

## EXISTENCE THEORY FOR NONLINEAR STURM–LIOUVILLE PROBLEMS WITH NON-LOCAL BOUNDARY CONDITIONS

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*Abstract.* In this work we provide conditions for the existence of solutions to nonlinear Sturm-Liouville problems of the form

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(x(t))$$

subject to non-local boundary conditions

$$ax(0) + bx'(0) = \eta_1(x) \text{ and } cx(1) + dx'(1) = \eta_2(x).$$

Our approach will be topological, utilizing Schaefer's fixed point theorem and the Lyapunov-Schmidt procedure.

### 1. Introduction

In this paper we provide criteria for the solvability of nonlinear Sturm-Liouville problems of the form,

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(x(t)) \quad t \in [0, 1], \quad (1)$$

subject to non-local boundary conditions

$$ax(0) + bx'(0) = \eta_1(x) \text{ and } cx(1) + dx'(1) = \eta_2(x). \quad (2)$$

There are several standard ways in which one may define a solution to problem (1)–(2), and so to maintain completeness, we mention that in this paper we will be interested in proving the existence of classical solutions to (1)–(2). Formally, by a solution to (1)–(2) we mean a function  $x : [0, 1] \rightarrow \mathbb{R}$  such that  $px'$  is continuously differentiable and satisfies (1)–(2).

Throughout our analysis, we will assume that  $p, q : [0, 1] \rightarrow \mathbb{R}$  are continuous,  $p(t) > 0$  for all  $t \in [0, 1]$ ,  $a^2 + b^2 > 0$  and  $c^2 + d^2 > 0$ ,  $\lambda$  is an eigenvalue of the associated linear Sturm-Liouville problem,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, and for  $i = 1, 2$ ,  $\eta_i(x) = \int_{[0,1]} g_i(x) d\mu_i$ , where  $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$  are continuous and  $\mu_1$  and  $\mu_2$  are finite Borel measures on  $[0, 1]$ .

The focus of this paper is the analysis of nonlinear Sturm-Liouville problems at resonance subject to non-local boundary conditions, where by resonance we mean that the linear homogeneous problem (7)–(8) has nontrivial solutions. Since the pioneering work of Landesman-Lazer, [12], much has been written about resonant nonlinear

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Sturm-Liouville boundary value problems with linear boundary conditions. Pertinent references from the point of view of this paper are [2, 4, 5, 6, 8, 11, 12, 14, 15, 18, 19]. Less has been said in regard to problems with nonlocal boundary conditions, even for the case of nonresonance; readers interested in results in this direction may consult [1, 7, 10, 13, 17, 20, 21, 22, 23].

The novelty of this work is due in large part to the generality of the nonlinear boundary conditions  $\eta_1$  and  $\eta_2$ . As an important special case we point out that, by taking  $\mu_1$  and  $\mu_2$  to be point-supported measures, our integral boundary conditions allow for nonlinear multipoint boundary conditions of the form

$$\eta_1(x) = \sum_{k=1}^n f_k(x(t_k)),$$

$$\eta_2(x) = \sum_{j=1}^m h_j(x(t_j)),$$

where each  $f_k, h_j$  is a continuous function and each  $t_k, t_j \in [0, 1]$ .

Our main result, Theorem 3.1, provides conditions for the existence of solutions to (1)–(2) under a suitable interaction of the eigenspace of the linear Sturm-Liouville problem and the nonlinearities in both the differential equation and the boundary conditions. We would like to remark that the result we obtain in Theorem 3.1 constitutes a significant extension of the work found in [15] by allowing for much more generality in the boundary conditions, (2).

## 2. Preliminaries

The nonlinear boundary value problem (1)–(2) will be viewed as an operator equation. We let  $C := C[0, 1]$  denote the space of real-valued continuous functions topologized by the supremum norm,  $\|\cdot\|_C$ . As usual,  $L^2 := L^2[0, 1]$  will denote the space of real-valued square-integrable functions defined on  $[0, 1]$ . The topology on  $L^2$  will be that induced by the standard  $L^2$ -norm,  $\|\cdot\|_{L^2}$ . We use  $H^2$  to denote the Sobelov space of functions with two weak derivatives in  $L^2$ ; that is,

$$H^2 = \{x \in L^2 \mid x' \text{ is absolutely continuous and } x'' \in L^2\}.$$

Unless otherwise stated, the topology on  $H^2$  will be the subspace topology inherited from  $L^2$ . However, we will, on several occasions, topologize  $H^2$  with the Sobelov norm,

$$\|x\|_{H^2} = \|x\|_{L^2} + \|x'\|_{L^2} + \|x''\|_{L^2}.$$

On occasion, we may also view  $H^2$  as a subspace of  $C$ . We will use  $|\cdot|$  to denote the Euclidean norm on  $\mathbb{R}^2$  and  $\langle \cdot, \cdot \rangle_2, \langle \cdot, \cdot \rangle_S$ , and  $\langle \cdot, \cdot \rangle_{\mathbb{R}}$  will denote the inner products on  $L^2, H^2$ , and  $\mathbb{R}^2$ , respectively. Weak convergence in  $L^2$  will be denoted by  $\xrightarrow{L^2}$  and weak convergence in the Sobelov space  $H^2$  will be denoted by  $\xrightarrow{S}$ . We make  $L^2 \times \mathbb{R}^2$

an inner product space with inner product

$$\left\langle \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} g \\ v_1 \\ v_2 \end{bmatrix} \right\rangle := m \left( \langle h, g \rangle_2 + \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle_{\mathbb{R}} \right), \tag{3}$$

where  $m$  is a positive constant which will be chosen later, and we will use  $\|\cdot\|_{L^2 \times \mathbb{R}^2}$  to denote the norm generated by this inner product. Lastly, we give  $C \times \mathbb{R}^2$  the product topology, and we will use  $\|\cdot\|_{C \times \mathbb{R}^2}$  to denote the standard product norm on this space.

