Differential
Equations
& Applications
Volume 10, Number 2 (2018), 209–218 doi:10.7153/dea-2018-10-13

LOWER BOUNDS FOR THE FIRST ZERO FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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(*Communicated by Mervan Pašić*)

Abstract. We consider establishing lower bounds for the first zero of the solution of the nonlinear second order initial value problem

$$(p(x)y'(x))' + f(x,y(x)) = 0, \ \ x \geqslant 0$$
$$y(0) = a > 0, \ \ y'(0) = 0.$$

Using the linear case as a starting point, we prove several of these theorems, comparing them by considering several examples.

We consider the initial value problem (IVP):

$$(p(x)y'(x))' + f(x,y(x)) = 0, \ \ x \geqslant 0$$
$$y(0) = a > 0, \ \ y'(0) = 0,$$

where $p : [0,\infty) \to (0,\infty)$ and $f : [0,\infty) \times (0,\infty) \to \mathbb{R}$ is allowed to be nonlinear. We assume a solution y exists and has a zero to the right of 0. We let b represent the smallest zero of y greater than 0. We define a solution y to satisfy i) $p(x)y'(x)$ is absolutely continuous on $[0,b]$, ii) $(p(x)y'(x))' + f(x,y(x)) = 0$ a.e. on $[0,b]$ and iii) $y(0) = a$, $y'(0) = 0$. (Note that $y(x) > 0$ on $[0,b)$.) The goal of this article is to construct a lower bound for b. This question was originally motivated by [1] in which a similar problem was studied for the linear differential equation $(p(x)y'(x))' + q(x)y(x) = 0$. (The linear case for this and similar problems can also be found in [2], [3], [4], [5], [6], and [7].) The nonlinear problem, while similar in some respects, involves an additional difficulty which we explain as follows. Consider the linear problem

$$y'' + y = 0$$
$$y(0) = a > 0, \ \ y'(0) = 0$$

Elementary techniques yield the solution $y(x) = a\cos x$ and therefore the exact value of b can be found, which is $\pi/2$ and is independent of a. However, consider the nonlinear problem

$$y'' + y^2 = 0$$
$$y(0) = a > 0, \ \ y'(0) = 0.$$

Mathematics subject classification (2010): 34C10, 34A34, 34A36.
Keywords and phrases: Second order differential equations, nonlinear differential equations, estimation of a zero.

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There is not a simple closed-form solution, but by varying a and applying the Runge-Kutta method, it appears that b depends strongly on a. In fact, it appears that there is an inverse relationship between a and b and it also appears that as $a \to 0^+$, we have $b \to \infty$.

The proof of our first theorem is based on the proof found in [1].

THEOREM 1. *Let y be a solution of (IVP). Assume that*
1) $\int_0^b \frac{1}{p(x)} dx$ exists,
2) $f(x,y(x)) \geqslant 0$ a.e. on $[0,b)$,
3) $f(x,y(x))/y(x) \leqslant f(x,y(0))/y(0)$ a.e. on $[0,b)$,
4) $\int_0^b f(x,y(0))dx$ exists.
Then,

$$\frac{\pi}{2} \leqslant \int_0^b \max\left\{\frac{f(x,y(0))}{y(0)}, \frac{1}{p(x)}\right\} dx.$$

Proof. Define $\Phi : [0,b) \to \mathbb{R}$ by

$$\Phi(x) = \arctan \frac{p(x)y'(x)}{y(x)}.$$

(Recall $y > 0$ on $[0,b)$.) Then,

$$\begin{aligned}\sec^2\Phi(x)\Phi'(x) &= \frac{y(x)[p(x)y'(x)]' - p(x)y'(x)y'(x)}{[y(x)]^2} \\ &= \frac{-y(x)f(x,y(x))}{[y(x)]^2} - \frac{\tan^2\Phi(x)}{p(x)}\end{aligned}$$

and hence

$$\Phi'(x) = \frac{-f(x,y(x))}{y(x)} \cos^2\Phi(x) - \frac{1}{p(x)} \sin^2\Phi(x). \tag{0.1}$$

