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EXISTENCE AND UNIQUENESS OF MONOTONE POSITIVE SOLUTIONS FOR A THIRD–ORDER THREE–POINT BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we study the existence and uniqueness of monotone positive solutions for a class of nonlinear third-order three-point boundary value problem. The main tool is a fixed point theorem of generalized concave operators in ordered Banach spaces. An example is given to illustrate the main result.

1. Introduction

In this article, we discuss the existence and uniqueness of monotone positive solutions for the following third-order differential equation

$$u'''(t)+f(t,u(t),u'(t))=0,\ \ t\in(0,1), \tag{1.1}$$

under three-point boundary conditions

$$u(0)=u'(0)=0,\ \ u'(1)=\alpha u'(\eta), \tag{1.2}$$

where $\eta\in(0,1)$, $\alpha>0$, $\alpha\eta<1$, $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous. Here, the solution $u^*(t)$ of the problem (1.1), (1.2) is called positive if $u^*(t)>0$, $t\in(0,1)$. A monotone positive solution means increasing positive solution.

In several decades, third-order ordinary differential equations have extensive applications in mechanics and engineering. So the results on the existence of solutions or positive solutions for nonlinear third-order ordinary differential equations with three-point boundary conditions have been obtained continuously in the literature, see [1–26, 28, 29] and references therein. For example, Guo *et al.* [8, 9] gave the existence of at least one or three positive solutions for the problem (1.1), (1.2) by using the Krasnosel'skii fixed point theorem and the Leggett-Williams fixed point theorem, respectively. Based upon the upper and lower solutions and the maximum principle, Yao, Feng [24] and Feng, Liu [4] established the existence of solutions for the problem (1.1), (1.2) with $\alpha=0$, respectively. From literature, we can see that there are many results

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on the existence of positive solutions, see [5, 8, 9, 11, 12, 18–21, 25] for instance. On the other hand, there are some papers which were concerned with the uniqueness of positive solutions, see [15, 16, 26] for details.

Different from the papers mentioned above, in this paper we will discuss the existence and uniqueness of monotone positive solutions for the problem (1.1), (1.2). The method used here is a fixed point theorem of generalized concave operators in ordered Banach spaces. As we know, there has no papers considered monotone positive solutions for nonlinear third-order differential equation boundary value problems.

2. Preliminaries

In this section, we give some definitions and preliminary facts.

Let E be a real Banach space which is partially ordered by a cone $P \subset E$, i.e., $x \leqslant y$ if and only if $y - x \in P$. By θ we denote the zero element of E. A non-empty closed convex set $P \subset E$ is called a cone if it satisfies

(i) $x \in P,\ \lambda > 0 \Longrightarrow \lambda x \in P$;

(ii) $x \in P,\ -x \in P \Rightarrow x = \theta$.

P is called normal if there is a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$; in this case N is called the normality constant of P.

We say that an operator $A : E \to E$ is increasing if $x \leqslant y$ implies $Ax \leqslant Ay$.

For $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E | x \sim h\}$. Clearly, $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$.

Our main tool is the following fixed point theorem of generalized concave operator, which will be used in the latter proof. See [27] for further information.

THEOREM 2.1. (see [27, Lemma 2.1] *and* [27, Theorem 2.1]) *Let* $h > \theta$ *and* P *be a normal cone. Suppose:*

(d_1) $A : P \to P$ *is increasing and* $Ah \in P_h$*;*

(d_2) *for any* $x \in P$ *and* $t \in (0,1)$*, there exists* $\alpha(t) \in (t,1)$ *such that* $A(tx) \geqslant \alpha(t)Ax$.

Then

(i) *there exist* $u_0,\ v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \leqslant u_0 < v_0$, $u_0 \leqslant Au_0 \leqslant Av_0 \leqslant v_0$*;*

(ii) *operator equation* $x = Ax$ *has a unique solution in* P_h.

REMARK 2.1. An operator A is said to be generalized concave if A satisfies condition (d_2).

