Differential
Equations
& Applications
Volume 11, Number 1 (2019), 1–85
doi:10.7153/dea-2019-11-01

COUPLED AND MIXED COUPLED HYBRID FIXED POINT PRINCIPLES IN A PARTIALLY ORDERED BANACH ALGEBRA AND PBVPS OF NONLINEAR COUPLED QUADRATIC DIFFERENTIAL EQUATIONS

BAPURAO C. DHAGE

This paper is dedicated in the loving memory
of my late parents Kasubai and Chandrabhan Dhage
who inculcated in me the hard work, honesty and love for other.

(*Communicated by Michal Fečkan*)

Abstract. In this paper we prove some coupled and mixed coupled hybrid fixed point theorems involving different algebraic combinations of three operators and coupled operators in a partially ordered Banach algebra by an application of a coupled hybrid fixed point principle for partially condensing coupled mappings developed in Dhage [J. Fixed Point Theory Appl. **19** (2017), 2541–2575]. Our approach is based on the partial Kuratowskii measure of noncompactness with maximum property and is somewhat different from the approach of coupled hybrid fixed point theorems presented in Dhage [J. Fixed Point Theory Appl. **19** (2017), 3231–3264]. We apply our newly developed abstract mixed coupled hybrid fixed point theorems along with algorithms to a couple of nonlinear first and second order coupled quadratically perturbed hybrid differential equations with the periodic boundary conditions for proving the existence and approximation theorems under certain mixed hybrid conditions from algebra, analysis and topology. The abstract existence and approximation results of the coupled quadratic periodic boundary value problems of first and second order ordinary differential equations are also illustrated by presenting a few numerical examples. We claim that the results of this paper are new to the literature on nonlinear analysis applications.

1. Introduction

Throughout this paper, unless otherwise mentioned, let $(E,\leqslant,\|\cdot\|)$ denote a partially ordered Banach algebra with the order relation $\leqslant$ and the norm $\|\cdot\|$ defined on it. Given a nonlinear operator $\mathscr{T}: E\times E\to E$, consider a pair of operator equations

$$x=\mathscr{T}(x,y) \tag{1.1}$$

Mathematics subject classification (2010): 47H07, 47H10, 34A12, 34A45.
Keywords and phrases: Partially ordered Banach algebra, partial measure of noncompactness, Dhage monotone iteration principle, coupled hybrid fixed point theorem, coupled quadratic periodic boundary value problems, existence and approximation theorem.

Paper DEA-11-01

and

$$y = \mathscr{T}(y,x) \tag{1.2}$$

which together are called the nonlinear *coupled operator equations* and the nonlinear operator $\mathscr{T}$ involved in them is called the *coupled operator* on $E \times E$ into E.

A pair (x^*,y^*) of elements in E is called a *coupled fixed point* of the coupled operator $\mathscr{T}$ or a *coupled solution* of the coupled operator equations (1.1) and (1.2) if

$$x^* = \mathscr{T}(x^*,y^*) \quad \text{and} \quad y^* = \mathscr{T}(y^*,x^*). \tag{1.3}$$

A coupled fixed point (x^*,y^*) is called *unique comparable* if there does not exist another coupled fixed point (u^*,v^*) which is comparable to it. A coupled fixed point (x^*,y^*) is called *unique* if it is the only coupled solution of the coupled operator equations (1.1)–(1.2) in the space $E \times E$. Finally, a point (x^*,y^*) is called a *fixed point* if $x^* = y^*$, i.e., $x^* = \mathscr{T}(x^*,x^*)$.

The coupled hybrid fixed point theorems for mixed monotone condensing operators using the properties of cones in an ordered Banach space have been proved by Chang and Ma [6], Nistri *et.al* [36], Sun [38] and references therein. Similarly coupled hybrid fixed point theorems for mixed monotone partially condensing coupled mappings in a partially ordered metric space guaranteeing the existence of coupled fixed points have been proved in Dhage [19] which include the coupled fixed point theorems of Bhaskar and Lakshmikantham [4], Berinde [3], Dhage and Dhage [27] and Dhage [18] as special cases. Bhaskar and Lakshmikatham [4] used a partial contraction type condition on the mixed monotone coupled operator $\mathscr{T}$ which is further generalized by Berinde [3] by generalizing the partial contraction condition to symmetric partial contraction condition for getting the same conclusion via constructive method. However, Dhage [18] used a compactness type topological arguments on the mixed monotonic coupled operator $\mathscr{T}$ and obtained an algorithm for the coupled solutions for the coupled operator equations (1.1)–(1.2). Sometimes it may happen that the mixed monotone operator $\mathscr{T}$ is neither contraction nor satisfies the compactness type condition, but the splitting of the coupled operator $\mathscr{T}$ into three operators or coupled operators $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ into the form $\mathscr{T} = \mathscr{F}\mathscr{G} + \mathscr{H}$ satisfy the above criteria separately, where the product of coupled operators $\mathscr{F}\mathscr{G} : E \times E \to E$ is defined by $\big(\mathscr{F}\mathscr{G}\big)(x,y) = \mathscr{F}(x,y)\mathscr{G}(x,y)$. See Dhage [10, 12, 13] and the references therein. So in this case it is interesting to establish the coupled hybrid fixed point theorems involving the sum and product of three operators in a partially ordered Banach algebra (cf. Dhage [15, 16, 17]).

The rest of the paper is organized as follows: Section 2 deals with the preliminaries and auxiliary results concerning the Janhavi sets and the regularity of the partially ordered Banach space which will be used in the subsequent part of the paper. Section 3 deals with the main coupled hybrid fixed point theorems and their various consequences. The mixed coupled hybrid fixed point theorems are presented in Section 4. Section 5 consists of coupled hybrid quadratic periodic boundary value problems (in short QPBVPs) of first order nonlinear differential equations and the related results to be used in the subsequent section of the paper. The existence and approximation results for coupled hybrid QPBVPs of first order nonlinear differential equations are given in

Section 6. Section 7 deals with the coupled hybrid QPBVPs of second order nonlinear differential equations and related existence and approximation results. Some illustrative numerical examples of the first and second order QBVPs are given in Section 8. Finally some concluding remarks are given in Section 9. We claim that the results of this paper are new to the literature on nonlinear analysis and applications.

2. Preliminaries and auxiliary results

Throughout this section, unless otherwise mentioned, let $(E,\leqslant,d)$ denote a partially ordered metric space with partial order $\leqslant$ and the metric d on X. The partially ordered normed linear spaces and the order Banach spaces are the examples of a partially ordered metric space. Two elements x and y in E are said to be *comparable* if either the relation $x\leqslant y$ or $y\leqslant x$ holds. A non-empty subset C of E is called a *chain* or *totally ordered* if all the elements of C are comparable. It is known that E is *regular* if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E and $x_n\to x^*$ as $n\to\infty$, then $x_n\leqslant x^*$ (resp. $x_n\geqslant x^*$) for all $n\in\mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Guo and Lakshmikantham [30] and the references therein. Similarly a few details of a partially ordered normed linear space are given in Dhage [14] while orderings defined by different order cones are given in Deimling [7], Guo and Lakshmikantham [30], Heikkilä and Lakshmikantham [31], Carl and Heikkilä [5], Zeidler [40] and references therein.

We need the following definitions (see Dhage [13, 14, 15, 16, 17] and the references therein) in what follows.

A mapping $\mathscr{T}:E\to E$ is called *isotone* or *monotone nondecreasing* if it preserves the order relation $\leqslant$, that is, if $x\leqslant y$ implies $\mathscr{T}x\leqslant\mathscr{T}y$ for all $x,y\in E$. Similarly, $\mathscr{T}$ is called *monotone nonincreasing* if $x\leqslant y$ implies $\mathscr{T}x\geqslant\mathscr{T}y$ for all $x,y\in E$. Finally, $\mathscr{T}$ is called *monotonic* or simply *monotone* if it is either monotone nondecreasing or monotone nonincreasing on E. A mapping $\mathscr{T}:E\to E$ is called *partially continuous* at a point $a\in E$ if for given $\varepsilon>0$ there exists a $\delta>0$ such that $d(\mathscr{T}x,\mathscr{T}a)<\varepsilon$ whenever x is comparable to a and $d(x,a)<\delta$. $\mathscr{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa. A non-empty subset S of the partially ordered metric space E is called *partially bounded* if every chain C in S is bounded. A mapping $\mathscr{T}$ on a partially ordered metric space E into itself is called *partially bounded* if $\mathscr{T}(E)$ is a partially bounded subset of E. $\mathscr{T}$ is called *uniformly partially bounded* if all chains C in $\mathscr{T}(E)$ are bounded by a unique constant. A non-empty subset S of the partially ordered metric space E is called *partially compact* if every chain C in S is a compact subset of E. A mapping $\mathscr{T}:E\to E$ is called *partially compact* if every chain C in $\mathscr{T}(E)$ is a relatively compact subset of E. $\mathscr{T}$ is called *uniformly partially compact* if $\mathscr{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathscr{T}$ is called *partially totally bounded* if for any bounded subset S of E, $\mathscr{T}(S)$ is a partially totally bounded subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on E.

REMARK 2.1. Suppose that $\mathscr{T}$ is a monotone nondecreasing operator on E into itself. Then $\mathscr{T}$ is a partially bounded or partially compact on E if $\mathscr{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.

DEFINITION 2.1. (Dhage [15, 16], Dhage and Dhage [26]) The order relation $\leqslant$ and the metric d on a non-empty set E are said to be *$\mathscr{D}$-compatible* if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \leqslant, \|\cdot\|)$, the order relation $\leqslant$ and the norm $\|\cdot\|$ are said to be $\mathscr{D}$-compatible if $\leqslant$ and the metric d defined through the norm $\|\cdot\|$ are $\mathscr{D}$-compatible. A subset S of E is called *Janhavi* if the order relation $\leqslant$ and the metric d or the norm $\|\cdot\|$ are $\mathscr{D}$-compatible in it. In particular, if $S = E$, then E is called a *Janhavi metric* or *Janhavi Banach space*.

There do exist several examples of the regular and Janhavi Banach spaces in the literature. In fact, every finite dimensional Euclidean space $\mathbb{R}^n$ is regular as well as Janhavi with respect to the usual componentwise order relation and the standard norm in $\mathbb{R}^n$. The following results are of fundamental importance concerning the regularity of a partially ordered Banach space and the Janhavi sets whereby which it is possible to extend the utility or applicability of the abstract coupled hybrid fixed point theorems of this paper to the variety of nonlinear problems in a natural way.