From (0.1) and our assumptions, we have that $\Phi'(x) \leqslant 0$, that is, Φ is nonincreasing. Note that $\Phi(0) = 0$, and we also have $\lim_{x \uparrow b} \tan\Phi(x) = -\infty$ and hence we can let $\Phi(b) = -\pi/2$. We then have from (0.1):

$$\begin{aligned}\frac{\pi}{2} &= \Phi(0) - \Phi(b) \\ &= \int_0^b \left[\frac{f(x,y(x))}{y(x)} \cos^2\Phi(x) + \frac{1}{p(x)} \sin^2\Phi(x)\right] dx \\ &\leqslant \int_0^b \left[\frac{f(x,y(0))}{y(0)} \cos^2\Phi(x) + \frac{1}{p(x)} \sin^2\Phi(x)\right] dx. \end{aligned}\tag{0.2}$$

Now, applying the inequality $c_1 \cos^2\theta + c_2 \sin^2\theta \leqslant \max\{c_1, c_2\}$, we have that the right-hand side in (0.2) is less than or equal to

$$\int_0^b \max\left\{\frac{f(x,y(0))}{y(0)}, \frac{1}{p(x)}\right\} dx. \quad \square$$

REMARKS.

1. We made the decision to state the hypotheses of these results in the weakest form possible, but these assumptions typically follow from much more easily verified, stronger assumptions. For example, if we assume $f(x,w)/w$ is nondecreasing in w for each x and f is nonnegative valued, then 2) and 3) follow (note that the solution is nonincreasing in that event) – an example is $f(x,w) = w^n$ for $n \in \mathbb{N}$. The reader should be able to identify simpler, more easily verifiable conditions for the theorems below.

2. For the linear initial value problem

$$\begin{gathered} y'' + y = 0 \\ y(0) = 1, \;\; y'(0) = 0, \end{gathered}$$

Theorem 1 yields the optimum value $b = \pi/2$.

By following the proof in [1] a little more closely, we obtain the next result. We first need two lemmas that are special cases of Theorem 1 from [1].

LEMMA 1. *Let α, β be piecewise-continuous on $[0,b]$ and let β be nondecreasing and nonnegative-valued. Then,*

$$\frac{\int_0^b \alpha(x)\beta(x)dx}{\int_0^b \beta(x)dx} \leqslant \frac{\int_0^b \max\{\alpha(s) : x \leqslant s \leqslant b\}\, dx}{b}.$$

LEMMA 2. *Let α, β be piecewise-continuous on $[0,b]$ and let β be nonincreasing and nonnegative-valued. Then,*

$$\frac{\int_0^b \alpha(x)\beta(x)dx}{\int_0^b \beta(x)dx} \leqslant \frac{\int_0^b \max\{\alpha(s) : 0 \leqslant s \leqslant x\}\, dx}{b}.$$

We can now prove:

THEOREM 2. *Let y be a solution of (IVP). Assume that*

1) $\frac{1}{p}$ is piecewise continuous on $[0,b]$,

2) $f(x,y(x)) \geqslant 0$ a.e. on $[0,b)$,

3) $f(\cdot,y(0))$ is piecewise continuous on $[0,b]$,

4) $f(x,y(x))/y(x) \leqslant f(x,y(0))/y(0)$ a.e. on $[0,b)$,

5) $\int_0^b \max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}\, dx > 0$.

Then,

$$\frac{\pi}{2} \leqslant \sqrt{\int_0^b \max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}\, dx}\sqrt{\int_0^b \max\left\{\frac{1}{p(s)} : x \leqslant s \leqslant b\right\} dx} \tag{0.3}$$

Proof. Define the positive constant A by

$$A = \sqrt{\frac{\int_0^b \max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}\,dx}{\int_0^b \max\left\{\frac{1}{p(s)} : x \leqslant s \leqslant b\right\}dx}}$$

and $\Phi : [0,b) \to \mathbb{R}$ by

$$\Phi(x) = \arctan \frac{p(x)y'(x)}{Ay(x)}.$$

Proceeding as in Theorem 1, we obtain

$$\frac{\pi}{2} \leqslant \int_0^b \left[\frac{1}{A}\frac{f(x,y(0))}{y(0)}\cos^2\Phi(x) + \frac{A}{p(x)}\sin^2\Phi(x)\right]dx. \tag{0.4}$$