Linear boundary operators  $B_1$  and  $B_2$  will be defined as follows:

$B_1 : H^2 \rightarrow \mathbb{R}$  is given by

$$B_1x = ax(0) + bx'(0)$$

and  $B_2 : H^2 \rightarrow \mathbb{R}$  is given by

$$B_2x = cx(1) + dx'(1).$$

We define  $\mathcal{L} : H^2 \rightarrow L^2 \times \mathbb{R}^2$

$$\mathcal{L}x = \begin{bmatrix} \mathcal{A}x \\ B_1x \\ B_2x \end{bmatrix},$$

where  $\mathcal{A} : H^2 \rightarrow L^2$  is defined by

$$\mathcal{A}x(t) = (p(t)x'(t))' + (q(t) + \lambda)x(t).$$

Similarly, we define a nonlinear operator  $\mathcal{G} : H^2 \rightarrow L^2 \times \mathbb{R}^2$  by

$$\mathcal{G}(x) = \begin{bmatrix} \mathcal{F}(x) \\ \eta_1(x) \\ \eta_2(x) \end{bmatrix},$$

where  $\mathcal{F}(x)(t) = f(x(t))$  and, as before, for  $i = 1, 2$ ,  $\eta_i(x) = \int_{[0,1]} g_i(x) d\mu_i$ . Solving the nonlinear boundary value problem (1)–(2) is now equivalent to solving

$$\mathcal{L}x = \mathcal{G}(x). \tag{4}$$

The study of the nonlinear boundary value problem (1)–(2) will be intimately related to the linear nonhomogeneous boundary value problem

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = h(t), \quad t \in [0, 1] \tag{5}$$

$$ax(0) + bx'(0) = w_1 \quad \text{and} \quad cx(1) + dx'(1) = w_2, \tag{6}$$

where  $h$  is an element of  $L^2$  and  $w_1$  and  $w_2$  are elements of  $\mathbb{R}$ . Using our notation from above, we have that solving (5)–(6) is equivalent to solving

$$\mathcal{L}x = \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}.$$

We begin our study of the nonlinear boundary value problem (1)–(2) by analyzing (5)–(6). To aid in this analysis, we first recall some well-known facts regarding the linear homogeneous Sturm-Liouville problem

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = 0 \tag{7}$$

$$ax(0) + bx'(0) = 0 \text{ and } cx(1) + dx'(1) = 0. \tag{8}$$

For those readers interested in a more detailed introduction to linear Sturm-Liouville problems, we suggest [9].

It is well known that  $\lambda$  is a simple eigenvalue; that is,  $\text{Ker}(\mathcal{L})$  is one-dimensional. We may therefore choose a vector,  $\psi$ , which forms a basis for  $\text{Ker}(\mathcal{L})$ . Without loss of generality, we will assume  $\|\psi\|_{L^2} = 1$ . Since (7) is a second-order linear homogeneous differential equation, we may choose  $\phi$  satisfying (7) so that  $\{\psi, \phi\}$  forms a basis for the solution space of this linear homogeneous problem. We will assume  $\langle \psi, \phi \rangle_2 = 0$ .

For  $u, v \in H^2$ , let  $wr(u, v)$  denote the Wronskian of  $u$  and  $v$ ; that is,  $wr(u, v) = uv' - vu'$ . It follows from standard ode theory that if  $u$  and  $v$  are linearly independent solutions to (7), then  $p \cdot wr(u, v)$  is a nonzero constant. We will assume that  $\phi$  has been chosen so that  $p \cdot wr(\psi, \phi) = 1$  and define  $\omega : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  by

$$\omega(t, s) = \begin{cases} \psi(t)\phi(s) & \text{if } 0 \leq t \leq s \leq 1 \\ \psi(s)\phi(t) & \text{if } 0 \leq s \leq t \leq 1 \end{cases}. \tag{9}$$

As a reminder to the reader,  $\omega$  is often referred to as a fundamental solution of (7).

If we define  $K : L^2 \rightarrow H^2$  by

$$Kh(t) = \int_0^1 \omega(t, s)h(s)ds, \tag{10}$$

then it is easy to verify that  $K$  is self-adjoint, compact, and satisfies  $\mathcal{L}Kh = h$  for every  $h \in L^2$ . Differentiating under the integral symbol, one easily establishes that for every  $h \in L^2$ ,  $B_1Kh = \langle h, \phi \rangle_2 B_1\psi = 0$  and  $B_2Kh = \langle h, \psi \rangle_2 B_2\phi$ . Let

$$v_1 = B_1\phi \text{ and } v_2 = B_2\phi.$$

Since  $\phi$  satisfies (7) and is linearly independent of  $\psi$ , we must have  $B_1\phi \neq 0$  and  $B_2\phi \neq 0$ ; this is a consequence of the uniqueness of solutions to initial value problems and that fact the linear Sturm-Liouville boundary conditions can be thought of as an orthogonality condition.