2.1. Regularity and Janhavi sets

We need often the concepts of regularity and Janhavi sets in a partially ordered and ordered Banach space in the development of coupled hybrid fixed point theory and applications. In the following we obtain some basic results in this direction.

We recall that a non-empty closed and convex subset K of the Banach algebra E is called a *cone* if i) $K + K \subseteq K$, ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$, and iii) $\{-K\} \bigcap K = \{\theta\}$, where θ is a zero element of E. The cone K in E is called *positive* if iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplicative composition in E. The details of cones and their properties may be found in Guo and Lakshmikantham [30], Heikkilä and Lakshmikantham [31] and references therein. We define an order relation $\leqslant$ in the Banach algebra E by

$$x \leqslant y \iff y - x \in K \tag{2.1}$$

for all $x, y \in E$, where K is a positive cone in E. The Banach algebra E together with the order relation $\leqslant$ becomes a partially ordered or simply ordered Banach algebra and it is denoted by (E, K). We observe that every ordered Banach algebra (E, K) is not necessarily a Janhavi Banach algebra as against the claim made in Yang *et.al* [39]. The following two useful lemmas are recently proved in Dhage [22, 23] play a crucial role in this connection. Since the proofs of these lemmas are not well-known, we give the details of proof or completeness and ready reference.

LEMMA 2.1. (Dhage [22, 23]) *Every ordered Banach space (E, K) is regular.*

Proof. Let $\{x_n\}$ be a monotone nondecreasing sequence of points in a partially ordered Banach space (E,K). By monotonic nature, we have

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots. \qquad (*)$$

Suppose that the sequence $\{x_n\}$ converges to a point x^*, that is, $x_n \to x^*$ as $n \to \infty$. Then, every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ also converges to the same limit point x^*, that is, $x_{n_k} \to x^*$ as $k \to \infty$. Since $\{x_n\}$ is nondecreasing, for any given positive integer n, we have $x_n \leqslant x_{n_k}$ for each $k \geqslant n \in \mathbb{N}$. This further by definition of the order relation $\leqslant$ implies that $x_{n_k} - x_n \in K$. As the cone K is closed and convex set in E, one has

$$\lim_{k\to\infty} \left(x_{n_k} - x_n \right) = x^* - x_n \in K$$

for each $n \in \mathbb{N}$. Therefore, $x_n \leqslant x^*$ for all $n \in \mathbb{N}$. Similarly, if $\{x_n\}$ is monotone noincreasing sequence of points in E, then using the similar arguments, it can be proved that $x^* \leqslant x_n$ for all $n \in \mathbb{N}$. As a result, (E,K) is a regular ordered Banach space and the proof of the lemma is complete. □

LEMMA 2.2. (Dhage [21, 22]) *Every partially compact subset S of an ordered Banach space (E,K) is Janhavi.*

Proof. Let C be an arbitrary chain in a partially compact subset S of an ordered Banach space E. Then $C = \overline{C}$ is a compact set in E. Let $\{x_n\}$ be a monotone nondecreasing sequence of points in the chain C, that is,

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots. \qquad (2.2)$$

Then $\{x_n\}$ is a relatively compact set in E. Therefore, $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ converging to a point x^*. We show that $\{x_n\}$ also converges to x^*. Suppose not. Then for $\varepsilon > 0$ there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\|x_{n_i} - x^*\| \geqslant \varepsilon \quad \text{for each } i = 1,2,\ldots. \qquad (2.3)$$

Now, by relative compactness of $\{x_{n_i}\}$, there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \to x'$ as $j \to \infty$. Hence for any given positive integer k, by nondecreasing nature of $\{x_n\}$ it follows that when j is large enough ($j \geqslant k$), we have that $x_{n_k} \leqslant x_{n_{i_j}}$. Then $x_{n_{i_j}} - x_{n_k} \in K$. As K is closed and convex, taking the limit first as $j \to \infty$ and then as $k \to \infty$, we obtain

$$x' - x^* \in K \quad \Longrightarrow \quad x^* \leqslant x'.$$

Similarly, it can be shown that $x' \leqslant x^*$. As a result, we have $x' = x^*$ and that $x_{n_{i_j}} \to x^*$ as $j \to \infty$. Therefore, we get

$$\|x_{n_{i_j}} - x^*\| < \varepsilon \qquad (2.4)$$

for large j. This is a contradiction to (2.3) and the proof of the lemma is complete. □

The above two lemmas, Lemmas 2.1 and 2.2 are very much useful in the study of nonlinear differential and integral equations in ordered Banach spaces for approximation and algorithms of the solutions. Next, we discuss some more information about the regularity and Janhavi sets in the partially ordered normed linear product spaces. The results so obtained in this field are useful in the development of hybrid fixed point theory in nonlinear analysis and applications to the systems of nonlinear equations. For that we consider the following definitions in what follows.

DEFINITION 2.2. A mapping $f:\mathbb{R}_+^n \to \mathbb{R}_+$ is called *sublinear* if

(i) $f(x+y) \leqslant f(x)+f(y)$ (subadditivity), and

(ii) $f(\lambda x) = \lambda f(x)$ (homogeneity)

for all $x,y \in \mathbb{R}_+^n$ and $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$.

DEFINITION 2.3. A continuous mapping $f:\mathbb{R}_+^n \to \mathbb{R}_+$ is called *Kasu function* if

(i) f is sublinear,

(ii) $f(r_1,\dots,r_n) = 0$ if and only if $r_i = 0$ for all i, $i = 1,2,\dots,n$, and

(iii) $f(r_1,\dots,r_n)$ is nondecreasing in each of its co-ordinate variables.

The class of Kasu functions is denoted by $\mathfrak{K}$.

EXAMPLE 2.1. Define a mapping $f_s:\mathbb{R}_+^n \to \mathbb{R}_+$ by

$$f_s(r_1,\dots,r_n) = \sum_{i=1}^{n} a_i r_i, \tag{2.5}$$

where $a_i \in \mathbb{R}$, $a_i > 0$ for all $i = 1,\dots,n$. Then f_s is a Kasu function.

EXAMPLE 2.2. Let the mapping $f_m:\mathbb{R}_+^n \to \mathbb{R}_+$ be defined by

$$f_m(r_1,\dots,r_n) = a \max\{r_1,\dots,r_n\}, \tag{2.6}$$

where $a \in \mathbb{R}$, $a > 0$. Then f_m is a Kasu function.

PROPOSITION 2.1. *Let $\|\cdot\|_1,\dots,\|\cdot\|_n$ be the norms on n vector spaces $E_1,\dots,E_n$ respectively and let $E = E_1 \times \cdots \times E_n$. Then the function $\|\cdot\| : E \to \mathbb{R}_+$ defined by*

$$\|x\| = f\left(\|x_1\|_1,\dots,\|x_n\|_n \right) \tag{2.7}$$

is a norm on E, where $x = \left(x_1,\dots,x_n \right) \in E_1 \times \cdots \times E_n$ and $f \in \mathfrak{K}$.

Proof. Clearly, E is a vector space with respect to the co-ordinatewise addition and scalar multiplication in it. We show that the function $\|\cdot\|$ defined by (2.7) satisfies all the properties of a norm on E.

(i) By definition of the of Kasu function f, we obtain

$$\|x\| = f\left(\|x_1\|_1, \ldots, \|x_n\|_n\right) \geqslant 0$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}_+$. Furthermore, we have

$$\|x\| = 0 \iff f\left(\|x_1\|_1, \ldots, \|x_n\|_n\right) = 0 \iff x_i = 0 \text{ for } i = 1, \ldots, n,$$

and therefore, $\|x\| = 0$ if and only if $x = 0$ in view of property (ii) of the Kasu function f.

(ii) Let $\lambda \in \mathbb{R}$ be arbitrary. Then, by sublinearity of Kasu function f, we obtain

$$\begin{aligned}
\|\lambda x\| &= \|(\lambda x_1, \ldots, \lambda x_n)\| \\
&= f\left(\|\lambda x_1\|_1, \ldots, \|\lambda x_n\|_n\right) \\
&= f\left(|\lambda|\, \|x_1\|_1, \ldots, |\lambda|\, \|x_n\|_n\right) \\
&= |\lambda| f\left(\|x_1\|_1, \ldots, \|x_n\|_n\right) \\
&= |\lambda|\, \|x\|.
\end{aligned}$$

(iii) Next, we prove the triangle inequality for the function $\|\cdot\|$ on E. Let $x, y \in E$ be such that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then, we have

$$\begin{aligned}
\|x + y\| &= \left\| (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \right\| \\
&= \left\| \left(x_1 + y_1, \ldots, x_n + y_n\right) \right\| \\
&= f\left(\|x_1 + y_1\|_1, \ldots, \|x_n + y_n\|_n\right) \\
&\leqslant f\left(\|x_1\|_1 + \|y_1\|_1, \ldots, \|x_n\|_n + \|y_n\|_n\right) \\
&\leqslant f\left(\|x_1\|_1, \ldots, \|x_n\|_n\right) + f\left(\|y_1\|_1, \ldots, \|y_n\|_n\right) \\
&= \|x\| + \|y\|.
\end{aligned}$$

and so, the function $\|\cdot\|$ satisfies the triangle inequality.

(iv) Finally, we show that the function $\|\cdot\|$ is a continuous function on E. To see this, let $\{x^m\}$ be a sequence of points in E converging to a point x in E, where $x^m = (x_1^m, \ldots, x_n^m)$ and $x = (x_1, \ldots, x_n)$. Then, by definition of the function $\|\cdot\|$, we have

$$\begin{aligned}
\big|\, \|x^m\| - \|x\| \,\big| &\leqslant \|x^m - x\| \\
&= \left\| \left(x_1^m, \ldots, x_n^m\right) - \left(x_1, \ldots, x_n\right) \right\| \\
&= \left\| \left(x_1^m - x_1, \ldots, x_n^m - x_n\right) \right\| \\
&= f\Big(\|x_1^m - x_1\|_1, \ldots, \|x_n^m - x_n\|_n\Big).
\end{aligned}$$

Taking the limit as $m \to \infty$ in the above expression, we obtain

$$\begin{aligned}\lim_{m\to\infty} \big|\, \|x^m\| - \|x\| \,\big| &\leqslant \lim_{m\to\infty} f\Big(\|x_1^m - x_1\|_1, \ldots, \|x_n^m - x_n\|_n\Big) \\ &= f\Big(\lim_{m\to\infty} \|x_1^m - x_1\|_1, \ldots, \lim_{m\to\infty} \|x_n^m - x_n\|_n\Big) \\ &= f(0,\ldots,0) \\ &= 0\end{aligned}$$

and so, $\|x^m\| \to \|x\|$ as $m \to \infty$. This shows that $\|\cdot\|$ is a continuous function on the product vector space E.