We first note that, applying Lemma 1,

$$\begin{aligned}
&\int_0^b \frac{A}{p(x)}\sin^2\Phi(x)dx \\
&= \frac{\int_0^b \frac{A}{p(x)}\sin^2\Phi(x)dx}{\int_0^b \sin^2\Phi(x)dx}\int_0^b \sin^2\Phi(x)dx \\
&\leqslant \frac{A}{b}\int_0^b \max\left\{\frac{1}{p(s)} : x \leqslant s \leqslant b\right\}dx \int_0^b \sin^2\Phi(x)dx \\
&= \frac{1}{b}\sqrt{\int_0^b [\max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}\cos^2\Phi(x)]\,dx} \\
&\quad \times \sqrt{\int_0^b \max\left\{\frac{1}{p(s)} : x \leqslant s \leqslant b\right\}dx}\int_0^b \sin^2\Phi(x)dx.
\end{aligned}$$

Similarly, applying Lemma 2,

$$\begin{aligned}
&\frac{1}{A}\int_0^b \left[\frac{f(x,y(0))}{y(0)}\cos^2\Phi(x)\right]dx \\
&= \frac{1}{A}\frac{\int_0^b \left[\frac{f(x,y(0))}{y(0)}\cos^2\Phi(x)\right]dx}{\int_0^b \cos^2\Phi(x)dx}\int_0^b \cos^2\Phi(x)dx \\
&\leqslant \frac{1}{Ab}\int_0^b \max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}\,dx \int_0^b \cos^2\Phi(x)dx \\
&\leqslant \frac{\frac{1}{b}\int_0^b \max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}\,dx}{\sqrt{\frac{\int_0^b \max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}dx}{\int_0^b \max\left\{\frac{1}{p(s)} : x \leqslant s \leqslant b\right\}dx}}}\int_0^b \cos^2\Phi(x)dx \\
&\leqslant \frac{1}{b}\sqrt{\int_0^b \max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}\,dx} \\
&\quad \times \sqrt{\int_0^b \max\left\{\frac{1}{p(s)} : x \leqslant s \leqslant b\right\}dx}\int_0^b \cos^2\Phi(x)dx.
\end{aligned}$$

Applying this to (0.4), we have

$$\frac{\pi}{2} \leqslant \frac{1}{b}\sqrt{\int_0^b \max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}\,dx}\sqrt{\int_0^b \max\left\{\frac{1}{p(s)} : x \leqslant s \leqslant b\right\}dx}$$
$$\times\left(\int_0^b \cos^2\Phi(x)dx + \int_0^b \sin^2\Phi(x)dx\right)$$
$$= \sqrt{\int_0^b \max\{f(s,y(0))/y(0) : 0 \leqslant s \leqslant x\}\,dx}\sqrt{\int_0^b \max\left\{\frac{1}{p(s)} : x \leqslant s \leqslant b\right\}dx. \quad \square$$

REMARKS.

1. For the linear initial value problem

$$y'' + y = 0$$
$$y(0) = 1, \;\; y'(0) = 0,$$

   Theorem 2 yields $\pi/2 \leqslant \sqrt{b\int_0^b dx}$, the optimum value $b = \pi/2$.

2. For the case $f(x,y) = y^n$, $n \in \{2,3,\ldots\}$, (0.3) becomes $\pi/2 \leqslant \sqrt{\frac{b}{y(0)}\int_0^b (y(0))^n\,dx}$, i.e., $b \geqslant \pi/\left[2\,(y(0))^{n/2-1/2}\right]$, and hence as $y(0) \to 0^+$, it follows that $b \to \infty$, as conjectured above.

3. For the initial value problem

$$y'' + xy^2 = 0$$
$$y(0) = 1, \;\; y'(0) = 0,$$

   Theorem 1 yields $\pi/2 \leqslant \int_0^b \max\{x,1\}\,dx$. It is easy to see $b \geqslant 1$ (integrating the differential equation twice we have $1 = \int_0^b\int_0^t x[y(x)]^2dxdt$ and then using the fact that y is nonincreasing, if $b < 1$ we also have $\int_0^b\int_0^t x[y(x)]^2dxdt \leqslant \int_0^b\int_0^t 1dxdt = b^2/2 < 1/2$, a contradiction), so we have $\pi/2 \leqslant \int_0^1 1\,dx + \int_1^b x\,dx$ and hence $b \geqslant \sqrt{\pi - 1} \approx 1.46$. Theorem 2 gives $\pi/2 \leqslant \sqrt{b\int_0^b x\,dx}$ and hence $b \geqslant \left(\pi/\sqrt{2}\right)^{2/3} \approx 1.70$, so Theorem 2 gives more information in this example than Theorem 1.