With the above ideas in hand, we are now in a position characterize the range of  $\mathcal{L}$ . We have the following result.

PROPOSITION 2.1. *Let  $h \in L^2$  and  $w_1, w_2 \in \mathbb{R}$ . Then  $\vec{h} := \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix} \in \text{Im}(\mathcal{L})$  if*

*and only if  $\langle \vec{h}, \vec{\psi} \rangle = 0$ , where  $\vec{\psi} := \begin{bmatrix} \psi \\ v_1^{-1} \\ -v_2^{-1} \end{bmatrix}$ . That is, in  $L^2 \times \mathbb{R}^2$ ,  $\text{Im}(\mathcal{L}) = \{\vec{\psi}\}^\perp$ .*

*Proof.*  $\mathcal{L}x = \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}$  if and only if  $\mathcal{A}x = h$ ,  $B_1x = w_1$ , and  $B_2x = w_2$ . However,

$\mathcal{A}x = h$  if and only if  $x = c_1\psi + c_2\phi + Kh$ , for some real numbers  $c_1, c_2$ . Applying the boundary map  $B_1$  and recalling  $B_1Kh = 0$ , we get  $B_1(c_1\psi + c_2\phi + Kh) = c_2v_1$ . Similarly, using  $B_2Kh = \langle h, \psi \rangle_2 B_2\phi$ , we get  $B_2(c_1\psi + c_2\phi + Kh) = (c_2 + \langle h, \psi \rangle_2)v_2$ .

Now,

$$c_2v_1 = w_1 \quad \text{and} \quad (c_2 + \langle h, \psi \rangle_2)v_2 = w_2$$

if and only if

$$c_2 = \frac{w_1}{v_1} \quad \text{and} \quad \langle h, \psi \rangle_2 = \frac{w_2}{v_2} - \frac{w_1}{v_1} = \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} -v_1^{-1} \\ v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}},$$

which happens if and only if  $\langle \vec{h}, \vec{\psi} \rangle = 0$ .  $\square$

With this characterization of the  $Im(\mathcal{L})$  in hand, we make the following definitions which will play a crucial role in our ability to analyze the nonlinear Sturm-Liouville problem, (1)–(2), using a projection scheme.

DEFINITION 2.2. Define  $P : L^2 \rightarrow L^2$  by  $Px = \langle x, \psi \rangle_2 \psi$ .

It is clear that  $P$  is the orthogonal projection onto  $Ker(\mathcal{L})$ .

Now, choose  $m$ , see (3), to be  $\frac{1}{1 + \left\| \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\|^2}$ . With this choice of  $m$ ,  $\vec{\psi}$  is a unit

vector in  $L^2 \times \mathbb{R}^2$ .

DEFINITION 2.3. Define  $Q : L^2 \times \mathbb{R}^2 \rightarrow L^2 \times \mathbb{R}^2$  by

$$Q \left( \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix} \right) = \left\langle \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}, \vec{\psi} \right\rangle \vec{\psi}.$$

From Proposition 2.1, we have that  $Q$  is the orthogonal projection of  $L^2 \times \mathbb{R}^2$  on  $Im(\mathcal{L})^\perp$ . Thus,  $I - Q$ , is a projection onto the  $Im(\mathcal{L})$ .

In our analysis of the nonlinear Sturm-Liouville problem we will use a projection scheme often referred to as the Lyapunov-Schmidt procedure. The use of the Lyapunov-Schmidt reduction will allow us to write the operator equation (4) as an equivalent equation in which a fixed point argument may be applied to prove the existence of solutions. Interested readers may consult [3, 16] for a more detailed account of these ideas.

PROPOSITION 2.4. Solving  $\mathcal{L}x = \mathcal{G}(x)$  is equivalent to solving the system

$$\left\{ \begin{array}{l} (I - P)x - M(I - Q)\mathcal{G}(x) = 0 \\ \text{and} \\ \left( \langle \mathcal{F}(x), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \psi = 0 \end{array} \right.,$$

where  $M$  denotes  $(L|_{H^2 \cap \text{Ker}(\mathcal{L})^\perp})^{-1}$ .