Thus, the function $\|\cdot\|$ satisfies all the properties of the norm on E and hence $(E, \|\cdot\|)$ is a normed linear space. □

REMARK 2.2. The norm $\|\cdot\|$ defined by (2.7) is called a *Kasu norm* on the product linear or vector space $E_1 \times \cdots \times E_n$.

PROPOSITION 2.2. *Let $(E_1, \|\cdot\|_1), \ldots, (E_n, \|\cdot\|_n)$ be n normed linear spaces and let $E = E_1 \times \cdots \times E_n$. Suppose that the Kasu norm $\|\cdot\|$ is defined by (2.7). If each of the normed linear spaces $E_1, \ldots, E_n$ is complete, then so is also $(E, \|\cdot\|)$.*

Proof. We show that every Cauchy sequence of points in E converges to a point in E. Let $\{x^m\} = \big\{(x_1^m, \ldots, x_n^m)\big\}$ be a Cauchy sequence in E. Then, we have

$$\lim_{m,p\to\infty} \|x^m - x^p\| = 0.$$

Now, by definition of the Kasu norm $\|\cdot\|$, we have that

$$\lim_{m,p\to\infty} f\Big(\|x_1^m - x_1^p\|_1, \ldots, \|x_n^m - x_n^p\|_n\Big) = 0$$

which further yields

$$\lim_{m,p\to\infty} \|x_i^m - x_i^p\|_i = 0$$

for each i, $i = 1, 2, \ldots, n$. This shows that $\{x_i^m\}$ is a Cauchy sequence in E_i for $i = 1, 2, \ldots, n$. Since each E_i is complete, the sequence $\{x_i^m\}$ converges to a point, say $x_i^* \in E_i$ for $i = 1, 2, \ldots, n$. As a result, we have

$$\lim_{m\to\infty} \|x_i^m - x_i^*\|_i = 0,\ i = 1, 2, \ldots, n.$$

Now, by definition of the norm $\|\cdot\|$ we obtain

$$\begin{aligned}\lim_{m\to\infty} \|x^m - x^*\| &= \lim_{m\to\infty} \big\|\, (x_1^m, \ldots, x_n^m) - (x_1^*, \ldots, x_n^*) \,\big\| \\ &= \lim_{m\to\infty} \big\|\, (x_1^m - x_1^*, \ldots, x_n^m - x_n^*) \,\big\| \\ &= \lim_{m\to\infty} f\Big(\|x_1^m - x_1^*\|_1, \ldots, \|x_n^m - x_n^*\|_n\Big) \\ &= f\Big(\lim_{m\to\infty} \|x_1^m - x_1^*\|_1, \ldots, \lim_{m\to\infty} \|x_n^m - x_n^*\|_n\Big) \\ &= 0.\end{aligned}$$

As a result every Cauchy sequence in E is convergent and converges to a point in E. Hence, E is a complete normed linear space. □

Next, we introduce the binary operation multiplication "$\cdot$" in $\mathbb{R}^n_+$ as follows. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two elements of $\mathbb{R}^n_+$. Then the multiplication $a \cdot b$ is defined by the co-ordinatewise multiplication as

$$a \cdot b = \big(a_1, \dots, a_n\big) \cdot \big(b_1, \dots, b_n\big) = \big(a_1 b_1, \dots, a_n b_n\big).$$

Similarly, the multiplication "$\cdot$" in the product Banach space $E = E_1 \times \cdots \times E_n$ is defined as the co-ordinatewise multiplication,

$$x \cdot y = \big(x_1, \dots, x_n\big) \cdot \big(y_1, \dots, y_n\big) = \big(x_1 y_1, \dots, x_n y_n\big). \tag{2.8}$$

for $x = \big(x_1, \dots, x_n\big), y = \big(y_1, \dots, y_n\big) \in E = E_1 \times \cdots \times E_n$, where each of $E_1, \dots, E_n$ is a Banach algebra.

DEFINITION 2.4. A Kasu function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is called submultiplicative if $f(a \cdot b) \leqslant f(a) f(b)$ for $a, b \in \mathbb{R}^n_+$.

LEMMA 2.3. *Let $\|\cdot\|_1, \dots, \|\cdot\|_n$ be the norms in the Banach algebras $E_1, \dots, E_n$ respectively and let $E = E_1 \times \cdots \times E_n$. Let $\|\cdot\|$ and "$\cdot$" be the Kasu norm and co-ordinatewise multiplication in E defined by the relations* (2.7) *and* (2.8) *respectively. If the Kasu function f is submultiplicative, then E is also a Banach algebra.*

Proof. Let $x = \big(x_1, \dots, x_n\big), y = \big(y_1, \dots, y_n\big)$ be two elements of the product Banach space $E = E_1 \times \cdots \times E_n$. Then, by (2.8), we obtain

$$x \cdot y = \big(x_1, \dots, x_n\big) \cdot \big(y_1, \dots, y_n\big) = \big(x_1 y_1, \dots, x_n y_n\big).$$

Therefore, by submultiplicativity, we get

$$\begin{aligned}
\|x \cdot y\| &= f\big(\|x_1 y_1\|_1, \dots, \|x_n y_n\|_n\big) \\
&\leqslant f\big(\|x_1\|_1 \|y_1\|_1, \dots, \|x_n\|_n \|y_n\|_n\big) \\
&= f\Big(\big(\|x_1\|_1, \dots, \|x_n\|_n\big) \cdot \big(\|y_1\|_1, \dots, \|y_n\|_n\big)\Big) \\
&\leqslant f\big(\|x_1\|_1, \dots, \|x_n\|_n\big) \cdot f\big(\|y_1\|_1, \dots, \|y_n\|_n\big) \\
&= \|x\| \, \|y\|.
\end{aligned}$$

Hence, $(E, \|\cdot\|)$ is a Banach algebra and the proof of the lemma is complete. □

REMARK 2.3. We remark that Lemma 2.3 is useful in the study of coupled hybrid fixed point theory in Banach algebras and applications on the lines of Dhage [20].

EXAMPLE 2.3. Let $\|\cdot\|_E$ be a norm in a Banach space E. Then the functions $\|\cdot\|_s$ and $\|\cdot\|_m$ defined by

$$\|u\|_s = \|x\|_E + \|y\|_E \tag{2.9}$$

and

$$\|u\|_m = \max\left\{\|x\|_E, \|y\|_E\right\} \tag{2.10}$$

are the norms in E^2 in view of the expressions (2.5) and (2.6), where $u=(x,y)\in E^2$. Moreover, $(E^2,\|\cdot\|_s)$ is also a Banach algebra with respect to the co-ordinatewise multiplication in $E^2 = E\times E$, provided E is a Banach algebra.

Now, we introduce an order relation α in the product metric space $E=E_1\times\cdots\times E_n$. An order relation $\preceq$ is a binary relation which is reflexive, antisymmetric and transitive. Note that a vector or linear space X together with the order relation $\preceq$ is called partially ordered if the following conditions are satisfied.

(i) $x\preceq y \implies \lambda x \preceq \lambda y$ for all $x,y\in X$ and $\lambda\in\mathbb{R}$, $\lambda\geqslant 0$, and

(ii) $x\preceq y \implies x+z\preceq y+z$ for $x,y,z\in X$.

A few details of a partially ordered vector space appear in Dhage [14] and references therein. If a partial order $\preceq$ is introduced in a normed linear space X and which is also complete with respect to the norm, then it is called a partially ordered Banach space.

Let $\alpha_1,\dots,\alpha_n$ be the partial order relations in the partially ordered Banach space $E_1,\dots,E_n$ respectively. Denote $E=E_1\times\cdots\times E_n$ and $\alpha=\alpha_1\times\cdots\times\alpha_n$. We define a partial order α in the product space E as follows. Let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ be two elements of E. Then,

$$x\,\alpha\,y \iff x_i\,\alpha_i\,y_i \text{ for } i=1,2,\dots,n. \tag{2.11}$$

The order relation α defined by the above expression (2.11) is called the *Kasu partial order* on the product Banach space $E=E_1\times\cdots\times E_n$. The product space E together with the above Kasu partial order α becomes a partially ordered linear space and if the norm $\|\cdot\|$ in E is defined by (2.7), then $(E,\alpha,\|\cdot\|)$ becomes a partially ordered Banach space.

EXAMPLE 2.4. Let $\leqslant$ be the partial order relation in a partially ordered Banach space E and let $E^2=E\times E$. Let $Z=(x,y)$ and $W=(u,v)$ be two elements of E^2. Then the binary relations $\preceq_s$ and $\preceq_m$ defined in E^2 by

$$Z\preceq_s W \iff x\leqslant u \wedge y\leqslant v \tag{2.12}$$

and

$$Z\preceq_m W \iff x\leqslant u \wedge y\geqslant v \tag{2.13}$$

are the Kasu partial order relations and called the *simple* and *mixed* Kasu partial order relations in E^2 respectively.

Now, we are equipped with all the necessary details to state the significant results concerning the Janhavi sets and regularity of the partially ordered product Banach space $(E,\alpha,\|\cdot\|)$.