4. For the initial value problem

$$y'' + 2e^{-x}y^2 = 0$$
$$y(0) = 1, \;\; y'(0) = 0,$$

   Theorem 1 yields $\pi/2 \leqslant \int_0^b \max\{2e^{-x},1\}\,dx$. Proceeding as before, we have $b \geqslant 1$, so we have $\pi/2 \leqslant \int_0^{\ln 2} 2e^{-x}dx + \int_{\ln 2}^b 1\,dx$ and hence $b \geqslant \pi/2 - 1 + \ln 2 \approx$

1.26. Theorem 2 yields $\pi/2 \leqslant \sqrt{b\int_0^b \max\{2e^{-s} : 0 \leqslant s \leqslant x\}\,dx}$ which gives us $b \geqslant \pi/(2\sqrt{2}) \approx 1.1$, hence Theorem 1 gives more information in this example than Theorem 2.

Following the approach of [6], we can prove the following for the nonlinear case.

THEOREM 3. *Let y be a solution of (IVP). Assume that*
1) $f(x,y(x)) \leqslant f(x,y(0))$ a.e. on $[0,b)$,
2) $\int_0^x f(t,y(0))\,dt$ exists for every $x \in [0,b)$,
3) $\int_0^b \frac{\int_0^x f(t,y(0))\,dt}{p(x)}dx$ exists.
Then,

$$1 \leqslant \frac{1}{y(0)} \int_0^b \frac{\int_0^x f(t,y(0))\,dt}{p(x)}dx.$$

Proof. Let $x \in [0,b]$. Then,

$$\int_0^x (p(t)y'(t))'dt + \int_0^x f(t,y(t))\,dt = 0 \Longrightarrow$$

$$p(x)y'(x) - p(0)y'(0) = -\int_0^x f(t,y(t))\,dt \Longrightarrow$$

$$-p(x)y'(x) = \int_0^x f(t,y(t))\,dt \leqslant \int_0^x f(t,y(0))\,dt \Longrightarrow$$

$$-\int_0^b y'(x)\,dx \leqslant \int_0^b \frac{\int_0^x f(t,y(0))\,dt}{p(x)}dx \Longrightarrow$$

$$y(0) \leqslant \int_0^b \frac{\int_0^x f(t,y(0))\,dt}{p(x)}dx \Longrightarrow$$

$$1 \leqslant \frac{1}{y(0)} \int_0^b \frac{\int_0^x f(t,y(0))\,dt}{p(x)}dx. \quad \square$$

REMARKS.

1. Consider the problem

$$y'' + 2y^2 = 0$$
$$y(0) = 1, \ \ y'(0) = 0.$$

Theorem 3 yields $1 \leqslant \int_0^b \int_0^x 2\,dt\,dx$ from which it follows that $b \geqslant 1$. Theorem 1 yields $\pi/2 \leqslant \int_0^b \max\{2,1\}\,dx$ and hence tells us that $b \geqslant \pi/4 \approx 0.79$. Theorem 2 yields $\pi/2 \leqslant \sqrt{b\int_0^b 2\,ds}$ and hence gives $b \geqslant \pi/(2\sqrt{2}) \approx 1.11$.

2. Interestingly, for the linear problem

$$y'' + y = 0$$
$$y(0) = 1, \; y'(0) = 0,$$

Theorem 3 yields $1 \leqslant \int_0^b \int_0^x 1\, dt\, dx$ and hence $b \geqslant \sqrt{2}$, which is less than the optimum value of $\pi/2$.

We can improve Theorem 3 by using the iteration trick of [3], as illustrated in the following proof:

THEOREM 4. *Let y be a solution of (IVP). Assume that*

1) $\int_x^b \frac{\int_0^\tau f(r,y(0))\,dr}{p(\tau)} d\tau$ *exists for all* $x \in [0,b)$,

2) $f\left(x, \int_x^b \frac{\int_0^\tau f(r,y(r))\,dr}{p(\tau)} d\tau\right) \leqslant f\left(x, \int_x^b \frac{\int_0^\tau f(r,y(0))\,dr}{p(\tau)} d\tau\right)$ *a.e. on* $[0,b)$,

3) $\int_0^x f\left(s, \int_s^b \frac{\int_0^\tau f(r,y(0))\,dr}{p(\tau)} d\tau\right) ds$ *exists for all* $x \in [0,b]$.