*Proof.*

$$\begin{aligned}
 \mathcal{L}x = \mathcal{G}(x) &\iff \begin{cases} (I-Q)(\mathcal{L}x - \mathcal{G}(x)) = 0 \\ \text{and} \\ Q(\mathcal{L}x - \mathcal{G}(x)) = 0 \end{cases} \\
 &\iff \begin{cases} \mathcal{L}x - (I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ Q\mathcal{G}(x) = 0 \end{cases} \\
 &\iff \begin{cases} M\mathcal{L}x - M(I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ Q\mathcal{G}(x) = 0 \end{cases} \\
 &\iff \begin{cases} (I-P)x - M(I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ \left\langle \begin{bmatrix} \mathcal{F}(x) \\ \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} \psi \\ v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \bar{\psi} = 0 \end{cases} \\
 &\iff \begin{cases} (I-P)x - M(I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ \left\langle \begin{bmatrix} \mathcal{F}(x) \\ \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} \psi \\ v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \psi = 0 \end{cases} \\
 &\iff \begin{cases} (I-P)x - M(I-Q)\mathcal{G}(x) = 0 \\ \text{and} \\ (\langle \mathcal{F}(x), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}) \psi = 0 \end{cases} . \quad \square
 \end{aligned}$$

### 3. Main results

We now come to our main result. In what follows, we will assume that the nonlinear integral boundary operators  $\eta_1$  and  $\eta_2$  are induced by bounded continuous functions  $g_1$  and  $g_2$ .

To simplify the statement of the theorem, we introduce the following notation. For  $i = 1, 2$ , we let

$$g_{i,+}(+\infty) := \limsup_{x \rightarrow \infty} g_i(x),$$

$$g_{i,-}(+\infty) := \liminf_{x \rightarrow \infty} g_i(x),$$

$$g_{i,+}(-\infty) := \limsup_{x \rightarrow -\infty} g_i(x),$$

and

$$g_{i,-}(-\infty) := \liminf_{x \rightarrow -\infty} g_i(x).$$

We define  $\mathcal{O}_0 := \{t \mid \psi(t) = 0\}$ ,  $\mathcal{O}_+ := \{t \mid \psi(t) > 0\}$ , and  $\mathcal{O}_- := \{t \mid \psi(t) < 0\}$ . From Standard Sturm-Liouville theory, we have that  $\mathcal{O}_0$  is a finite set consisting of simple zeros. In what follows, this fact will be used several times, possibly without explicit mention. Finally, for  $i = 1, 2$ , we let

$$J_{i,\pm} = g_{i,\pm}(+\infty)\mu_i(\mathcal{O}_+) + g_{i,\pm}(-\infty)\mu_i(\mathcal{O}_-).$$

**THEOREM 3.1.** *Suppose that the following conditions hold:*

- C1. *The function  $f$  is “sublinear”; that is, there exists real numbers  $M_1, M_2$  and  $\beta$ , with  $0 \leq \beta < 1$ , such that for every  $x \in \mathbb{R}$ ,  $|f(x)| \leq M_1|x|^\beta + M_2$ ;*
- C2. *There exist positive real numbers  $\hat{z}$  and  $J$  such that for all  $z > \hat{z}$ ,*

$$f(-z) \leq -J < 0 < J \leq f(z);$$

- C3. *For  $i = 1, 2$ ,  $\mu_i(\mathcal{O}_0) = 0$ , where again  $\mu_i$  is the Borel measure in the definition of the boundary operator  $\eta_i$ ;*

C4.  $-J \int_0^1 |\psi| dt < \left\langle \begin{bmatrix} J_{1,\text{sgn}(-v_1)} \\ J_{2,\text{sgn}(v_2)} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}$ , where for a real number,  $v$ ,  $\text{sgn}(v) = +$  if  $v > 0$  and  $\text{sgn}(v) = -$  if  $v < 0$ ;

then, there exists a solution to (1)–(2).

*Proof.* We start by defining  $T : L^2 \rightarrow H^2$  by

$$T(x) = Px - \left( \langle \mathcal{F}(x), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \psi + M(I - Q)\mathcal{G}(x).$$

From Proposition 2.4, we have that the solutions to (1)–(2) are the fixed points of  $T$ . Since  $M$  is an integral mapping from  $L^2$  into  $H^2$ , it is compact, and thus so is  $T$ . We will show that

$$FP := \{x \in H^2 \mid x = \gamma T(x) \text{ for some } \gamma \in (0, 1)\}$$

is a priori bounded in  $L^2$ . A fixed point will then follow from an application of Schaefer’s fixed point theorem.

To this end, suppose that there exist sequences  $\{x_n\}_{n \in \mathbb{N}}$  and  $\{\gamma_n\}_{n \in \mathbb{N}}$  in  $H^2$  and  $(0, 1)$ , respectively, with  $\|x_n\|_{L^2} \rightarrow \infty$  and  $x_n = \gamma_n T(x_n)$ . Let  $y_n = \frac{x_n}{\|x_n\|_{H^2}}$ . Since

the closed unit ball in the Sobelov space  $H^2$  is weakly compact, by going to a subsequence if necessary, we may assume that  $y_n \xrightarrow{S} y$ , for some  $y \in H^2$ . Again, going to a subsequence if necessary, we may assume that  $\gamma_n$  converges to some  $\gamma \in [0, 1]$ .