THEOREM 2.1. *Assume that each of the partially ordered Banach spaces* $(E_1,\alpha_1,\|\cdot\|_1),\dots,(E_n,\alpha_n,\|\cdot\|_n)$ *is regular. Suppose that* $E = E_1\times\cdots\times E_n$ *and* $\alpha = \alpha_1\times\cdots\times\alpha_n$. *If the norm* $\|\cdot\|$ *in* E *is defined by Kasu function* (2.7)*, then the partially ordered Banach space* $(E,\alpha,\|\cdot\|)$ *is regular.*

Proof. Suppose first that $\{x^m\}$ is a monotone nondecreasing sequence of points in E. Then $x^m\,\alpha\,x^{m+1}$ for each $m\in\mathbb{N}$. By definition of the partial order α, we obtain $x_i^m\,\alpha_i\,x_i^{m+1}$ for each i, $i=1,2,\dots,n$. Next, we assume that $x^m\to x^*$. Then,

$$\lim_{m\to\infty}\|x^m-x^*\|=0.$$

Now, by definition of the Kasu norm (2.5), we obtain

$$\lim_{m\to\infty}\|x_i^m-x_i^*\|_i=0$$

for each $i=1,\dots,n$. Thus the sequence $\{x_i^m\}$ is monotone nondecreasing and converges to a point x_i^* for $i=1,\dots,n$. Since each $\left(E_i,\alpha_i,\|\cdot\|_i\right)$ is a regular partially ordered Banach space, one has $x_i^m\,\alpha_i\,x_i^*$ for all $m\in\mathbb{N}$ and for each i, $i=1,\dots,n$. Hence, by definition of α, we get $x^m\,\alpha\,x^*$ for all $m\in\mathbb{N}$. Similarly, if $\{x^m\}$ is monotone nonincreasing sequence of points in E, that is, $x^{m+1}\,\alpha\,x^m$ for all $m\in\mathbb{N}$ and if $\{x^m\}$ converges to a point x^*, then it can be shown that $x^*\,\alpha\,x^m$ for all $m\in\mathbb{N}$. As a result $(E,\alpha,\|\cdot\|)$ is a partially ordered regular Banach space. This completes the proof. □

COROLLARY 2.1. *Let* $E_1=(E,\preceq,\|\cdot\|_E)$ *and* $E_2=(E,\succeq,\|\cdot\|_E)$ *be two partially ordered Banach spaces, where* $\succeq$ *is the inverse or reverse of the order relation* $\preceq$ *and let* $E=E_1\times E_2$. *If one of the partially ordered Banach spaces* E_1 *or* E_2 *is regular, then,* $(E,\alpha,\|\cdot\|)$ *is also a regular partially ordered Banach space, where* $\alpha={\preceq}\times{\succeq}$ *is a mixed Kasu order relation and* $\|\cdot\|$ *is a Kasu norm in* E.

Proof. Suppose that $E_1=(E,\preceq,\|\cdot\|_E)$ is a regular partially ordered Banach space and suppose that $\{x_n\}$ is a monotone nondecreasing sequence of points in E_1 with respect to the order relation $\preceq$ converging to a point x^* in E. Then, $\{x_n\}$ is a nonincreasing sequence of points in E with respect to the order relation $\succeq$ converging to the point x^*. As E_1 is regular, we have that $x_n\preceq x^*$ for all $n\in\mathbb{N}$. By definition of $\succeq$ which implies that $x^*\succeq x_n$ for all $n\in\mathbb{N}$. Similarly, if $\{x_n\}$ is a monotone nonincreasing sequence of points in E_1 converging to the point x_*, then $\{x_n\}$ is also a monotone nondecreasing sequence of point in E_2 with respect to the order relation $\succeq$ converging to the same limit point, x_*. As E_1 is regular one has, $x_*\preceq x_n$ for all $n\in\mathbb{N}$. By definition of $\succeq$ which implies that $x^*\succeq x_n$ for all $n\in\mathbb{N}$. As a result, $(E_1,\succeq,\|\cdot\|_E)$ is also a regular partially ordered Banach space. Now the desired conclusion follows by an application of Theorem 2.1. □

COROLLARY 2.2. *Let* $(E,\preceq,\|\cdot\|_E)$ *a regular partially ordered Banach spaces and let* $E^2=E\times E$. *Then,* $(E^2,\preceq_s,\|\cdot\|)$ *is also a regular partially ordered Banach space, where* $\preceq_s$ *is a simple Kasu order relation and* $\|\cdot\|$ *is a Kasu norm in* E.

THEOREM 2.2. *Let* $(E_1, K_1, \|\cdot\|_1), \dots, (E_n, K_n, \|\cdot\|_n)$ *be n ordered Banach spaces. Suppose that* $E = E_1 \times \cdots \times E_n$ *and* $K = K_1 \times \cdots \times K_n$. *If the norm* $\|\cdot\|$ *in* E *is defined by Kasu function* (2.7) *and the Kasu order relation* α *is defined by* (2.11), *then the ordered Banach space* $(E, K, \|\cdot\|)$ *is regular.*

Proof. By Lemma 2.1, each of the ordered Banach spaces $(E_1, K_1, \|\cdot\|_1), \dots, (E_n, K_n, \|\cdot\|_n)$ is regular. Now the desired conclusion follows by an application of Theorem 2.1. □

THEOREM 2.3. *Assume that each of the partially ordered Banach spaces* $(E_1, \alpha_1, \|\cdot\|_1), \dots, (E_n, \alpha_n, \|\cdot\|_n)$ *is Janhavi. Suppose that* $E = E_1 \times \cdots \times E_n$ *and* $\alpha = \alpha_1 \times \cdots \times \alpha_n$. *If the norm* $\|\cdot\|$ *in* E *is defined by Kasu function* (2.7) *and the Kasu order relation* α *is defined by* (2.11), *then partially ordered Banach space* $(E, \alpha, \|\cdot\|)$ *is also Janhavi.*

Proof. Let $\{x^m\}$ be a monotone sequence of points in E and let a subsequence $\{x^{m_k}\}$ of $\{x^m\}$ be convergent converging to the point x^*. Then, from the nature of the sequence $\{x^m\}$, it follows that the sequence $\{x_i^m\}$ is monotone and has a convergent subsequence $\{x_i^{m_k}\}$ converging to a point x_i^* in E_i for $i = 1, \dots, n$. As each partially ordered Banach space $(E_i, \alpha_i, \|\cdot\|_i)$ is Janhavi, we have that $x_i^m \to x_i^*$ as $m \to \infty$ for $i = 1, 2, \dots, n$. Finally, from the definition of the Kasu function it follows that $x^m \to x^*$ as $n \to \infty$. As a result the partially ordered Banach space $(E, \alpha, \|\cdot\|)$ is Janhavi. □

COROLLARY 2.3. *Let* $E_1 = (E, \preceq, \|\cdot\|_E)$ *and* $E_2 = (E, \succeq, \|\cdot\|_E)$ *be two partially ordered Banach spaces, where* $\succeq$ *is the inverse or reverse of the order relation* $\preceq$ *and let* $E^2 = E_1 \times E_2$. *If every partially compact subset of one of the partially ordered Banach spaces* E_1 *or* E_2 *is Janhavi, then, every partially compact subset* S *of* $(E^2, \alpha, \|\cdot\|)$ *is Janhavi, where* $\alpha = \preceq \times \succeq$ *is a mixed Kasu order relation and* $\|\cdot\|$ *is a Kasu norm in* E.

Proof. Assume that every partially compact subset of the partially ordered Banach space E_1 is Janhavi. Let S be an arbitrary partially compact subset of the partially ordered Banach space E_2 and let C be a compact chain in S. Let $\{x_n\}$ be a monotone sequence of points in C with respect to the order relation $\succeq$ in E_2. Then $\{x_n\}$ is also a monotone sequence of points in C with respect to the order relation $\preceq$ in E_1. By our assumption, the convergence of a subsequence of the sequence $\{x_n\}$ to the point x^* implies the convergence of the original sequence $\{x_n\}$ to the point x^*. As a result, the compact chain C is Janhavi. Consequently, every partially compact subset of the partially ordered Banach space E_2 is Janhavi. Now, the desired conclusion follows by an application of Theorem 2.3. □

COROLLARY 2.4. *Let* $(E, \preceq, \|\cdot\|_E)$ *a partially ordered Banach space and let* $E^2 = E \times E$. *If every compact chain in* E *is Janhavi, then every compact chain in* $(E^2, \preceq_s, \|\cdot\|)$ *is also Janhavi, where* $\preceq_s$ *is a simple Kasu order relation and* $\|\cdot\|$ *is a Kasu norm in* E.

THEOREM 2.4. *Let* $\big(E_1,K_1,\|\cdot\|_1\big),\dots,\big(E_n,K_n,\|\cdot\|_n\big)$ *be* n *Janhavi ordered Banach spaces. Suppose that* $E=E_1\times\cdots\times E_n$ *and* $K=K_1\times\cdots\times K_n$*. If the norm* $\|\cdot\|$ *in* E *is defined by the Kasu function* (2.7) *and the Kasu order relation* α *is defined by* (2.11)*, then the ordered Banach space* $(E,K,\|\cdot\|)$ *is also Janhavi.*

Proof. The proof is similar to Theorem 2.3 and hence we omit the details. □

THEOREM 2.5. *Assume that every partially compact subset of each of the partially ordered Banach spaces* $(E_1,\alpha_1,\|\cdot\|_1),\dots,(E_n,\alpha_n,\|\cdot\|_n)$ *is Janhavi. Suppose that* $E=E_1\times\cdots\times E_n$ *and* $\alpha=\alpha_1\times\cdots\times\alpha_n$*. If the norm* $\|\cdot\|$ *in* E *is defined by Kasu function* (2.7) *and the Kasu order relation* α *is defined by* (2.11)*, then every partially compact subset of the partially ordered Banach space* $(E,\alpha,\|\cdot\|)$ *is also Janhavi.*

Proof. Suppose that S is a partially compact subset of the partially ordered Banach space $(E,\alpha,\|\cdot\|)$. Then $S=S_1\times\cdots\times S_n$, where $S_1,\dots,S_n$ are partially compact natural projections of S on $E_1,\dots,E_n$ respectively. Let $\mathscr{C}$ be a chain in S which is compact by virtue of partial compactness of S. Then $\mathscr{C}=C_1\times\cdots\times C_n$, where $C_1,\dots,C_n$ are compact chains and natural projections of $\mathscr{C}$ on $S_1,\dots,S_n$ respectively. Let $\{x^m\}$ be any monotone sequence of points in $\mathscr{C}$. Then, by compactness of $\mathscr{C}$, it has a convergent subsequence $\{x^{m_k}\}$ converging to a point, say $x^*\in\mathscr{C}$. Now, $x^m=\big(x_1^m,\dots,x_n^m\big)$, so that there are monotone sequences $\{x_i^m\}$ in C_i for $i=1,\dots,n$ and subsequences $\{x_i^{m_k}\}$ converging to the points x_i^* in view of the definition of the Kasu norm in E. Since every partially compact subset of the partially ordered Banach spaces $(E_i,\alpha_i,\|\cdot\|_i)$ is Janhavi, the sequence $\{x_i^m\}$ converges to x_i^* for each i, $i=1,\dots,n$. From definition of the norm $\|\cdot\|$ it follows that the original sequence $\{x^m\}$ converges to x^*. This shows that the partially compact subset S of the partially ordered Banach space E is Janhavi. This completes the proof. □

