{G}(x) = 0 \\ \text{and} \\ \left(\langle \mathcal{F}(x), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \psi = 0 \end{array} \right.,$$

where M denotes $(L|_{H^2 \cap \text{Ker}(\mathcal{L})^\perp})^{-1}$.

Proof.

$$\begin{aligned}
 \mathcal{L}x = \mathcal{G}(x) &\iff \begin{cases} (I-Q)(\mathcal{L}x - \mathcal{G}(x)) = 0 \\ \text{and} \\ Q(\mathcal{L}x - \mathcal{G}(x)) = 0 \end{cases} \\
 &\iff \begin{cases} \mathcal{L}x - (I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ Q\mathcal{G}(x) = 0 \end{cases} \\
 &\iff \begin{cases} M\mathcal{L}x - M(I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ Q\mathcal{G}(x) = 0 \end{cases} \\
 &\iff \begin{cases} (I-P)x - M(I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ \left\langle \begin{bmatrix} \mathcal{F}(x) \\ \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} \psi \\ v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \bar{\psi} = 0 \end{cases} \\
 &\iff \begin{cases} (I-P)x - M(I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ \left\langle \begin{bmatrix} \mathcal{F}(x) \\ \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} \psi \\ v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \psi = 0 \end{cases} \\
 &\iff \begin{cases} (I-P)x - M(I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ (\langle \mathcal{F}(x), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}) \psi = 0 \end{cases} . \quad \square
 \end{aligned}$$

3. Main results

We now come to our main result. In what follows, we will assume that the nonlinear integral boundary operators η_1 and η_2 are induced by bounded continuous functions g_1 and g_2 .

To simplify the statement of the theorem, we introduce the following notation. For $i = 1, 2$, we let

$$g_{i,+}(+\infty) := \limsup_{x \rightarrow \infty} g_i(x),$$

$$g_{i,-}(+\infty) := \liminf_{x \rightarrow \infty} g_i(x),$$

$$g_{i,+}(-\infty) := \limsup_{x \rightarrow -\infty} g_i(x),$$

and

$$g_{i,-}(-\infty) := \liminf_{x \rightarrow -\infty} g_i(x).$$

We define $\mathcal{O}_0 := \{t \mid \psi(t) = 0\}$, $\mathcal{O}_+ := \{t \mid \psi(t) > 0\}$, and $\mathcal{O}_- := \{t \mid \psi(t) < 0\}$. From Standard Sturm-Liouville theory, we have that \mathcal{O}_0 is a finite set consisting of simple zeros. In what follows, this fact will be used several times, possibly without explicit mention. Finally, for $i = 1, 2$, we let

$$J_{i,\pm} = g_{i,\pm}(+\infty)\mu_i(\mathcal{O}_+) + g_{i,\pm}(-\infty)\mu_i(\mathcal{O}_-).$$

THEOREM 3.1. *Suppose that the following conditions hold:*

- C1. *The function f is “sublinear”; that is, there exists real numbers M_1, M_2 and β , with $0 \leq \beta < 1$, such that for every $x \in \mathbb{R}$, $|f(x)| \leq M_1|x|^\beta + M_2$;*
- C2. *There exist positive real numbers \hat{z} and J such that for all $z > \hat{z}$,*

$$f(-z) \leq -J < 0 < J \leq f(z);$$

- C3. *For $i = 1, 2$, $\mu_i(\mathcal{O}_0) = 0$, where again μ_i is the Borel measure in the definition of the boundary operator η_i ;*

C4. $-J \int_0^1 |\psi| dt < \left\langle \begin{bmatrix} J_{1,\text{sgn}(-v_1)} \\ J_{2,\text{sgn}(v_2)} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}$, where for a real number, v , $\text{sgn}(v) = +$ if $v > 0$ and $\text{sgn}(v) = -$ if $v < 0$;

then, there exists a solution to (1)–(2).

Proof. We start by defining $T : L^2 \rightarrow H^2$ by

$$T(x) = Px - \left(\langle \mathcal{F}(x), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \psi + M(I - Q)\mathcal{G}(x).$$

From Proposition 2.4, we have that the solutions to (1)–(2) are the fixed points of T . Since M is an integral mapping from L^2 into H^2 , it is compact, and thus so is T . We will show that

$$FP := \{x \in H^2 \mid x = \gamma T(x) \text{ for some } \gamma \in (0, 1)\}$$

is a priori bounded in L^2 . A fixed point will then follow from an application of Schaefer’s fixed point theorem.