In what follows, we shall consider the Banach space $E = C^1[0,1]$ equipped with the norm

$$\|u\| = \max\{\max_{0 \leqslant t \leqslant 1} |u(t)|, \max_{0 \leqslant t \leqslant 1} |u'(t)|\}.$$

LEMMA 2.1. (see [19]) *Assume that* $\alpha\eta \neq 1$, $y \in C[0,1]$, *then the unique solution of the following equation*

$$u'''(t) + h(t) = 0, \ \ t \in (0,1), \tag{2.1}$$

with boundary conditions (1.2) can be given by

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{2.2}$$

where

$$G(t,s) = K(t,s) + \frac{\alpha t^2}{2(1-\alpha\eta)} K_1(\eta,s), \tag{2.3}$$

$$K(t,s) = \frac{1}{2}\begin{cases} t^2(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(-t^2+2t-s), & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \tag{2.4}$$

and

$$K_1(t,s) := \frac{\partial K(t,s)}{\partial t} = \begin{cases} (1-s)t, & 0 \leqslant t \leqslant s \leqslant 1, \\ (1-t)s, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

To establish the existence and uniqueness of monotone positive solutions for the problem (1.1), (1.2), we give some properties of functions $K(t,s)$, $K_1(t,s)$.

LEMMA 2.2. *For all* $(t,s) \in [0,1]\times[0,1]$, *we have*
(i) $0 \leqslant K_1(t,s) \leqslant t \leqslant 1$;
(ii) $0 \leqslant 1/2t^2(1-s)s \leqslant K(t,s) \leqslant t^2 \leqslant 1$.

Proof. The conclusion (i) is obvious. So we only need to prove the conclusion (ii). For all $t,s \in [0,1]$, if $s \leqslant t$, it follows from (2.4) that

$$K(t,s) = \frac{1}{2}(2t-t^2-s)s \leqslant \frac{1}{2}(2t-t^2)t \leqslant t^2 \leqslant 1,$$

and

$$\begin{aligned} K(t,s) &= \frac{1}{2}(2t-t^2-s)s = \frac{1}{2}(2t-t^2-s+t^2-t^2+t^2s-t^2s)s \\ &= \frac{1}{2}t^2(1-s)s + \frac{1}{2}(1-t)[(t-s)+(1-s)t]s \\ &\geqslant \frac{1}{2}t^2(1-s)s \geqslant 0. \end{aligned}$$

If $t \leqslant s$, then from (2.4) we have

$$0 \leqslant \frac{1}{2}t^2(1-s)s \leqslant K(t,s) = \frac{1}{2}t^2(1-s) \leqslant t^2 \leqslant 1.$$

The proof is completed. □

3. Monotone positive solutions

For seek monotone positive solutions, we consider the closed convex cone of non-negative increasing functions $P=\{u\in E|u(t)\geqslant 0, u'(t)\geqslant 0, \forall t\in[0,1]\}$. Note that this induces an order relation $\dot{\leqslant}$ in E by defining $u\dot{\leqslant}v$ if and only if $u-v\in P$. It is easy to prove that this cone is normal. Namely, if $u\dot{\leqslant}v$, then $u(t)\leqslant v(t)$, $u'(t)\leqslant v'(t)$, $t\in[0,1]$. Therefore, $\|u\|\leqslant\|v\|$ and the normality constant is 1.

Define an operator

$$Au(t)=\int_0^1 K(t,s)f(s,u(s),u'(s))ds+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,u(s),u'(s))ds,\ t\in[0,1].$$

Then

$$(Au)'(t)=\int_0^1 K_1(t,s)f(s,u(s),u'(s))ds+\frac{\alpha t}{1-\alpha\eta}\int_0^1 K_1(\eta,s)f(s,u(s),u'(s))ds,\ t\in[0,1].$$

THEOREM 3.1. *Assume that*

(H_1) $f(t,x,y):[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *is continuous with* $f(t,0,0)\not\equiv 0$, $t\in[0,1]$*;*

(H_2) $f(t,x,y)$ *are increasing in* $x,y\in[0,+\infty)$ *for fixed* $t\in[0,1]$ *respectively;*

(H_3) *for any* $\lambda\in(0,1)$ *and* $x,y\geqslant 0$*, there exists* $\varphi(\lambda)\in(\lambda,1)$ *such that*

$$f(t,\lambda x,\lambda y)\geqslant\varphi(\lambda)f(t,x,y).$$

Then:

(i) *there are* $u_0,v_0\in P_h$ *such that*

$$u_0(t)\leqslant\int_0^1 K(t,s)f(s,u_0(s),u_0'(s))ds+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,u_0(s),u_0'(s))ds,\ t\in[0,1],$$

$$u_0'(t)\leqslant\int_0^1 K_1(t,s)f(s,u_0(s),u_0'(s))ds+\frac{\alpha t}{1-\alpha\eta}\int_0^1 K_1(\eta,s)f(s,u_0(s),u_0'(s))ds,\ t\in[0,1],$$

$$v_0(t)\geqslant\int_0^1 K(t,s)f(s,v_0(s),v_0'(s))ds+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,v_0(s),v_0'(s))ds,\ t\in[0,1],$$

$$v_0'(t)\geqslant\int_0^1 K_1(t,s)f(s,v_0(s),v_0'(s))ds+\frac{\alpha t}{1-\alpha\eta}\int_0^1 K_1(\eta,s)f(s,v_0(s),v_0'(s))ds,\ t\in[0,1];$$