Now,

$$\begin{aligned} y_n &= \frac{x_n}{\|x_n\|_{H^2}} \\ &= \gamma_n \frac{T(x_n)}{\|x_n\|_{H^2}} \\ &= \gamma_n \frac{Px_n - \left( \langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \Psi + M(I - Q)\mathcal{G}(x_n)}{\|x_n\|_{H^2}}. \end{aligned}$$

Since  $f$  is sublinear (See C1) and  $g_1$  and  $g_2$  are bounded, it follows that

$$\|\mathcal{G}(x)\|_{L^2 \times \mathbb{R}^2} \leq K_1 \|x\|_{L^2}^\beta + K_2, \tag{11}$$

and

$$\|\mathcal{G}(x)\|_{C \times \mathbb{R}^2} \leq K_1 \|x\|_C^\beta + K_2, \tag{12}$$

for some positive real numbers  $K_1$  and  $K_2$  and every  $x \in H^2$ . Thus, from (11),

$$\gamma_n \frac{\left( \langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \Psi + M(I - Q)\mathcal{G}(x_n)}{\|x_n\|_{H^2}} \xrightarrow{2} 0,$$

so that

$$\gamma_n \frac{Px_n - \left( \langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \Psi + M(I - Q)\mathcal{G}(x_n)}{\|x_n\|_{H^2}} \xrightarrow{2} \gamma Py.$$

Since  $y_n \xrightarrow{S} y$ ,  $y_n \xrightarrow{2} y$ , so that we conclude  $y = \gamma Py$ . Applying  $P$  gives

$$Py = \gamma P^2 y = \gamma Py,$$

from which we deduce that  $\gamma = 1$  or  $Py = 0$ . Since  $\|y\|_{H^2} = 1$ , it follows that  $\gamma = 1$ . Thus,  $Py = y$  and we deduce that  $y = \pm \frac{1}{\|\Psi\|_{H^2}} \Psi$ . We will assume that  $y = \frac{1}{\|\Psi\|_{H^2}} \Psi$ , as the other case is similar.

Now, by the compact embedding of  $H^2$  in  $C$ , we have, since  $y_n \xrightarrow{S} y$ , that  $y_n \rightarrow y$  in  $C$ . Using the fact that  $y_n \xrightarrow{2} \frac{1}{\|\Psi\|_{H^2}} \Psi$ , we have that

$$\langle y_n, \Psi \rangle_2 \rightarrow \frac{1}{\|\Psi\|_{H^2}} \langle \Psi, \Psi \rangle_2 = \frac{1}{\|\Psi\|_{H^2}}. \tag{13}$$



However,  $\langle x_n, \Psi \rangle_2 = \|x_n\|_{H^2} \langle y_n, \Psi \rangle_2$ , so that  $\langle x_n, \Psi \rangle_2 \rightarrow \infty$ , since  $\|x_n\|_{H^2}$  does (recall  $\|x_n\|_{L^2} \rightarrow \infty$ ). Without loss of generality, we will assume from now on that  $\langle x_n, \Psi \rangle_2 > 0$  for each  $n$ .

From  $x_n = \gamma_n T(x_n)$ , it follows that for each  $n$

$$(I - P)x_n = \gamma_n M(I - Q)\mathcal{G}(x_n)$$

and

$$Px_n = \gamma_n Px_n - \gamma_n \left( \langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \Psi.$$

This is equivalent to

$$(I - P)x_n = \gamma_n M(I - Q)\mathcal{G}(x_n) \tag{14}$$

and

$$(1 - \gamma_n)\langle x_n, \Psi \rangle_2 + \gamma_n \left( \langle \mathcal{F}(x_n), \Psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) = 0. \tag{15}$$

Let  $v_n$  denote  $(I - P)x_n$ . From (12) and (14), we have that

$$\begin{aligned} \|v_n\|_C &\leq |\gamma_n| \|M(I - Q)\| (K_1 \|x_n\|_C^\beta + K_2) \\ &\leq D_1 \|x_n\|_C^\beta + D_2, \end{aligned}$$

where  $\|M(I - Q)\|$  denotes the operator norm of  $M(I - Q)$  and for  $i = 1, 2$ ,  $D_i = \|M(I - Q)\| K_i$ . Applying the compact embedding theorem again, we may assume, by scaling each  $D_i$ , that

$$\|v_n\|_C \leq D_1 \|x_n\|_{H^2}^\beta + D_2.$$

However, from (13) we have that  $\frac{\langle x_n, \Psi \rangle_2}{\|x_n\|_{H^2}} \rightarrow \frac{1}{\|\Psi\|_{H^2}}$ , so that by rescaling one more time, we may assume

$$\|v_n\|_C \leq D_1 \langle x_n, \Psi \rangle_2^\beta + D_2. \tag{16}$$

For the moment, fix  $t \in \mathcal{O}_+ \cup \mathcal{O}_-$ . Since

$$\begin{aligned} |x_n(t)| &\geq \langle x_n, \Psi \rangle_2 |\Psi(t)| - |v_n(t)| \\ &\geq \langle x_n, \Psi \rangle_2 |\Psi(t)| - \|v_n\|_C, \end{aligned}$$

we have, using (16), that

$$\lim_{n \rightarrow \infty} x_n(t) = \pm\infty, \text{ whenever } t \in \mathcal{O}_\pm. \tag{17}$$

Define

$$E_n = \{t \mid |\Psi(t)| \geq \varepsilon_n\},$$

where  $\varepsilon_n = \frac{\hat{z} + \|v_n\|_C}{\langle x_n, \Psi \rangle_2}$ . If  $t \in E_n$ , then

$$\begin{aligned} |x_n(t)| &\geq \langle x_n, \Psi \rangle_2 |\Psi(t)| - |v_n(t)| \\ &\geq \langle x_n, \Psi \rangle_2 |\Psi(t)| - \|v_n\|_C, \\ &\geq \langle x_n, \Psi \rangle_2 \left( \frac{\hat{z} + \|v_n\|_C}{\langle x_n, \Psi \rangle_2} \right) - \|v_n\|_C \\ &= \hat{z}. \end{aligned}$$