To this end, suppose that there exist sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{\gamma_n\}_{n \in \mathbb{N}}$ in H^2 and $(0, 1)$, respectively, with $\|x_n\|_{L^2} \rightarrow \infty$ and $x_n = \gamma_n T(x_n)$. Let $y_n = \frac{x_n}{\|x_n\|_{H^2}}$. Since

the closed unit ball in the Sobelov space H^2 is weakly compact, by going to a subsequence if necessary, we may assume that $y_n \xrightarrow{S} y$, for some $y \in H^2$. Again, going to a subsequence if necessary, we may assume that γ_n converges to some $\gamma \in [0, 1]$.

Now,

$$\begin{aligned} y_n &= \frac{x_n}{\|x_n\|_{H^2}} \\ &= \gamma_n \frac{T(x_n)}{\|x_n\|_{H^2}} \\ &= \gamma_n \frac{Px_n - \left(\langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \Psi + M(I - Q)\mathcal{G}(x_n)}{\|x_n\|_{H^2}}. \end{aligned}$$

Since f is sublinear (See C1) and g_1 and g_2 are bounded, it follows that

$$\|\mathcal{G}(x)\|_{L^2 \times \mathbb{R}^2} \leq K_1 \|x\|_{L^2}^\beta + K_2, \tag{11}$$

and

$$\|\mathcal{G}(x)\|_{C \times \mathbb{R}^2} \leq K_1 \|x\|_C^\beta + K_2, \tag{12}$$

for some positive real numbers K_1 and K_2 and every $x \in H^2$. Thus, from (11),

$$\gamma_n \frac{\left(\langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \Psi + M(I - Q)\mathcal{G}(x_n)}{\|x_n\|_{H^2}} \xrightarrow{2} 0,$$

so that

$$\gamma_n \frac{Px_n - \left(\langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \Psi + M(I - Q)\mathcal{G}(x_n)}{\|x_n\|_{H^2}} \xrightarrow{2} \gamma Py.$$

Since $y_n \xrightarrow{S} y$, $y_n \xrightarrow{2} y$, so that we conclude $y = \gamma Py$. Applying P gives

$$Py = \gamma P^2 y = \gamma Py,$$

from which we deduce that $\gamma = 1$ or $Py = 0$. Since $\|y\|_{H^2} = 1$, it follows that $\gamma = 1$. Thus, $Py = y$ and we deduce that $y = \pm \frac{1}{\|\Psi\|_{H^2}} \Psi$. We will assume that $y = \frac{1}{\|\Psi\|_{H^2}} \Psi$, as the other case is similar.

Now, by the compact embedding of H^2 in C , we have, since $y_n \xrightarrow{S} y$, that $y_n \rightarrow y$ in C . Using the fact that $y_n \xrightarrow{2} \frac{1}{\|\Psi\|_{H^2}} \Psi$, we have that

$$\langle y_n, \Psi \rangle_2 \rightarrow \frac{1}{\|\Psi\|_{H^2}} \langle \Psi, \Psi \rangle_2 = \frac{1}{\|\Psi\|_{H^2}}. \tag{13}$$

However, $\langle x_n, \Psi \rangle_2 = \|x_n\|_{H^2} \langle y_n, \Psi \rangle_2$, so that $\langle x_n, \Psi \rangle_2 \rightarrow \infty$, since $\|x_n\|_{H^2}$ does (recall $\|x_n\|_{L^2} \rightarrow \infty$). Without loss of generality, we will assume from now on that $\langle x_n, \Psi \rangle_2 > 0$ for each n .