THEOREM 2.6. *Let* $\big(E_1,K_1,\|\cdot\|_1\big),\dots,\big(E_n,K_n,\|\cdot\|_n\big)$ *be* n *ordered Banach spaces. Suppose that* $E=E_1\times\cdots\times E_n$ *and* $K=K_1\times\cdots\times K_n$*. If the norm* $\|\cdot\|$ *in* E *is defined by Kasu function* (2.7) *and the Kasu order relation* α *is defined by* (2.11)*, then every partially compact subset of the ordered Banach space* $(E,K,\|\cdot\|)$ *is also Janhavi.*

Proof. Suppose that S is a partially compact subset of the ordered Banach space E and suppose that $S_1,\dots,S_n$ be the natural projections of S on the ordered Banach spaces $E_1,\dots,E_n$ respectively. Then the sets $S_1,\dots,S_n$ are also partially compact subsets of $E_1,\dots,E_n$ respectively. By Lemma 2.2, each compact chain of the sets $S_1,\dots,S_n$ is Janhavi. Now the desired conclusion follows by an application of Theorem 2.5. This completes the proof. □

DEFINITION 2.5. An element u of the partially ordered set $(E,\preceq)$ is called a lower bound for a pair $\{x,y\}$ of elements in E if $u\preceq x$ and $u\preceq y$. Similarly, an upper bound for a pair of elements in the partially set E is defined. If every pair of elements in E have a lower as well as an upper bound, then the partially ordered set $(E,\preceq)$ is called a lattice. Moreover, if E is a Banach space, then it is called a Banach lattice.

The following results are sometimes useful for proving the uniqueness of fixed point for nonlinear operators and coupled operators on a partially ordered product Banach space satisfying certain partial contraction condition along with the applications to simultaneous nonlinear equations.

LEMMA 2.4. *Let* $(E_1,\alpha_1,\|\cdot\|_1),\dots,(E_n,\alpha_n,\|\cdot\|_n)$ *be* n *partially ordered Banach spaces and let* $E=E_1\times\cdots\times E_n$. *Suppose that* $\|\cdot\|$ *and* α *are respectively the Kasu norm and Kasu partial order in* E *defined by* (2.7) *and* (2.11) *respectively. If every pair of elements in each of* $E_1,\dots,E_n$ *have a lower bound or an upper bound, then every pair of elements in* E *have a lower bound or an upper bound. In particular, the above conclusion holds if each of* $E_1,\dots,E_n$ *is a Banach lattice.*

Proof. Let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ be any two elements of E. Then $x_i,y_i\in E_i$ for each $i=1,\dots,n$. Suppose that each pair of elements in each E_i have a lower bound, say $z_i\in E_i$. Then, we have $z_i\,\alpha_i\,x_i$ and $z_i\,\alpha_i\,y_i$ for each $i=1,\dots,n$. Therefore, the elements $z=(z_1,\dots,z_n)$ serves as a lower bound for the pair of elements $\{x,y\}$ in E. Similarly, if each pair of elements in each E_i have an upper bound for each i, $i=1,\dots,n$, then it can be proved that every pair of elements of E have an upper bound z' in E. Again, if each of $E_1,\dots,E_n$ is a Banach lattice, then the partially ordered product Banach space E is also a Banach lattice and a fortiori, the above conclusion holds for all elements of E. This completes the proof. □

LEMMA 2.5. *If every pair of elements in a partially ordered Banach space* $(E,\preceq,\|\cdot\|_E)$ *have a lower bound or an upper bound, then every pair of elements in the partially ordered product Banach space* $(E^2,\alpha,\|\cdot\|)$ *have a lower bound or an upper bound in* E^2, *where* α *and* $\|\cdot\|$ *are respectively the Kasu partial order and Kasu norm defined* E^2. *In particular, the above conclusion holds if each of* E, *is a Banach lattice.*

Proof. Here, $E_1=E_2$. Hence, the proof of the lemma follows by an application of Lemma 2.4. We omit the details. □

REMARK 2.4. The assertions of Lemma 2.4 remains true if we replace the partially ordered Banach spaces $(E_i,\alpha_i,\|\cdot\|_{E_i})$ with the ordered Banach spaces (E_i,K_i), $i=1,\dots,n$. Similarly the assertion of Lemma 2.5 also remains true if we replace the partially ordered Banach space E with the ordered Banach space (E,K).

2.2. Partial measure of noncompactness

The second most important concept that will be used in the development of coupled hybrid fixed point theory and applications is the partial measure of noncompactness in the partially ordered Banach spaces. A few details concerning the partial measures of noncompactness along with their applications to nonlinear differential and integral equations appear in Dhage [14, 15, 16, 17] and the references therein. For ready reference, we describe in the following some basic facts about the partial measures of noncompactness in a partially ordered Banach space E.

If C is a chain in E, then C' denotes the set of all limit points of C in E. The symbol $\overline{C}$ stands for the closure of C in E defined by $\overline{C} = C \cup C'$. The set $\overline{C}$ is called a closed chain in E. Thus, $\overline{C}$ is the intersection of all closed chains containing C. Clearly, $\inf C, \sup C \in \overline{C}$ provided $\inf C$ and $\sup C$ exist. The $\sup C$ is an element $z \in E$ such that for every $\varepsilon > 0$ there exists a $c \in C$ such that $d(c,z) < \varepsilon$ and $x \leqslant z$ for all $x \in C$. Similarly, $\inf C$ is defined essentially in an analogous way.

In what follows, let $\mathscr{P}_p(E)$ denote the class of all subsets of E with property p. In particular, we denote by $\mathscr{P}_{cl}(E)$, $\mathscr{P}_{bd}(E)$, $\mathscr{P}_{rcp}(E)$, $\mathscr{P}_{cn}(E)$, $\mathscr{P}_{bd,cn}(E)$, $\mathscr{P}_{rcp,cn}(E)$ the family of all nonempty and closed, bounded, relatively compact, chains, bounded chains and relatively compact chains of E respectively. Now we introduce the concept of a partial measure of noncompactness of the chains in E on the lines of Dhage [15, 16, 17]. The related idea of classical measure of noncompactness may be found in Appell [1], Banas and Goebel [2] and references therein.

DEFINITION 2.6. A mapping $\mu_p : \mathscr{P}_{bd,cn}(E) \to \mathbb{R}_+ = [0,\infty)$ is said to be a partial measure of noncompactness in E if it satisfies the following properties:

(P_1) $\emptyset \neq (\mu_p)^{-1}(\{0\}) \subset \mathscr{P}_{rcp,cn}(E)$. (kernel compactivity)

(P_2) $\mu_p(\overline{C}) = \mu_p(C)$. (closure invariance)

(P_3) μ_p is nondecreasing, i.e., if $C \subset D \Rightarrow \mu_p(C) \leqslant \mu_p(D)$. (monotonicity)

(P_4) $\mu_p(\lambda C) = |\lambda|\, \mu_p(C)$. (scalar multiplicativity)

(P_5) $\mu_p(C+D) \leqslant \mu_p(C) + \mu_p(D)$. (subadditivity)

(P_6) If $\{C_n\}$ is a sequence of closed chains from $\mathscr{P}_{bd,cn}(E)$ such that $C_{n+1} \subset C_n$, $n \in \mathbb{N}$ and if $\lim\limits_{n\to\infty} \mu_p(C_n) = 0$, then $\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty. (limit intersection property)

The family of sets described in (P_1) is said to be the *kernel of the partial measure of noncompactness* μ_p and is defined as

$$\ker \mu_p = \big\{ C \in \mathscr{P}_{bd,cn}(E) \,\big|\, \mu_p(C) = 0 \big\}. \tag{2.14}$$

Clearly, $\ker \mu_p \subset \mathscr{P}_{rcp,cn}(E)$. Observe that the intersection set C_∞, from condition (P_3) is a member of the family $\ker \mu_p$. In fact, since $\mu_p(C_\infty) \leqslant \mu_p(C_n)$ for any n, we infer that $\mu_p(C_\infty) = 0$. This yields that $C_\infty \in \ker \mu_p$. This simple observation will be essential in our further investigations.

The partial measure μ_p of noncompactness is called *full* or *complete* if it satisfies

(P_7) $\ker \mu_p = \mathscr{P}_{rcp,cn}(E)$.

Finally, μ_p is said to satisfy *maximum property* if

(P_8) $\mu_p\Big(C_1 \bigcup C_2\Big) = \max\big\{\mu_p(C_1), \mu_p(C_2)\big\}$.

EXAMPLE 2.5. Define three functions $\alpha_p, \beta_p, \delta_p : \mathscr{P}_{bd,cn}(E) \to \mathbb{R}_+$ by

$$\alpha_p(C) = \inf\left\{ r > 0 \;\middle|\; C = \bigcup_{i=1}^{n} C_i,\ \mathrm{diam}\,(C_i) \leqslant r \right\}, \tag{2.15}$$

where $C \in \mathscr{P}_{bd,cn}(E)$ and $\mathrm{diam}\,(C_i) = \sup\{d(x,y) : x, y \in C_i\}$,

$$\beta_p(C) = \inf\left\{ r > 0 \;\middle|\; C \subset \bigcup_{i=1}^{n} \mathscr{B}(x_i, r) \ \text{ for some } x_i \in E \right\}, \tag{2.16}$$

where $\mathscr{B}(x_i, r) = \{x \in E : d(x_i, x) < r\}$, and

$$\delta_p(C) = \mathrm{diam}\,(C) = \sup\left\{\ d(x,y)\ :\ x, y \in C\ \right\}. \tag{2.17}$$

It is easy to prove that α_p, β_p and δ_p are partial measures of noncompactness and are respectively called the partial Kuratowskii, partial ball and partial diametric measures of noncompactness in E. Note that partial measures α_p and β_p are full or complete and enjoy the maximum property in E but the partial measure δ_p is not full as well as does not satisfy the maximum property.