Then,

$$1 \leqslant \frac{1}{y(0)} \int_0^b \int_0^t f\left(s, \int_s^b \frac{\int_0^\tau f(r,y(0))\,dr}{p(\tau)} d\tau\right) ds dt.$$

Proof. Let $\tau, s \in [0,b]$. Then,

$$\int_0^\tau (p(r)y'(r))' dr + \int_0^\tau f(r,y(r))\,dr = 0 \Longrightarrow$$

$$p(\tau)y'(\tau) - p(0)y'(0) = -\int_0^\tau f(r,y(r))\,dr \Longrightarrow$$

$$-p(\tau)y'(\tau) = \int_0^\tau f(r,y(r))\,dr \Longrightarrow$$

$$-\int_s^b y'(\tau)\,d\tau = \int_s^b \frac{\int_0^\tau f(r,y(r))\,dr}{p(\tau)} d\tau \Longrightarrow$$

$$y(s) = \int_s^b \frac{\int_0^\tau f(r,y(r))\,dr}{p(\tau)} d\tau.$$

Let $t \in [0,b]$. Substituting this expression for $y(s)$ into $\int_0^t y''(s)\,ds + \int_0^t f(s,y(s))\,ds = 0$, we have

$$-\int_0^t y''(s)\,ds = \int_0^t f\left(s, \int_s^b \frac{\int_0^\tau f(r,y(r))\,dr}{p(\tau)} d\tau\right) ds \Longrightarrow$$

$$-y'(t) \leqslant \int_0^t f\left(s, \int_s^b \frac{\int_0^\tau f(r,y(0))\,dr}{p(\tau)} d\tau\right) ds \Longrightarrow$$

$$-\int_0^b y'(t)\,dt \leqslant \int_0^b \int_0^t f\left(s, \int_s^b \frac{\int_0^\tau f(r,y(0))\,dr}{p(\tau)} d\tau\right) dsdt \Longrightarrow$$

$$y(0) \leqslant \int_0^b \int_0^t f\left(s, \int_s^b \frac{\int_0^\tau f(r,y(0))\,dr}{p(\tau)} d\tau\right) dsdt. \quad \square$$

REMARKS.

1. We note that hypothesis 2 holds if $f(x,\cdot)$ is nonnegative and is nondecreasing for each $x \in [0,b]$.

2. Consider the problem

$$y'' + y^2 = 0$$
$$y(0) = 1, \;\; y'(0) = 0.$$

Theorem 3 yields $b \geqslant \sqrt{2} \approx 1.41$. For Theorem 4, we observe that

$$\int_s^b \int_0^\tau f(r,y(0))\,dr\,d\tau = \left(b^2 - s^2\right)/2$$

and then

$$\int_0^b \int_0^t \left[\left(b^2 - s^2\right)^2/4\right] dsdt = \int_0^b \left(b^4 t/4 - b^2 t^3/6 + t^5/20\right) dt$$
$$= (11/120) b^6$$

and hence $b \geqslant \sqrt[6]{(120/11)} \approx 1.49$.

We can improve Theorems 3 and 4 further by exploiting the fact that in many cases y is concave down on $[0,b]$ as discussed in [5].

THEOREM 5. *Let y be a solution of (IVP). Assume that*

1) $f(x,y_1(x)) \leqslant f(x,y(x))$ a.e. on $[0,b)$, where $y_1(x) = \frac{y(0)(b-x)}{b}$,

2) $\int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)} d\tau$ exists for $s \in [0,b)$,

3) $f(s,y(s)) \leqslant f\left(s, y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)} d\tau\right)$ a.e. on $[0,b)$,

4) $\int_0^b \frac{\int_0^t f\left(s, y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)} d\tau\right) ds}{p(t)} dt$ exists.