(ii) *the problem (1.1), (1.2) has a unique monotone positive solution* u^* *in* P_h*, where* $h(t)=t^2$*,* $t\in[0,1]$*.*

Proof. We prove that all the conditions of Theorem 2.1 are satisfied. The proof is divided into several steps.

Step 1. We show that $A:P\to P$ is increasing. For $u\in P$, we know that $u(t)\geqslant 0$, $u'(t)\geqslant 0$, $t\in[0,1]$. From (H_1), (H_2) and Lemmas 2.1, 2.2, we have $Au(t)\geqslant 0$, $(Au)'(t)\geqslant 0$, $t\in[0,1]$. Therefore, $Au\in P$. For any u_1, $u_2\in P$ with $u_1\dot{\leqslant}u_2$, we know that $u_1(t) \leqslant u_2(t)$, $u_1'(t) \leqslant u_2'(t)$, $t \in [0,1]$. Also from (H_1), (H_2), we have $Au_1(t) \leqslant Au_2(t)$, $(Au_1)'(t) \leqslant (Au_2)'(t)$, $t \in [0,1]$. Then $Au_1 \dot{\leqslant} Au_2$, that is: $A : P \to P$ is an increasing operator.

Step 2. We prove that $A : P \to P$ is generalized concave. For any $\lambda \in (0,1)$ and $u \in P$, from (H_2), (H_3), we have

$$\begin{aligned}
&A(\lambda u)(t)\\
&= \int_0^1 K(t,s) f(s, \lambda u(s), (\lambda u)'(s)) ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 K_1(\eta,s) f(s, \lambda u(s), (\lambda u)'(s)) ds\\
&= \int_0^1 K(t,s) f(s, \lambda u(s), \lambda u'(s)) ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 K_1(\eta,s) f(s, \lambda u(s), \lambda u'(s)) ds\\
&\geqslant \varphi(\lambda) \left[\int_0^1 K(t,s) f(s, u(s), u'(s)) ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 K_1(\eta,s) f(s, u(s), u'(s)) ds \right]\\
&= \varphi(\lambda) Au(t),
\end{aligned}$$

and

$$\begin{aligned}
&(A(\lambda u))'(t)\\
&= \int_0^1 K_1(t,s) f(s, \lambda u(s), \lambda u'(s)) ds + \frac{\alpha t}{1-\alpha\eta} \int_0^1 K_1(\eta,s) f(s, \lambda u(s), \lambda u'(s)) ds\\
&\geqslant \varphi(\lambda) \left[\int_0^1 K_1(t,s) f(s, u(s), u'(s)) ds + \frac{\alpha t}{1-\alpha\eta} \int_0^1 K_1(\eta,s) f(s, u(s), u'(s)) ds \right]\\
&= \varphi(\lambda) (Au)'(t).
\end{aligned}$$

Hence, $A(\lambda u)(t) \geqslant \varphi(\lambda) Au(t)$, $(A(\lambda u))'(t) \geqslant \varphi(\lambda)(Au)'(t)$. So $A(\lambda u) \dot{\geqslant} \varphi(\lambda) Au$, $\forall\, \lambda \in (0,1)$, $u \in P$.

Step 3. We show that $Ah \in P_h$. That is, we need to prove that there exist two constants l_1, $l_2 > 0$ such that $l_1 h \dot{\leqslant} Ah \dot{\leqslant} l_2 h$. From (H_2) and Lemma 2.2,

$$\begin{aligned}
Ah(t) &= \int_0^1 K(t,s) f(s, h(s), h'(s)) ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 K_1(\eta,s) f(s, h(s), h'(s)) ds\\
&= \int_0^1 K(t,s) f(s, s^2, 2s) ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 K_1(\eta,s) f(s, s^2, 2s) ds\\
&\geqslant \frac{1}{2} t^2 \int_0^1 s(1-s) f(s, s^2, 2s) ds\\
&\geqslant \frac{1}{2} \int_0^1 s(1-s) f(s,0,0) ds \cdot t^2,
\end{aligned}$$

and

$$\begin{aligned} Ah(t) &= \int_0^1 K(t,s)f(s,s^2,2s)ds + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,s^2,2s)ds \\ &\leqslant t^2\int_0^1 f(s,s^2,2s)ds + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 f(s,s^2,2s)ds \\ &\leqslant t^2\left[\int_0^1 f(s,1,2)ds + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 f(s,1,2)ds\right]. \end{aligned}$$