This gives, using C2, that

$$\begin{aligned} \int_0^1 f(x_n) \Psi dt &= \int_{E_n} f(x_n) \Psi dt + \int_{E_n^c} f(x_n) \Psi dt \\ &\geq J \int_{E_n} |\Psi| dt + \int_{E_n^c} f(x_n) \Psi dt \\ &\geq J \int_{E_n} |\Psi| dt - \int_{E_n^c} |f(x_n) \Psi| dt \end{aligned}$$

We claim that  $\int_{E_n^c} |f(x_n) \Psi| dt \rightarrow 0$ , so that by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_0^1 f(x_n) \Psi dt &\geq \liminf_{n \rightarrow \infty} J \int_{E_n} |\Psi| dt \\ &= J \int_0^1 |\Psi| dt. \end{aligned} \tag{18}$$

To see that  $\int_{E_n^c} |f(x_n) \Psi| dt \rightarrow 0$ , first note that for any  $t \in E_n^c$

$$\begin{aligned} |x_n(t)| &\leq \langle x_n, \Psi \rangle_2 \varepsilon_n + \|v_n\|_C \\ &\leq \langle x_n, \Psi \rangle_2 \left( \frac{\hat{z} + \|v_n\|_C}{\langle x_n, \Psi \rangle_2} \right) + \|v_n\|_C \\ &= \hat{z} + 2 \|v_n\|_C \\ &\leq \hat{z} + 2(D_1 \langle x_n, \Psi \rangle_2^\beta + D_2) \quad (\text{using (16)}). \end{aligned}$$

It then follows, from C1, that

$$\begin{aligned} |f(x_n)(t)| &\leq M_1 |x_n(t)|^\beta + M_2 \\ &\leq M_1 (\hat{z} + 2(D_1 \langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2, \end{aligned}$$

which gives that

$$\int_{E_n^c} |f(x_n) \Psi| dt \leq (M_1 (\hat{z} + 2(D_1 \langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2) \varepsilon_n \mu_L(E_n^c),$$

where  $\mu_L$  denotes Lebesgue measure on  $[0, 1]$ .

Since  $\frac{\|v_n\|_C}{\langle x_n, \Psi \rangle_2} \rightarrow 0$ , we have that  $E_n^c \rightarrow \mathcal{O}_0$ . Further, since  $\mathcal{O}_0$  consists of finitely many simple zeros, it follows from the Mean Value Theorem that there exists a positive constant, say  $L$ , with

$$\mu_L(E_n^c) \leq L\mathcal{E}_n.$$

We then have that

$$\begin{aligned} \int_{E_n^c} |f(x_n)\Psi| dt &\leq (M_1(\hat{z} + 2(D_1\langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2)L\mathcal{E}_n^2 \\ &= (M_1(\hat{z} + 2(D_1\langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2)L\left(\frac{\hat{z} + \|v_n\|_C}{\langle x_n, \Psi \rangle_2}\right)^2 \\ &\leq (M_1(\hat{z} + 2(D_1\langle x_n, \Psi \rangle_2^\beta + D_2))^\beta + M_2)L\left(\frac{\hat{z} + D_1\langle x_n, \Psi \rangle_2^\beta + D_2}{\langle x_n, \Psi \rangle_2}\right)^2, \end{aligned}$$

so that

$$\int_{E_n^c} |f(x_n)\Psi| dt \leq R \frac{\langle x_n, \Psi \rangle_2^{2\beta^2}}{\langle x_n, \Psi \rangle_2^2},$$

for some positive constant  $R$ . Letting  $n \rightarrow \infty$ , and using the fact that  $\beta < 1$ , we conclude that  $\int_{E_n^c} |f(x_n)\Psi| dt \rightarrow 0$ .

We now look to analyze  $\liminf_{n \rightarrow \infty} \int_0^1 g_i(x_n) d\mu_i$  and  $\limsup_{n \rightarrow \infty} \int_0^1 g_i(x_n) d\mu_i$ , for  $i = 1, 2$ . From (17), if  $t \in \mathcal{O}_+$ , then

$$g_{i,-}(+\infty) \leq \liminf_{n \rightarrow \infty} g_i(x_n)(t) \text{ and } \limsup_{n \rightarrow \infty} g_i(x_n)(t) \leq g_{i,+}(+\infty).$$

Similarly, for each  $t \in \mathcal{O}_-$  and each  $i, i = 1, 2$ ,

$$g_{i,-}(-\infty) \leq \liminf_{n \rightarrow \infty} g_i(x_n)(t) \text{ and } \limsup_{n \rightarrow \infty} g_i(x_n)(t) \leq g_{i,+}(-\infty).$$