Then,

$$1 \leqslant \frac{1}{y(0)} \int_0^b \frac{\int_0^t f\left(s, y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)} d\tau\right) ds}{p(t)} dt.$$

Proof. For $s, \tau \in [0,b]$,

$$\int_0^\tau (p(r)y'(r))'\,dr + \int_0^\tau f(r,y(r))\,dr = 0 \Longrightarrow$$

$$p(\tau)y'(\tau) - p(0)y'(0) = -\int_0^\tau f(r,y(r))\,dr \Longrightarrow$$

$$-p(\tau)y'(\tau) = \int_0^\tau f(r,y(r))\,dr \Longrightarrow$$

$$-\int_0^s y'(\tau)\,d\tau = \int_0^s \frac{\int_0^\tau f(r,y(r))\,dr}{p(\tau)}d\tau \Longrightarrow$$

$$-y(s)+y(0) = \int_0^s \frac{\int_0^\tau f(r,y(r))\,dr}{p(\tau)}d\tau \geqslant \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)}d\tau \Longrightarrow$$

$$y(s) \leqslant y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)}d\tau \tag{0.5}$$

Let $t \in [0,b]$. We have, from the original differential equation

$$-\int_0^t (p(s)y'(s))'ds = \int_0^t f(s,y(s))\,ds$$

and then applying (0.5) to the right-hand side

$$-\int_0^t (p(s)y'(s))'ds \leqslant \int_0^t f\left(s, y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)}d\tau\right)ds \Longrightarrow$$

$$-p(t)y'(t) \leqslant \int_0^t f\left(s, y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)}d\tau\right)ds \Longrightarrow$$

$$-\int_0^b y'(t)\,dt \leqslant \int_0^b \frac{\int_0^t f\left(s, y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)}d\tau\right)ds}{p(t)}dt \Longrightarrow$$

$$y(0) \leqslant \int_0^b \frac{\int_0^t f\left(s, y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)}d\tau\right)ds}{p(t)}dt. \quad \square$$

COROLLARY 1. *Let y be a solution of (IVP). Assume that*

1) p is differentiable and nonincreasing,

2) $f(x,y(x)) \geqslant 0$ a.e. on $[0,b)$

3) For each $x \in [0,b]$, $f(x,\cdot)$ is nondecreasing on $[0,a]$.

4) $\int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)}d\tau$ exists for $s \in [0,b)$,

5) $\int_0^b \frac{\int_0^t f\left(s, y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)}d\tau\right)ds}{p(t)}dt$ exists.

Then,

$$1 \leqslant \frac{1}{y(0)} \int_0^b \frac{\int_0^t f\left(s, y(0) - \int_0^s \frac{\int_0^\tau f(r,y_1(r))\,dr}{p(\tau)}d\tau\right)ds}{p(t)}dt.$$

Proof. Since $y'(x) = -\int_0^x f(s,y(s))ds/p(x) \leqslant 0$ and $y''(x) = p'(x)\int_0^x f(s,y(s))ds/[p(x)]^2 - f(x,y(x))/p(x) \leqslant 0$ on $[0,b]$ from our assumptions, y dominates the line from $(0,y(0))$ to $(b,0)$, i.e., $y(x) \geqslant y_1(x)$. From this and assumption 3 of our Corollary, assumption 1 of Theorem 5 follows. Assumption 3 of Theorem 5 follows from assumption 3 of the Corollary along with (0.5). □

REMARK. Returning to the problem from Remark 2 following Theorem 4, we have

$$\int_0^s \int_0^\tau f(r,y_1(r))\,dr\,d\tau = \int_0^s \int_0^\tau \left[(b-r)^2/b^2\right] dr\,d\tau = (6b^2s^2 - 4bs^3 + s^4)/(12b^2)$$

and then

$$\begin{aligned} &\int_0^b \int_0^t f\left(s, 1 - \int_0^s \int_0^\tau f(r,y_1(r))\,dr\,d\tau\right) ds\,dt \\ &= \int_0^b \int_0^t \left(1 - (6b^2s^2 - 4bs^3 + s^4)/(12b^2)\right)^2 ds\,dt \\ &= (1/630)\,(2b^6 - 35b^4 + 315b^2) \end{aligned}$$

and hence $b \geqslant 1.63$ from Theorem 5 (or its corollary). Incidentally, both Theorems 1 and 2 give $b \geqslant 1.57$, even using y_1 in place of $y(0)$.

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(Received July 26, 2017)

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