The following proposition is very much useful for obtaining the partial measures of noncompactness of the partially ordered product Banach spaces provided the partial measures of associated components in the partially ordered Banach spaces are known to us.

PROPOSITION 2.3. *Let $\mu_p^1, \ldots, \mu_p^n$ be the partial measures of noncompactness in the n partially ordered Banach spaces $E_1, \ldots, E_n$ respectively and let $E = E_1 \times \cdots \times E_n$. Suppose that $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is a Kasu function. Then the function $\mu_p : \mathscr{P}_{bd,cn}(E) \to \mathbb{R}_+$ defined by*

$$\mu_p(\mathscr{C}) = f\Big(\mu_p^1(C_1), \ldots, \mu_p^n(C_n)\Big) \tag{2.18}$$

is a partial measure of noncompactness in E, where $C_1, \ldots, C_n$ denote the natural projections of the chain $\mathscr{C}$ on $E_1, \ldots, E_n$ respectively.

Proof. We shall show that the function μ_p satisfies all the conditions (P_1) through (P_6) of a partial measure of noncompactness in the partially ordered Banach space $(E, \alpha, \|\cdot\|)$.

(i) *Kernel compactivity:*

Let $C_1, \ldots, C_n$ be the natural projections of the chain $\mathscr{C}$ in E on $E_1, \ldots, E_n$ respectively. Then, $\mu_p(\mathscr{C}) = f\Big(\mu_p^1(C_1), \ldots, \mu_p^n(C_n)\Big) = 0 \Rightarrow \mu_p^i(C_i) = 0$ for each $i = 1, \ldots, n$. Therefore, $C_1, \ldots, C_n$ are relatively compact chains in $E_1, \ldots, E_n$ respectively. As a result $\mathscr{C} = C_1 \times \cdots \times C_n$ is a relatively compact chain in the product Banach space E.

(ii) *Closure invariance:*

Now for any $\mathscr{C} = C_1 \times \cdots \times C_n$, we have that $\overline{C_1 \times \cdots \times C_n} = \overline{C_1} \times \cdots \times \overline{C_n}$. Therefore, we obtain

$$\mu_p(\overline{\mathscr{C}}) = f\Big(\mu_p^1(\overline{C_1}), \ldots, \mu_p^n(\overline{C_n})\Big) = f\Big(\mu_p^1(C_1), \ldots, \mu_p^n(C_n)\Big) = \mu_p(\mathscr{C}).$$

(iii) *Monotonicity:*

Let $\mathscr{C}$ and $\mathscr{D}$ be two chains in E with natural projections $C_1, \ldots, C_n$ and D_1, $\ldots$, D_n on $E_1, \ldots, E_n$ respectively. Suppose that $\mathscr{C} \subset \mathscr{D}$. Then, it follows that $C_i \subset D_i$ for each i, $i = 1, \ldots, n$. Now, by nondecreasing nature of Kasu function in each co-ordinate variable, we obtain

$$\mu_p(\mathscr{C}) = f\Big(\mu_p^1(C_1), \ldots, \mu_p^n(C_n)\Big) \leqslant f\Big(\mu_p^1(D_1), \ldots, \mu_p^n(D_n)\Big) = \mu_p(\mathscr{D}).$$

This shows that μ_p is nondecreasing in E.

(iv) *Scalar multiplicativity:*

Let $\mathscr{C}$ be a bounded chain in E with natural projections $C_1, \ldots, C_n$ on $E_1, \ldots, E_n$ respectively. Then, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\mu_p(\lambda \mathscr{C}) &= f\Big(\mu_p^1(\lambda C_1), \ldots, \mu_p^n(\lambda C_n)\Big) \\
&= f\Big(|\lambda| \mu_p^1(C_1), \ldots, |\lambda| \mu_p^n(C_n)\Big) \\
&= |\lambda| f\Big(\mu_p^1(C_1), \ldots, \mu_p^n(C_n)\Big) \\
&= |\lambda| \mu_p(\mathscr{C})
\end{aligned}$$

and so, μ_p is a scalar multiplicative function in E.

(v) *Subadditivity:*

Let $\mathscr{C}$ and $\mathscr{D}$ be two chains in E with natural projections $C_1, \ldots, C_n$ and $D_1, \ldots,$ C_n on $E_1, \ldots, E_n$ respectively. Then, $\mathscr{C} + \mathscr{D}$ is again a chain with natural projection $C_1 + D_1, \ldots, C_n + D_n$ on $E_1, \ldots, E_n$ respectively. Now, by sublinearity of the Kasu function, we obtain

$$\begin{aligned}
\mu_p(\mathscr{C} + \mathscr{D}) &= f\Big(\mu_p^1(C_1 + D_1), \ldots, \mu_p^n(C_n + D_n)\Big) \\
&\leqslant f\Big(\mu_p^1(C_1) + \mu_p^1(D_1), \ldots, \mu_p^n(C_n) + \mu_p^n(D_n)\Big) \\
&\leqslant f\Big(\mu_p^1(C_1), \ldots, \mu_p^n(C_n)\Big) + f\Big(\mu_p^1(D_1), \ldots, \mu_p^n(D_n)\Big) \\
&= \mu_p(\mathscr{C}) + \mu_p(\mathscr{D}).
\end{aligned}$$

which proves that μ_p is subadditive in E.

(vi) *Limit intersection property:*

Let $\{\mathscr{C}^m\}$ be a decreasing sequence of closed and bounded sets in the partially ordered set E, that is, $\mathscr{C}^1 \supset \cdots \supset \mathscr{C}^m \cdots$; and let us assume that $\lim_{m\to\infty} \mu_p(\mathscr{C}^m) = 0$.

Suppose that $C_1^m,\dots,C_n^m$ be the natural projections of the chain $\mathscr{C}^m$ on $E_1,\dots,E_n$ respectively. For the sake of convenience we write this as $\mathscr{C}=C_1\times\cdots\times C_n$. Then $\{C_i^m\}$ is also a decreasing sequence of closed and bounded chains in the partially ordered Banach space E_i for $i=1,\dots,n$. Now by definition of μ_p,

$$\mu_p(\mathscr{C}^m)=f\Big(\mu_p^1(C_1^m),\dots,\mu_p^n(C_n^m)\Big).$$

Therefore,

$$\begin{aligned}\lim_{m\to\infty}\mu_p(\mathscr{C}^m) &= \lim_{m\to\infty} f\Big(\mu_p^1(C_1^m),\dots,\mu_p^n(C_n^m)\Big)\\ &= f\Big(\lim_{m\to\infty}\mu_p^1(C_1^m),\dots,\lim_{m\to\infty}\mu_p^n(C_n^m)\Big)\\ &=0\end{aligned}$$

if and only if $\lim\limits_{m\to\infty}\mu_p^i(C_i^m)=0$ for $i=1,\dots,n$. As μ_p^i's are the partial measures of noncompactness in the partially ordered Banach spaces E_i, we have that $\bigcap_{m=1}^{\infty}C_i^m=C_i^{\infty}\neq\emptyset$ for each i, $i=1,\dots,n$. Therefore, we obtain

$$\bigcap_{m=1}^{\infty}\mathscr{C}^m=\overline{\mathscr{C}^{\infty}}=\overline{C_1^{\infty}}\times\cdots\times\overline{C_n^{\infty}}\neq\emptyset.$$

Thus the function μ_p satisfies all the properties (P_1) through (P_6) of the partial measure of noncompactness and hence it is a partial measure of noncompactness in E. This completes the proof. □

REMARK 2.5. The partial measure μ_p of noncompactness defined by (2.17) is called a *partial Kasu measure* in the product linear or vector space $E_1\times\cdots\times E_n$.

EXAMPLE 2.6. Let $\mu_p^1,\dots,\mu_p^n$ be the partial measures of noncompactness in the n partially ordered Banach spaces $E_1,\dots,E_n$ respectively and let $E=E_1\times\cdots\times E_n$. Define two functions μ_p^s and μ_p^m on $\mathscr{P}_{bd,cn}(E)$ by

$$\mu_p^s(\mathscr{C})=\sum_{i=1}^{n}a_i\,\mu_p^i(C_i),\ a_i>0\,\forall\, i, \tag{2.19}$$

and

$$\mu_p^m(\mathscr{C})=a\max\big\{\,\mu_p^1(C_1),\dots,\mu_p^n(C_n)\,\big\},\ a>0, \tag{2.20}$$

where $C_1,\dots,C_n$ are the natural projections of the chain $\mathscr{C}$ on $E_1,\dots,E_n$ respectively. Then the functions μ_p^s and μ_p^m are Kasu partial measures of noncompactness in E, because here the Kasu functions f_s and f_m are defined by (2.5) and (2.6) respectively.

EXAMPLE 2.7. Let μ_p be the partial measure in the partially ordered Banach space E. Then the partial measures μ_p^s and μ_p^m of noncompactness of a chain $\mathscr{C}=C\times D$ in $E^2=E\times E$ may be defined as

$$\mu_p^s(\mathscr{C})=\mu_p(C)+\mu_p(D) \tag{2.21}$$

and

$$\mu_p^m(\mathscr{C}) = \max\{\mu_p(C), \mu_p(D)\} \tag{2.22}$$

where, C and D are the natural projections or components of the chain $\mathscr{C}$ of E^2 in E.

We employ the partial measure $\widetilde{\mu_p}$ of noncompactness given by (2.21) in the study of coupled operators and coupled equations in the subsequent part of this paper. The following definition of partially condensing monotone mappings on a partially ordered Banach space is well-known and may found in Dhage [15, 16, 17], however it is new for the monotone mappings on a partially ordered Banach space of the product form $E\times E$ into itself.

DEFINITION 2.7. Let $(E^2,\alpha,\|\cdot\|)$ be a partially ordered Banach space, where α and $\|\cdot\|$ are Kasu partial order and Kasu norm in E^2 respectively. An operator $\mathscr{T}: E^2 \to E^2$ is called monotone nondecreasing if it preserves the order relation α in E^2, that is, $\mathscr{T}z\,\alpha\,\mathscr{T}w$ for all $z,w\in E^2$, $z\,\alpha\,w$. Similarly, an operator $\mathscr{T}$ on E^2 into itself is called monotone nonincreasing if $\mathscr{T}z\,\alpha'\,\mathscr{T}w$ for all $z,w\in E^2$, $z\,\alpha\,w$, where α' is the reverse of the order relation α in E. Finally, an operator $\mathscr{T}$ is called *monotone* if it is either monotone nondecreasing or monotone nonincreasing on E.