From $x_n = \gamma_n T(x_n)$, it follows that for each n

$$(I - P)x_n = \gamma_n M(I - Q)\mathcal{G}(x_n)$$

and

$$Px_n = \gamma_n Px_n - \gamma_n \left(\langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \Psi.$$

This is equivalent to

$$(I - P)x_n = \gamma_n M(I - Q)\mathcal{G}(x_n) \tag{14}$$

and

$$(1 - \gamma_n)\langle x_n, \Psi \rangle_2 + \gamma_n \left(\langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) = 0. \tag{15}$$

Let v_n denote $(I - P)x_n$. From (12) and (14), we have that

$$\begin{aligned} \|v_n\|_C &\leq |\gamma_n| \|M(I - Q)\| (K_1 \|x_n\|_C^\beta + K_2) \\ &\leq D_1 \|x_n\|_C^\beta + D_2, \end{aligned}$$

where $\|M(I - Q)\|$ denotes the operator norm of $M(I - Q)$ and for $i = 1, 2$, $D_i = \|M(I - Q)\| K_i$. Applying the compact embedding theorem again, we may assume, by scaling each D_i , that

$$\|v_n\|_C \leq D_1 \|x_n\|_{H^2}^\beta + D_2.$$

However, from (13) we have that $\frac{\langle x_n, \Psi \rangle_2}{\|x_n\|_{H^2}} \rightarrow \frac{1}{\|\Psi\|_{H^2}}$, so that by rescaling one more time, we may assume

$$\|v_n\|_C \leq D_1 \langle x_n, \Psi \rangle_2^\beta + D_2. \tag{16}$$

For the moment, fix $t \in \mathcal{O}_+ \cup \mathcal{O}_-$. Since

$$\begin{aligned} |x_n(t)| &\geq \langle x_n, \Psi \rangle_2 |\Psi(t)| - |v_n(t)| \\ &\geq \langle x_n, \Psi \rangle_2 |\Psi(t)| - \|v_n\|_C, \end{aligned}$$

we have, using (16), that

$$\lim_{n \rightarrow \infty} x_n(t) = \pm\infty, \text{ whenever } t \in \mathcal{O}_\pm. \tag{17}$$

Define

$$E_n = \{t \mid |\Psi(t)| \geq \varepsilon_n\},$$

where $\varepsilon_n = \frac{\hat{z} + \|v_n\|_C}{\langle x_n, \Psi \rangle_2}$. If $t \in E_n$, then

$$\begin{aligned} |x_n(t)| &\geq \langle x_n, \Psi \rangle_2 |\Psi(t)| - |v_n(t)| \\ &\geq \langle x_n, \Psi \rangle_2 |\Psi(t)| - \|v_n\|_C, \\ &\geq \langle x_n, \Psi \rangle_2 \left(\frac{\hat{z} + \|v_n\|_C}{\langle x_n, \Psi \rangle_2} \right) - \|v_n\|_C \\ &= \hat{z}. \end{aligned}$$

This gives, using C2, that

$$\begin{aligned} \int_0^1 f(x_n) \Psi dt &= \int_{E_n} f(x_n) \Psi dt + \int_{E_n^c} f(x_n) \Psi dt \\ &\geq J \int_{E_n} |\Psi| dt + \int_{E_n^c} f(x_n) \Psi dt \\ &\geq J \int_{E_n} |\Psi| dt - \int_{E_n^c} |f(x_n) \Psi| dt \end{aligned}$$

We claim that $\int_{E_n^c} |f(x_n) \Psi| dt \rightarrow 0$, so that by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_0^1 f(x_n) \Psi dt &\geq \liminf_{n \rightarrow \infty} J \int_{E_n} |\Psi| dt \\ &= J \int_0^1 |\Psi| dt. \end{aligned} \tag{18}$$

To see that $\int_{E_n^c} |f(x_n) \Psi| dt \rightarrow 0$, first note that for any $t \in E_n^c$

$$\begin{aligned} |x_n(t)| &\leq \langle x_n, \Psi \rangle_2 \varepsilon_n + \|v_n\|_C \\ &\leq \langle x_n, \Psi \rangle_2 \left(\frac{\hat{z} + \|v_n\|_C}{\langle x_n, \Psi \rangle_2} \right) + \|v_n\|_C \\ &= \hat{z} + 2 \|v_n\|_C \\ &\leq \hat{z} + 2(D_1 \langle x_n, \Psi \rangle_2^\beta + D_2) \quad (\text{using (16)}). \end{aligned}$$

It then follows, from C1, that

$$\begin{aligned} |f(x_n)(t)| &\leq M_1 |x_n(t)|^\beta + M_2 \\ &\leq M_1 (\hat{z} + 2(D_1 \langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2, \end{aligned}$$

which gives that

$$\int_{E_n^c} |f(x_n) \Psi| dt \leq (M_1 (\hat{z} + 2(D_1 \langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2) \varepsilon_n \mu_L(E_n^c),$$

where μ_L denotes Lebesgue measure on $[0, 1]$.