Moreover, also from (H_2) and Lemma 2.2, we have

$$\begin{aligned} (Au)'(t) &= \int_0^1 K_1(t,s)f(s,u(s),u'(s))ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 K_1(\eta,s)f(s,u(s),u'(s))ds \\ &= \int_0^1 K_1(t,s)f(s,s^2,2s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 K_1(\eta,s)f(s,s^2,2s)ds \\ &\geqslant 2t\left[\frac{\alpha}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,0,0)ds\right] \\ &= \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,0,0)ds\cdot h'(t), \end{aligned}$$

and

$$\begin{aligned} (Au)'(t) &= \int_0^1 K_1(t,s)f(s,s^2,2s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 K_1(\eta,s)f(s,s^2,2s)ds \\ &\leqslant 2t\left[\frac{1}{2}\int_0^1 f(s,1,2)ds + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,1,2)ds\right] \\ &= \left[\frac{1}{2}\int_0^1 f(s,1,2)ds + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,1,2)ds\right]\cdot h'(t). \end{aligned}$$

Let

$$\begin{aligned} c_1 &= \frac{1}{2}\int_0^1 s(1-s)f(s,0,0)ds, \\ c_2 &= \int_0^1 f(s,1,2)ds + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 f(s,1,2)ds, \\ c_3 &= \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,0,0)ds, \\ c_4 &= \frac{1}{2}\int_0^1 f(s,1,2)ds + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,1,2)ds. \end{aligned}$$

By (H_1), (H_2), we can easily get $c_2 > c_1 > 0$, $c_4 > c_3 > 0$. Let $l_1 = \min\{c_1,c_3\}$, $l_2 = \max\{c_2,c_4\}$. We have $0 < l_1 < l_2$, and then

$$l_1h(t) \leqslant c_1h(t) \leqslant Ah(t) \leqslant c_2h(t) \leqslant l_2h(t),$$

$$(l_1h)'(t) = l_1h'(t) \leqslant c_3h'(t) \leqslant (Ah)'(t) \leqslant c_4h'(t) \leqslant l_2h'(t) = (l_2h)'(t), \ \ t \in [0,1].$$

Thus, $l_1h \dot{\leqslant} Ah \dot{\leqslant} l_2h$. That is, $Ah \in P_h$.

Finally, an application of Theorem 2.1 implies that:

(i) there are u_0, $v_0 \in P_h$ such that $u_0 \dot{\leqslant} Au_0$, $Av_0 \dot{\leqslant} v_0$;

(ii) operator equation $u = Au$ has a unique solution u^* in P_h. That is, $u_0(t) \leqslant Au_0(t)$, $Av_0(t) \leqslant v_0(t)$, $u_0'(t) \leqslant (Au_0)'(t)$, $(Av_0)'(t) \leqslant v_0'(t)$ and

$$u_0(t) \leqslant \int_0^1 K(t,s)f(s,u_0(s),u_0'(s))ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 K_1(\eta,s)f(s,u_0(s),u_0'(s))ds, \ t \in [0,1],$$

$$u_0'(t) \leqslant \int_0^1 K_1(t,s)f(s,u_0(s),u_0'(s))ds + \frac{\alpha t}{1-\alpha\eta} \int_0^1 K_1(\eta,s)f(s,u_0(s),u_0'(s))ds, \ t \in [0,1],$$

$$v_0(t) \geqslant \int_0^1 K(t,s)f(s,v_0(s),v_0'(s))ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 K_1(\eta,s)f(s,v_0(s),v_0'(s))ds, \ t \in [0,1],$$

$$v_0'(t) \geqslant \int_0^1 K_1(t,s)f(s,v_0(s),v_0'(s))ds + \frac{\alpha t}{1-\alpha\eta} \int_0^1 K_1(\eta,s)f(s,v_0(s),v_0'(s))ds, \ t \in [0,1];$$

and the problem (1.1), (1.2) has a unique solution u^* in P_h. So $u^*(t) \geqslant 0$, $u^{*\prime}(t) \geqslant 0$, $t \in [0,1]$. Therefore, $u^*(t)$ is a monotone positive solution of the problem (1.1), (1.2). □

From Theorem 3.1, we can consider the uniqueness and existence of monotone positive solutions for the following third-order differential equation

$$u'''(t) + f(t,u(t)) = 0, \ \ t \in (0,1), \tag{3.1}$$

with the following three-point boundary conditions

$$u(0) = u'(0) = 0, \ \ u'(1) = \alpha u'(\eta), \tag{3.2}$$

where $\eta \in (0,1)$, $\alpha > 0$, $\alpha\eta < 1$.