DEFINITION 2.8. Let $(E^2,\alpha,\|\cdot\|)$ be a partially ordered Banach space, where α and $\|\cdot\|$ are Kasu partial order and Kasu norm in E^2 respectively. A monotone mapping $\mathscr{T}: E\times E\to E\times E$ is called *partially condensing* if

$$\widetilde{\mu_p}\,\big(\,\mathscr{T}(\mathscr{C})\,\big) < \widetilde{\mu_p}(\mathscr{C}) \tag{2.23}$$

for all bounded chains $\mathscr{C}$ in $E\times E$ for which $\widetilde{\mu_p}(\mathscr{C})>0$.

Note that monotone partially compact and monotone partially contractions operators on $E\times E$ are partially condensing, however the converse may not be true. Now we state a basic hybrid fixed point theorem for partially condensing monotone mappings in a higher dimensional partially ordered product space which is useful in the development of coupled hybrid fixed point theory and applications.

THEOREM 2.7. *Let $\big(E,\preceq,\|\cdot\|_E\big)$ be a regular partially ordered Banach space and let every compact chain C in E be Janhavi. Suppose that α and $\|\cdot\|$ be the Kasu partial order and Kasu norm defined in E^2 respectively and suppose that $\mathscr{Q}: E\times E\to E\times E$ is a monotone nondecreasing, partially continuous, partially bounded and partially condensing operator. If there exists an element $(x_0,y_0)\in E\times E$ such that $(x_0,y_0)\,\alpha\,\mathscr{Q}(x_0,y_0)$ or $\mathscr{Q}(x_0,y_0)\,\alpha\,(x_0,y_0)$, then $\mathscr{Q}$ has a fixed point $(x^*,y^*)\in E\times E$ and the sequence $\{\mathscr{Q}^n(x_0,y_0)\}$ of successive iterations converges monotonically to (x^*,y^*).*

Proof. Set $E^2=E\times E$. As α and $\|\cdot\|$ are respectively the Kasu order and Kasu norm in E^2, the triplet $(E^2,\alpha,\|\cdot\|)$ is a regular partially ordered Banach space and every compact chain $\mathscr{C}$ in E^2 is Janhavi in view of Corollarries 2.2 and 2.4. Furthermore, since the operator $\mathscr{Q}$ is a partially continuous, partially bounded, partially condensing and monotone nondecreasing on $(E^2,\alpha,\|\cdot\|)$ into itself and there exists an element $(x_0,y_0)\in E\times E$ such that $(x_0,y_0)\,\alpha\,\mathscr{Q}(x_0,y_0)$ or $\mathscr{Q}(x_0,y_0)\,\alpha\,(x_0,y_0)$, the desired conclusion follows by an application of a hybrid fixed point theorem for partial condensing mappings in a partially ordered Banach space proved in Dhage [15, 16, 17]. This completes the proof. □

COROLLARY 2.5. *Let $\big(E,\preceq,\|\cdot\|_E\big)$ be a regular partially ordered Banach space and let every compact chain C in E be Janhavi. Suppose that α and $\|\cdot\|$ are the Kasu partial order and Kasu norm defined in $E^2=E\times E$ respectively and suppose that $\mathscr{Q}:E^2\to E^2$ is a partially continuous, partially compact and monotone nondecreasing operator. If there exists an element $(x_0,y_0)\in E\times E$ such that $(x_0,y_0)\,\alpha\,\mathscr{Q}(x_0,y_0)$ or $\mathscr{Q}(x_0,y_0)\,\alpha\,(x_0,y_0)$, then $\mathscr{Q}$ has a fixed point $(x^*,y^*)\in E\times E$ and the sequence $\{\mathscr{Q}^n(x_0,y_0)\}$ of successive iterations converges monotonically to (x^*,y^*).*

REMARK 2.6. As mentioned in Dhage [17, 19] the condition

(A) every compact chain C in E is Janhavi,

of Theorem 2.7 may be replaced with a weaker condition that

(B) every compact chain $\mathscr{C}$ in $\mathscr{Q}(E\times E)$ is Janhavi.

We note that condition (A) $\Rightarrow$ condition (B), however the converse may not be true. To see this, let us assume that the condition (A) holds. Then, by Corollary 2.4, every chain $\mathscr{C}$ in $E\times E$ is Janhavi. As $\mathscr{Q}$ is partially continuous, it is continuous on $\mathscr{C}$ and consequently $\mathscr{Q}(\mathscr{C})$ is also again a compact chain in $E\times E$ and so it is Janhavi. As $\mathscr{C}$ is an arbitrary chain in $E\times E$, every compact chain in $\mathscr{Q}(E\times E)$ is Janhavi.

In view of the above remark, Remark 2.6 we obtain the following applicable coupled hybrid fixed point result as a corollary to Theorem 2.7.

COROLLARY 2.6. *Let $\big(E,\preceq,\|\cdot\|_E\big)$ be a regular partially ordered Banach space and let α and $\|\cdot\|$ be the Kasu partial order and Kasu norm defined in E^2 respectively. Suppose that $\mathscr{Q}:E^2\to E^2$ is a monotone nondecreasing operator satisfying the linear partial contraction condition*

$$\|\mathscr{Q}Z-\mathscr{Q}W\|\leqslant k\,\|Z-W\| \tag{2.24}$$

for all comparable elements $Z,W\in E^2$, where $0\leqslant k<1$. If there exists an element $Z_0=(x_0,y_0)\in E\times E$ such that $(x_0,y_0)\,\alpha\,\mathscr{Q}(x_0,y_0)$ or $\mathscr{Q}(x_0,y_0)\,\alpha\,(x_0,y_0)$, then $\mathscr{Q}$ has a unique comparable fixed point $(x^*,y^*)\in E\times E$ and the sequence $\{\mathscr{Q}^n(x_0,y_0)\}$ of successive iterations converges monotonically to (x^*,y^*). Moreover, the fixed point is unique if every pair of elements in E have a lower bound or an upper bound.

Proof. First we show that the condition (B) of Remark 2.6 holds. Let $\mathscr{C}$ be an arbitrary chain in $\mathscr{Q}(E\times E)$ and let $\{Z_n\}$ be a monotone sequence in $\mathscr{C}$. Since $\mathscr{C}$ is compact, it has a convergent subsequence $\{Z_{n_i}\}$ converging to a point, say Z^*. Without loss of generality, we may assume that $Zn = \mathscr{Q}^n(Z)$ for some $Z \in \mathscr{C}$. After simple computation, by condition (2.24), it can be shown that $\{Z_n\}$ is a Cauchy sequence of points in $\mathscr{C}$. As a result, the original sequence $\{Z_n\}$ converges to Z^* and that the compact chain $\mathscr{C}$ is Janhavi in $\mathscr{Q}(E \times E)$. Next, using the routine arguments, it can be shown that the operator $\mathscr{Q}$ is a k-set-contraction on $E \times E$ with respect to the partial Kuratowskii measure of noncompactness with a contraction constant $k < 1$. Now, by a direct application of Theorem 2.7 implies that operator $\mathscr{Q}$ has a fixed point $Z* = (x^*, y^*) \in E \times E$. If there is another fixed point $W^* = (u^*, v^*)$ of $\mathscr{Q}$ which is comparable to Z^*, then from the contraction condition (2.24) we get a contradiction. As a result, $\mathscr{Q}$ has a unique comparable fixed point.

To prove the uniqueness of fixed point, let $W^* = (u^*, v^*)$ be another fixed point of the operator $\mathscr{Q}$. Since given that every pair of elements of the partially ordered Banach space E have a lower or an upper bound, by Lemma 2.5, every pair of elements in E^2 also have a lower or an upper bound. Without loss of generality, we assume that there exists an upper bound U for the pair of elements $\{Z^*, W^*\}$ in E^2. Then, the elements Z^* and W^* are comparable to the element U. By nondecreasing nature of $\mathscr{Q}$, we obtain $Z^* = \mathscr{Q}^n Z^* \preceq \mathscr{Q}^n U$ and $W^* = \mathscr{Q}^n W^* \preceq \mathscr{Q}^n U$ for each $n \in \mathbb{N}$. Now, by contraction condition (2.24), we obtain

$$
\begin{aligned}
\|Z^* - W^*\| &= \|\mathscr{Q}^n Z^* - \mathscr{Q}^n W^*\| \\
&\leqslant \|\mathscr{Q}^n Z^* - \mathscr{Q}^n U\| + \|\mathscr{Q}^n U - \mathscr{Q}^n W^*\| \\
&\leqslant k^n \left[\|Z^* - U\| + \|U - W^*\|\right] \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Hence $Z^* = W^*$ and consequently $\mathscr{Q}$ has a unique fixed point. This completes the proof. □

Notice that Corollary 2.6 includes the main coupled fixed point theorem of Berinde [3] as a special case. Again, Theorem 2.7 and Corollaries 2.5 and 2.6 are useful for proving the existence and approximation of couple solutions to a system of a couple of nonlinear coupled differential and integral equations. Again, Theorem 2.7 and Corollaries 2.5 and 2.6 may be extended with appropriate modifications to nonlinear tripled, quadrupled and in general, n-tupled operators which have again nice applications to the systems of nonlinear tripled or quadrupled or n-tupled differential and integral equations for proving the existence as well as approximation of their solutions along with the algorithms.

Next, in the subsequent part of this paper, we shall discuss only the coupled or bivariate mixed monotone operators as well as mixed nonlinear operators and coupled mixed monotone operators and fixed points in a partially ordered Banach algebra along with applications of some of mixed coupled operators to the coupled periodic boundary problems of ordinary nonlinear first and second order ordinary differential equations. The following chain-contractive definition for coupled mixed monotone operators which plays a significant role in the study of nonlinear coupled operator equations and their appliations to nonlinear coupled integral equations has been introduced by Dhage [19, 20] in a partially ordered Banach space E.