Since  $g_1$  and  $g_2$  are bounded, we have, by Fatou’s lemma, that for each  $i$ ,

$$\begin{aligned} J_{i,-} &= g_{i,-}(+\infty)\mu_i(\mathcal{O}_+) + g_{i,-}(-\infty)\mu_i(\mathcal{O}_-) \tag{19} \\ &= \int_{\mathcal{O}_+} g_{i,-}(+\infty) d\mu_i + \int_{\mathcal{O}_-} g_{i,-}(-\infty) d\mu_i \\ &\leq \int_{\mathcal{O}_+ \cup \mathcal{O}_-} \liminf_{n \rightarrow \infty} g_i(x_n) d\mu_i \\ &= \int_{[0,1]} \liminf_{n \rightarrow \infty} g_i(x_n) d\mu_i \text{ (using C3)} \\ &\leq \liminf_{n \rightarrow \infty} \int_{[0,1]} g_i(x_n) d\mu_i \\ &\leq \limsup_{n \rightarrow \infty} \int_{[0,1]} g_i(x_n) d\mu_i \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{[0,1]} \limsup_{n \rightarrow \infty} g_i(x_n) d\mu_i \\
 &\leq \int_{\mathcal{O}_+ \cup \mathcal{O}_-} \limsup_{n \rightarrow \infty} g_i(x_n) d\mu_i \\
 &\leq \int_{\mathcal{O}_+} g_{i,+}(+\infty) d\mu_i + \int_{\mathcal{O}_-} g_{i,-}(-\infty) d\mu_i \\
 &= g_{i,+}(+\infty)\mu_i(\mathcal{O}_+) + g_{i,+}(-\infty)\mu_i(\mathcal{O}_-) \\
 &= J_{i,+}.
 \end{aligned}$$

Suppose for the moment that  $v_1 > 0$  and  $-v_2 > 0$  and let  $s$  and  $r$  be positive real numbers. Using the definitions of limit inferior and limit superior, see (18) and (19), there exists an  $n_s$  and an  $n_r$  such that if  $n \geq n_s$ , then

$$J \int_0^1 |\psi| dt - s < \langle f(x_n), \psi \rangle_2 = \langle \mathcal{F}(x_n), \psi \rangle_2, \tag{20}$$

and if  $n \geq n_r$ , then

$$J_{i,-} - r < \int_{[0,1]} g_i(x_n) d\mu_i < J_{i,+} + r. \tag{21}$$

Since  $v_1 > 0$  and  $-v_2 > 0$ , it follows that

$$\left\langle \begin{bmatrix} J_{1,-} - r \\ J_{2,-} - r \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \leq \left\langle \begin{bmatrix} \int_{[0,1]} g_1(x_n) d\mu_1 \\ \int_{[0,1]} g_2(x_n) d\mu_2 \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \leq \left\langle \begin{bmatrix} J_{1,+} + r \\ J_{2,+} + r \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}. \tag{22}$$

However,

$$\left\langle \begin{bmatrix} J_{1,-} \\ J_{2,-} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} = \left\langle \begin{bmatrix} J_{1,\text{sgn}(-v_1)} \\ J_{2,\text{sgn}(v_2)} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} > -J \int_0^1 |\psi| dt. \tag{23}$$

Thus, it follows, from (20),(21), (22), and (23), that we may choose  $r$  and  $s$  small enough so that

$$\langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} > 0, \tag{24}$$

for large enough  $n$ . The other cases for the sign of  $v_1$  and  $-v_2$  are similar. In each case, the conclusion in (24) holds. Recalling that  $\langle x_n, \psi \rangle_2 \rightarrow +\infty$ , we have that for large enough  $n$ ,

$$(1 - \gamma_n) \langle x_n, \psi \rangle_2 + \gamma_n \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) > 0.$$

However, this contradicts the fact that by (15),

$$(1 - \gamma_n) \langle x_n, \psi \rangle_2 + \gamma_n \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) = 0.$$

Thus,

$$FP := \{x \in H^2 \mid x = \gamma T(x) \text{ for some } \gamma \in (0, 1)\}$$

must be a priori bounded, and the proof is complete.  $\square$

REMARK 3.2. If  $\eta_1 = \eta_2 = 0$ , then by choosing for each  $i$ ,  $i = 1, 2$ ,  $g_i = 0$  and  $\mu_i$  to be Lebesgue measure on  $[0, 1]$ , we have that  $J_{i,\pm} = 0$ . Thus, condition C4 of Theorem 3.1 is trivially satisfied. This shows that Theorem 3.1 is a generalization of the result found in [15], where they analyze linear homogeneous boundary conditions.

The following corollary isolates the special case in which the boundary operators  $\eta_1$  and  $\eta_2$  are generated by bounded continuous function  $g_1$  and  $g_2$  for which we assume that for  $i = 1, 2$ ,  $g_i(\pm\infty) := \lim_{x \rightarrow \pm\infty} g_i(x)$  exists.