THEOREM 3.2. *Assume that*

(H_4) $f(t,x) : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous with* $f(t,0) \not\equiv 0$, $t \in [0,1]$;

(H_5) $f(t,x)$ *is increasing in* $x \in [0,+\infty)$ *for fixed* $t \in [0,1]$;

(H_6) *for any* $\lambda \in (0,1)$ *and* $x \geqslant 0$, *there exists* $\varphi(\lambda) \in (\lambda,1)$ *such that*

$$f(t,\lambda x) \geqslant \varphi(\lambda) f(t,x).$$

Then:

(i) *there are* $u_0, v_0 \in P_h$ *such that*

$$u_0(t) \leqslant \int_0^1 K(t,s)f(s,u_0(s))ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 K_1(\eta,s)f(s,u_0(s))ds, \ \ t \in [0,1],$$

$$u_0'(t) \leqslant \int_0^1 K_1(t,s)f(s,u_0(s))ds + \frac{\alpha t}{1-\alpha\eta} \int_0^1 K_1(\eta,s)f(s,u_0(s))ds, \ \ t \in [0,1],$$

$$v_0(t) \geqslant \int_0^1 K(t,s)f(s,v_0(s))ds + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 K_1(\eta,s)f(s,v_0(s))ds, \ \ t\in[0,1],$$

$$v_0'(t) \geqslant \int_0^1 K_1(t,s)f(s,v_0(s))ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 K_1(\eta,s)f(s,v_0(s))ds, \ \ t\in[0,1];$$

(ii) *the problem (3.1), (3.2) has a unique monotone positive solution* u^* *in* P_h*, where* $h(t)=t^2$*,* $t\in[0,1]$.

Now we can consider

$$u'''(t) + f(t,u(t),u'(t)) = 0, \ \ t\in(0,1), \tag{3.3}$$

under two-point boundary conditions

$$u(0) = u'(0) = 0, \ \ u'(1) = 0. \tag{3.4}$$

By Lemma 2.2 (ii) and Theorem 2.1, we can easily obtain the following result.

THEOREM 3.3. *Assume that* (H_1)–(H_3) *hold. Then:*
(i) *there are* $u_0, v_0 \in P_h$ *such that*

$$u_0(t) \leqslant \int_0^1 K(t,s)f(s,u_0(s),u_0'(s))ds, \ \ t\in[0,1],$$

$$u_0'(t) \leqslant \int_0^1 K_1(t,s)f(s,u_0(s),u_0'(s))ds, \ \ t\in[0,1],$$

$$v_0(t) \geqslant \int_0^1 K(t,s)f(s,v_0(s),v_0'(s))ds, \ \ t\in[0,1],$$

$$v_0'(t) \geqslant \int_0^1 K_1(t,s)f(s,v_0(s),v_0'(s))ds, \ \ t\in[0,1];$$

(ii) *the problem (3.3), (3.4) has a unique monotone positive solution* u^* *in* P_h*, where* $h(t)=t^2$*,* $t\in[0,1]$.

4. An example

Now, we present an example to illustrate the main result.

EXAMPLE 4.1. *Consider the following third-order three-point boundary value problem*

$$u'''(t) + [u(t)]^{1/2} + [u'(t)]^{1/3} + (1-t)t^{1/2} = 0, \ \ t\in(0,1), \tag{4.1}$$

$$u(0) = u'(0) = 0, \ \ u'(1) = 2u'(1/3). \tag{4.2}$$

We can show that the problem (4.1), (4.2) has a unique monotone positive solution in P_h*, where* $h(t)=t^2$*,* $t\in[0,1]$.

Proof. In this example, $\alpha = 2$, $\eta = 1/3$. Let $f(t,x,y) = x^{1/2} + y^{1/3} + (1-t)t^{1/2}$. It is not difficult to see that the conditions (H_1), (H_2) hold. In addition, let $\varphi(\lambda) = \lambda^{1/2}$. Then, the condition (H_3) of Theorem 3.1 holds. Hence, by Theorem 3.1, the conclusion follows, and the proof is complete. □

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