Since $\frac{\|v_n\|_C}{\langle x_n, \Psi \rangle_2} \rightarrow 0$, we have that $E_n^c \rightarrow \mathcal{O}_0$. Further, since \mathcal{O}_0 consists of finitely many simple zeros, it follows from the Mean Value Theorem that there exists a positive constant, say L , with

$$\mu_L(E_n^c) \leq L\mathcal{E}_n.$$

We then have that

$$\begin{aligned} \int_{E_n^c} |f(x_n)\Psi| dt &\leq (M_1(\hat{z} + 2(D_1\langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2)L\mathcal{E}_n^2 \\ &= (M_1(\hat{z} + 2(D_1\langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2)L\left(\frac{\hat{z} + \|v_n\|_C}{\langle x_n, \Psi \rangle_2}\right)^2 \\ &\leq (M_1(\hat{z} + 2(D_1\langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2)L\left(\frac{\hat{z} + D_1\langle x_n, \Psi \rangle_2^\beta + D_2}{\langle x_n, \Psi \rangle_2}\right)^2, \end{aligned}$$

so that

$$\int_{E_n^c} |f(x_n)\Psi| dt \leq R\frac{\langle x_n, \Psi \rangle_2^{2\beta^2}}{\langle x_n, \Psi \rangle_2^2},$$

for some positive constant R . Letting $n \rightarrow \infty$, and using the fact that $\beta < 1$, we conclude that $\int_{E_n^c} |f(x_n)\Psi| dt \rightarrow 0$.

We now look to analyze $\liminf_{n \rightarrow \infty} \int_0^1 g_i(x_n) d\mu_i$ and $\limsup_{n \rightarrow \infty} \int_0^1 g_i(x_n) d\mu_i$, for $i = 1, 2$. From (17), if $t \in \mathcal{O}_+$, then

$$g_{i,-}(+\infty) \leq \liminf_{n \rightarrow \infty} g_i(x_n)(t) \text{ and } \limsup_{n \rightarrow \infty} g_i(x_n)(t) \leq g_{i,+}(+\infty).$$

Similarly, for each $t \in \mathcal{O}_-$ and each $i, i = 1, 2$,

$$g_{i,-}(-\infty) \leq \liminf_{n \rightarrow \infty} g_i(x_n)(t) \text{ and } \limsup_{n \rightarrow \infty} g_i(x_n)(t) \leq g_{i,+}(-\infty).$$

Since g_1 and g_2 are bounded, we have, by Fatou’s lemma, that for each i ,

$$\begin{aligned} J_{i,-} &= g_{i,-}(+\infty)\mu_i(\mathcal{O}_+) + g_{i,-}(-\infty)\mu_i(\mathcal{O}_-) \tag{19} \\ &= \int_{\mathcal{O}_+} g_{i,-}(+\infty) d\mu_i + \int_{\mathcal{O}_-} g_{i,-}(-\infty) d\mu_i \\ &\leq \int_{\mathcal{O}_+ \cup \mathcal{O}_-} \liminf_{n \rightarrow \infty} g_i(x_n) d\mu_i \\ &= \int_{[0,1]} \liminf_{n \rightarrow \infty} g_i(x_n) d\mu_i \text{ (using C3)} \\ &\leq \liminf_{n \rightarrow \infty} \int_{[0,1]} g_i(x_n) d\mu_i \\ &\leq \limsup_{n \rightarrow \infty} \int_{[0,1]} g_i(x_n) d\mu_i \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{[0,1]} \limsup_{n \rightarrow \infty} g_i(x_n) d\mu_i \\
 &\leq \int_{\mathcal{O}_+ \cup \mathcal{O}_-} \limsup_{n \rightarrow \infty} g_i(x_n) d\mu_i \\
 &\leq \int_{\mathcal{O}_+} g_{i,+}(+\infty) d\mu_i + \int_{\mathcal{O}_-} g_{i,-}(-\infty) d\mu_i \\
 &= g_{i,+}(+\infty)\mu_i(\mathcal{O}_+) + g_{i,+}(-\infty)\mu_i(\mathcal{O}_-) \\
 &= J_{i,+}.
 \end{aligned}$$