COROLLARY 3.3. *Suppose that the following conditions hold:*

C1\*. *The function  $f$  is “sublinear”; that is, there exists real numbers  $M_1, M_2$  and  $\beta$ , with  $0 \leq \beta < 1$ , such that for every  $x \in \mathbb{R}$ ,  $|f(x)| \leq M_1|x|^\beta + M_2$ ;*

C2\*. *There exist positive real numbers  $\hat{z}$  and  $J$  such that for all  $z > \hat{z}$ ,*

$$f(-z) \leq -J < 0 < J \leq f(z);$$

C3\*. *For  $i = 1, 2$ ,  $\mu_i(\mathcal{O}_0) = 0$ , where again  $u_i$  is the Borel measure in the definition of the boundary operator  $\eta_i$ ;*

C4\*. *For  $i = 1, 2$ ,  $g_i(\pm\infty) := \lim_{x \rightarrow \pm\infty} g_i(x)$  exists;*

$$C5*. \quad -J \int_0^1 |\psi| dt < \left\langle \begin{bmatrix} J_{1,+} \\ J_{2,+} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}};$$

*then, there exists a solution to (1)–(2).*

*Proof.* If for  $i = 1, 2$ ,  $g_i(\pm\infty) := \lim_{x \rightarrow \pm\infty} g_i(x)$  exist, then for each of these  $i$ ,  $J_{i,-} = J_{i,+}$ .  $\square$

#### 4. Example

In this section we give a concrete example of the application of our main result, Theorem 3.1. We will use an interval of  $[0, \pi]$  to simplify calculations.

Consider

$$x'' + m^2x = f(x(t)) \tag{25}$$

subject to

$$x(0) = \int_{[0,\pi]} g_1(x) du_1 \text{ and } x(\pi) = \int_{[0,\pi]} g_2(x) du_2 \tag{26}$$

where  $f$ ,  $g_1$ , and  $g_2$  are real-valued continuous functions with  $g_1$  and  $g_2$  bounded.

It is well-known that the  $L^2$ -normalized eigenfunctions corresponding to the Dirichlet problem

$$x'' + m^2x = 0$$

subject to boundary conditions

$$x(0) = 0 \text{ and } x(\pi) = 0,$$

are  $\pm \frac{2}{\pi} \sin(mt)$ . We choose to take  $\psi(t) = \frac{2}{\pi} \sin(mt)$ . This gives that  $\phi$ , see (9), is  $-\frac{\pi}{2} \cos(mt)$ . Thus,  $v_1 = \phi(0) = -\frac{\pi}{2}$  and  $v_2 = \phi(\pi) = \frac{\pi}{2}$ . We also have that

$$\mathcal{O}_+ = \begin{cases} \bigcup_{i=0}^j \left( \frac{2i\pi}{m}, \frac{(2i+1)\pi}{m} \right) & \text{if } m = 2j + 1 \\ \bigcup_{i=0}^{j-1} \left( \frac{2i\pi}{m}, \frac{(2i+1)\pi}{m} \right) & \text{if } m = 2j \end{cases}$$

and

$$\mathcal{O}_- = \begin{cases} \bigcup_{i=0}^{j-1} \left( \frac{(2i+1)\pi}{m}, \frac{(2i+2)\pi}{m} \right) & \text{if } m = 2j + 1 \\ \bigcup_{i=0}^{j-1} \left( \frac{(2i+1)\pi}{m}, \frac{(2i+2)\pi}{m} \right) & \text{if } m = 2j \end{cases}.$$

Suppose for the moment that conditions C1-C3 hold, since these can be trivially satisfied by any number of choices for  $f$  and  $\mu_1, \mu_2$ . Condition C4 of Theorem 3.1 in this specific problem becomes

$$-\frac{4}{\pi} J < \left\langle \begin{bmatrix} J_{1,+} \\ J_{2,+} \end{bmatrix}, \begin{bmatrix} -\frac{2}{\pi} \\ -\frac{2}{\pi} \end{bmatrix} \right\rangle_{\mathbb{R}},$$

which is equivalent to  $(J_{1,+} + J_{2,+}) < 2J$ . It is clear that there are several bounded continuous functions  $g_1, g_2$  and Borel measures  $\mu_1, \mu_2$  which make the above inequality valid.

As a concrete example, let  $E_m = \{t \mid \sin(mt) = 0\}$  and fix  $t_0 \notin E_m$ . Take  $\mu := \mu_1 = \mu_2$  to be the measure point-supported at  $t_0$ ; that is, for a subset  $A$  of  $[0, 1]$ ,

$$\mu(A) = \begin{cases} 1 & \text{if } t_0 \in A \\ 0 & \text{if } t_0 \notin A \end{cases}.$$

Since  $t_0 \notin E_m$ , we have that  $t_0$  is in  $\mathcal{O}_+$  or  $\mathcal{O}_-$ . If for each  $i, i = 1, 2, g_i(\pm\infty) := \lim_{x \rightarrow \pm\infty} g_i(x)$  exists, then when  $t \in \mathcal{O}_+$ ,  $J_{i,+} = g_i(+\infty)$ . Similarly, when  $t \in \mathcal{O}_-$ , then  $J_{i,+} = g_i(-\infty)$ . Thus, if  $t_0 \in \mathcal{O}_{\pm}$ , then provided  $g_1(\pm\infty) + g_2(\pm\infty) < 2J$ , we have, from Corollary 3.3, that the nonlinear boundary value problem (25)–(26) has a solution. It is interesting to note that if  $t_0 \notin \cup_m E_m$ , and both  $g_1(+\infty) + g_2(+\infty) < 2J$  and  $g_1(-\infty) + g_2(-\infty) < 2J$ , then (25)–(26) has a solution for all eigenvalues  $m$ .

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