Suppose for the moment that $v_1 > 0$ and $-v_2 > 0$ and let s and r be positive real numbers. Using the definitions of limit inferior and limit superior, see (18) and (19), there exists an n_s and an n_r such that if $n \geq n_s$, then

$$J \int_0^1 |\psi| dt - s < \langle f(x_n), \psi \rangle_2 = \langle \mathcal{F}(x_n), \psi \rangle_2, \tag{20}$$

and if $n \geq n_r$, then

$$J_{i,-} - r < \int_{[0,1]} g_i(x_n) d\mu_i < J_{i,+} + r. \tag{21}$$

Since $v_1 > 0$ and $-v_2 > 0$, it follows that

$$\left\langle \begin{bmatrix} J_{1,-} - r \\ J_{2,-} - r \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \leq \left\langle \begin{bmatrix} \int_{[0,1]} g_1(x_n) d\mu_1 \\ \int_{[0,1]} g_2(x_n) d\mu_2 \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \leq \left\langle \begin{bmatrix} J_{1,+} + r \\ J_{2,+} + r \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}. \tag{22}$$

However,

$$\left\langle \begin{bmatrix} J_{1,-} \\ J_{2,-} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} = \left\langle \begin{bmatrix} J_{1,\text{sgn}(-v_1)} \\ J_{2,\text{sgn}(v_2)} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} > -J \int_0^1 |\psi| dt. \tag{23}$$

Thus, it follows, from (20),(21), (22), and (23), that we may choose r and s small enough so that

$$\langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} > 0, \tag{24}$$

for large enough n . The other cases for the sign of v_1 and $-v_2$ are similar. In each case, the conclusion in (24) holds. Recalling that $\langle x_n, \psi \rangle_2 \rightarrow +\infty$, we have that for large enough n ,

$$(1 - \gamma_n) \langle x_n, \psi \rangle_2 + \gamma_n \left(\langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) > 0.$$

However, this contradicts the fact that by (15),

$$(1 - \gamma_n) \langle x_n, \psi \rangle_2 + \gamma_n \left(\langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) = 0.$$

Thus,

$$FP := \{x \in H^2 \mid x = \gamma T(x) \text{ for some } \gamma \in (0, 1)\}$$

must be a priori bounded, and the proof is complete. \square

REMARK 3.2. If $\eta_1 = \eta_2 = 0$, then by choosing for each i , $i = 1, 2$, $g_i = 0$ and μ_i to be Lebesgue measure on $[0, 1]$, we have that $J_{i,\pm} = 0$. Thus, condition C4 of Theorem 3.1 is trivially satisfied. This shows that Theorem 3.1 is a generalization of the result found in [15], where they analyze linear homogeneous boundary conditions.

The following corollary isolates the special case in which the boundary operators η_1 and η_2 are generated by bounded continuous function g_1 and g_2 for which we assume that for $i = 1, 2$, $g_i(\pm\infty) := \lim_{x \rightarrow \pm\infty} g_i(x)$ exists.

COROLLARY 3.3. *Suppose that the following conditions hold:*

C1*. *The function f is “sublinear”; that is, there exists real numbers M_1, M_2 and β , with $0 \leq \beta < 1$, such that for every $x \in \mathbb{R}$, $|f(x)| \leq M_1|x|^\beta + M_2$;*

C2*. *There exist positive real numbers \hat{z} and J such that for all $z > \hat{z}$,*

$$f(-z) \leq -J < 0 < J \leq f(z);$$

C3*. *For $i = 1, 2$, $\mu_i(\mathcal{O}_0) = 0$, where again u_i is the Borel measure in the definition of the boundary operator η_i ;*

C4*. *For $i = 1, 2$, $g_i(\pm\infty) := \lim_{x \rightarrow \pm\infty} g_i(x)$ exists;*

$$C5*. \quad -J \int_0^1 |\psi| dt < \left\langle \begin{bmatrix} J_{1,+} \\ J_{2,+} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}};$$

then, there exists a solution to (1)–(2).

Proof. If for $i = 1, 2$, $g_i(\pm\infty) := \lim_{x \rightarrow \pm\infty} g_i(x)$ exist, then for each of these i , $J_{i,-} = J_{i,+}$. \square

4. Example

In this section we give a concrete example of the application of our main result, Theorem 3.1. We will use an interval of $[0, \pi]$ to simplify calculations.

Consider

$$x'' + m^2x = f(x(t)) \tag{25}$$

subject to

$$x(0) = \int_{[0,\pi]} g_1(x) du_1 \text{ and } x(\pi) = \int_{[0,\pi]} g_2(x) du_2 \tag{26}$$

where f , g_1 , and g_2 are real-valued continuous functions with g_1 and g_2 bounded.

It is well-known that the L^2 -normalized eigenfunctions corresponding to the Dirichlet problem

$$x'' + m^2x = 0$$

subject to boundary conditions

$$x(0) = 0 \text{ and } x(\pi) = 0,$$

are $\pm \frac{2}{\pi} \sin(mt)$. We choose to take $\psi(t) = \frac{2}{\pi} \sin(mt)$. This gives that ϕ , see (9), is $-\frac{\pi}{2} \cos(mt)$. Thus, $v_1 = \phi(0) = -\frac{\pi}{2}$ and $v_2 = \phi(\pi) = \frac{\pi}{2}$. We also have that

$$\mathcal{O}_+ = \begin{cases} \bigcup_{i=0}^j \left(\frac{2i\pi}{m}, \frac{(2i+1)\pi}{m} \right) & \text{if } m = 2j + 1 \\ \bigcup_{i=0}^{j-1} \left(\frac{2i\pi}{m}, \frac{(2i+1)\pi}{m} \right) & \text{if } m = 2j \end{cases}$$

and

$$\mathcal{O}_- = \begin{cases} \bigcup_{i=0}^{j-1} \left(\frac{(2i+1)\pi}{m}, \frac{(2i+2)\pi}{m} \right) & \text{if } m = 2j + 1 \\ \bigcup_{i=0}^{j-1} \left(\frac{(2i+1)\pi}{m}, \frac{(2i+2)\pi}{m} \right) & \text{if } m = 2j \end{cases}.$$

Suppose for the moment that conditions C1-C3 hold, since these can be trivially satisfied by any number of choices for f and μ_1, μ_2 . Condition C4 of Theorem 3.1 in this specific problem becomes

$$-\frac{4}{\pi} J < \left\langle \begin{bmatrix} J_{1,+} \\ J_{2,+} \end{bmatrix}, \begin{bmatrix} -\frac{2}{\pi} \\ -\frac{2}{\pi} \end{bmatrix} \right\rangle_{\mathbb{R}},$$

which is equivalent to $(J_{1,+} + J_{2,+}) < 2J$. It is clear that there are several bounded continuous functions g_1, g_2 and Borel measures μ_1, μ_2 which make the above inequality valid.

As a concrete example, let $E_m = \{t \mid \sin(mt) = 0\}$ and fix $t_0 \notin E_m$. Take $\mu := \mu_1 = \mu_2$ to be the measure point-supported at t_0 ; that is, for a subset A of $[0, 1]$,

$$\mu(A) = \begin{cases} 1 & \text{if } t_0 \in A \\ 0 & \text{if } t_0 \notin A \end{cases}.$$

Since $t_0 \notin E_m$, we have that t_0 is in \mathcal{O}_+ or \mathcal{O}_- . If for each $i, i = 1, 2, g_i(\pm\infty) := \lim_{x \rightarrow \pm\infty} g_i(x)$ exists, then when $t \in \mathcal{O}_+$, $J_{i,+} = g_i(+\infty)$. Similarly, when $t \in \mathcal{O}_-$, then $J_{i,+} = g_i(-\infty)$. Thus, if $t_0 \in \mathcal{O}_{\pm}$, then provided $g_1(\pm\infty) + g_2(\pm\infty) < 2J$, we have, from Corollary 3.3, that the nonlinear boundary value problem (25)–(26) has a solution. It is interesting to note that if $t_0 \notin \cup_m E_m$, and both $g_1(+\infty) + g_2(+\infty) < 2J$ and $g_1(-\infty) + g_2(-\infty) < 2J$, then (25)–(26) has a solution for all eigenvalues m .

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