DEFINITION 2.9. (Dhage [19]) A mixed monotone coupled operator $\mathscr{T} : E\times E \to E$ is called partially condensing if

$$\mu_p(\mathscr{T}(C\times D))+\mu_p(\mathscr{T}(D\times C)) < \mu_p(C)+\mu_p(D) \tag{2.25}$$

for all $C,D \in \mathscr{P}_{bd,cn}(E)$ for which $\mu_p(C)+\mu_p(D) > 0$, where μ_p is a full partial measure of noncompactness satisfying the maximum property on $\mathscr{P}_{bd,cn}(E)$.

The following coupled hybrid fixed point theorem for partially condensing mixed monotone coupled operators in a partially ordered Banach space E is proved in Dhage [19].

THEOREM 2.8. (Dhage [19]) *Let $(E,\leqslant,\|\cdot\|)$ be a complete and regular partially ordered normed linear space and let every compact chain C in E be Janhavi. Suppose that $\mathscr{T} : E^2 \to E$ is a partially continuous, partially bounded and partially condensing mixed monotone coupled operator. If there exists an element $(x_0,y_0)\in E\times E$ such that $x_0 \leqslant \mathscr{T}(x_0,y_0)$ and $y_0 \geqslant \mathscr{T}(y_0,x_0)$ or $x_0 \geqslant \mathscr{T}(x_0,y_0)$ and $y_0 \leqslant \mathscr{T}(y_0,x_0)$, then $\mathscr{T}$ has a coupled fixed point (x^*,y^*) and the sequences $\{\mathscr{T}^n(x_0,y_0)\}$ and $\{\mathscr{T}^n(y_0,x_0)\}$ of successive iterations converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled fixed points is compact.*

The above coupled hybrid fixed point theorem includes the coupled hybrid fixed point theorems for partially compact coupled operators proved in Dhage [18] and Dhage and Dhage [27]. Also note that Theorem 2.1 proved in Dhage [19] is only for the partial measures of noncompactness with maximum property, however the result is true for any arbitrary partial measure of noncompactness in the Banach space E. In this context we state the following corollary which is sometimes useful to obtain other coupled hybrid fixed point theorems involving the sum and product of two or three coupled operators in a partially ordered Banach space or Banach algebra.

COROLLARY 2.7. *Let $(E,\leqslant,\|\cdot\|)$ be a complete and regular partially ordered normed linear space and let every compact chain C in E be Janhavi. Suppose that $\mathscr{T} : E^2 \to E$ is a partially continuous, partially bounded and mixed monotone coupled operator satisfying*

$$\delta_p(\mathscr{T}(C\times D))+\delta_p(\mathscr{T}(D\times C)) < \delta_p(C)+\delta_p(D) \tag{2.26}$$

for all $C,D \in \mathscr{P}_{bd,cn}(E)$ for which $\delta_p(C)+\delta_p(D) > 0$. If there exists an element $(x_0,y_0)\in E\times E$ such that $x_0 \leqslant \mathscr{T}(x_0,y_0)$ and $y_0 \geqslant \mathscr{T}(y_0,x_0)$ or $x_0 \geqslant \mathscr{T}(x_0,y_0)$ and $y_0 \leqslant \mathscr{T}(y_0,x_0)$, then $\mathscr{T}$ has a coupled fixed point (x^*,y^*) and the sequences $\{\mathscr{T}^n(x_0,y_0)\}$ and $\{\mathscr{T}^n(y_0,x_0)\}$ of successive iterations converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled fixed points is compact.

Proof. The proof of the corollary is standard and so we omit the details. □

Now we are equipped with all the necessary details to deal with the nonlinear coupled operator equations and the related results. In the following section we prove our main coupled hybrid fixed point theorems of this paper under suitable natural mixed conditions from algebra, analysis and topology.

3. Coupled hybrid fixed point theorems

Given a partially ordered Banach algebra $(E,\preceq,\|\cdot\|)$ and given the three coupled operators $\mathscr{F},\mathscr{G},\mathscr{H}:E\times E\to E$, consider a couple of operator equations

$$x=\mathscr{F}(x,y)\mathscr{G}(x,y)+\mathscr{H}(x,y) \tag{3.1}$$

and

$$y=\mathscr{F}(y,x)\mathscr{G}(y,x)+\mathscr{H}(y,x) \tag{3.2}$$

for all $(x,y)\in E\times E$, where the coupled operators $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ are not necessarily continuous.

The coupled operators $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ involved in the coupled operator equations (3.1)–(3.2) satisfy different algebraic, geometric and topological properties on $E\times E$ into E. A pair of elements $(x^*,y^*)\in E\times E$ is called a *coupled fixed point* of the coupled operator equations (3.1) and (3.2) if

$$x^*=\mathscr{F}(x^*,y^*)\mathscr{G}(x^*,y^*)+\mathscr{H}(x^*,y^*) \tag{3.3}$$

and

$$y^*=\mathscr{F}(y^*,x^*)\mathscr{G}(y^*,x^*)+\mathscr{H}(y^*,x^*). \tag{3.4}$$

The existence and approximation of such coupled fixed points for coupled operators is generally obtained under certain monotonic condition of the coupled operator $\mathscr{T}$ on $E\times E$. See Heikkilä and Lakshmikantham [31], Chang and Ma [6], Bhaskar and Lakshmikantham [4] and Dhage and Dhage [26] and the references therein. A coupled operator $\mathscr{T}(x,y)$ is called *mixed monotone* if the map $x\mapsto\mathscr{T}(x,y)$ is nondecreasing for each $y\in E$ and the map $y\mapsto\mathscr{T}(x,y)$ is nonincreasing for each $x\in E$. A couple of coupled hybrid fixed point theorems for the coupled operator equations (3.3) and (3.4) are proved in Dhage [20] provided one of the coupled operators $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ is symmetric partial $\mathscr{D}$-Lipschitz, however there is a flaw in the arguments with that method. In this paper we improve the coupled hybrid fixed point theorems under slightly stronger partial $\mathscr{D}$-Lipschitz condition.

Before going to the main coupled hybrid fixed point theorems, we state some useful preliminary definitions and auxiliary results in what follows.

Let $(E,\leqslant,\|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$K=\big\{x\in E \mid x\geqslant\theta,\ \text{where } \theta \text{ is the zero element of } E\big\}$$

which is a closed and convex subset of E. The elements of K are called the positive vectors of the normed linear algebra E. Clearly, the set K is positive in view of the fact that it satisfies the relation "$u\cdot v\in K$ whenever $u,v\in K$". The next lemma follows immediately from the definition of the set K which is often used in the applications of hybrid fixed point theory in a partially ordered and an ordered Banach algebra E.

LEMMA 3.1. (Dhage [9, 12]) *If the elements* $u_1, u_2, v_1, v_2 \in K$ *are such that* $u_1 \preceq v_1$ *and* $u_2 \preceq v_2$, *then* $u_1 u_2 \preceq v_1 v_2$.

DEFINITION 3.1. An operator $\mathscr{T} : E \to E$ is said to be positive if the range $R(\mathscr{T})$ of $\mathscr{T}$ is such that $R(\mathscr{T}) \subseteq K$.

DEFINITION 3.2. (Dhage [8, 9, 10]) An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathscr{D}$-function if $\psi(0) = 0$. The class of all $\mathscr{D}$-functions is denoted by $\mathfrak{D}$.

DEFINITION 3.3. (Dhage [14, 15]) A monotone operator $\mathscr{T} : E \to E$ is called partial $\mathscr{D}$-Lipschitz if there exists a $\mathscr{D}$-function $\psi_{\mathscr{T}} \in \mathfrak{D}$ such that

$$\|\mathscr{T}x - \mathscr{T}y\| \leqslant \psi_{\mathscr{T}}\big(\|x-y\|\big) \tag{3.5}$$

for all comparable elements $x, y \in E$. If $\psi_{\mathscr{T}}(r) = kr$, $k > 0$, $\mathscr{T}$ is called a partial Lipschitz operator on $E \times E$ with the Lipschitz constant k. Again, if $0 \leqslant k < 1$, then $\mathscr{T}$ is called a partial contraction on E with contraction constant k. Furthermore, if $\psi_{\mathscr{T}}(r) < r$ for $r > 0$, then $\mathscr{T}$ is called a nonlinear partial $\mathscr{D}$- contraction on E.

DEFINITION 3.4. (Dhage [23]) A monotone coupled operator $\mathscr{T} : E \times E \to E$ is called nonlinear partial $\mathscr{D}$-Lipschitz if there exists a $\mathscr{D}$-function $\psi_{\mathscr{T}} \in \mathfrak{D}$ such that

$$\|\mathscr{T}(x,y) - \mathscr{T}(u,v)\| \leqslant \frac{1}{2} \cdot \psi_{\mathscr{T}}\big(\|x-u\| + \|v-y\|\big) \tag{3.6}$$

for all comparable elements $(x,y), (u,v) \in E \times E$. If $\psi_{\mathscr{T}}(r) = kr$, $\mathscr{T}$ is called a partial Lipschitz on $E \times E$ with the Lipschitz constant k. Again, if $0 \leqslant k < 1$, then $\mathscr{T}$ is called a partial contraction on $E \times E$ with contraction constant k. Furthermore, if $\psi_{\mathscr{T}}(r) < r$ for $r > 0$, then $\mathscr{T}$ is called a nonlinear partial $\mathscr{D}$- contraction on $E \times E$.

DEFINITION 3.5. (Dhage [19, 25]) A coupled operator $\mathscr{T} : E \times E \to E$ is called nonlinear symmetric partial $\mathscr{D}$-Lipschitz if there exists a $\mathscr{D}$-function $\psi_{\mathscr{T}} \in \mathfrak{D}$ such that

$$\|\mathscr{T}(x,y) - \mathscr{T}(u,v)\| + \|\mathscr{T}(y,x) - \mathscr{T}(v,u)\| \leqslant \psi_{\mathscr{T}}\big(\|x-u\| + \|v-y\|\big) \tag{3.7}$$

for all comparable elements $(x,y), (u,v) \in E \times E$. If $\psi_{\mathscr{T}}(r) = kr$, $\mathscr{T}$ is called a symmetric partial Lipschitz on $E \times E$ with the Lipschitz constant k. Furthermore, if $0 \leqslant k < 1$, then $\mathscr{T}$ is called a symmetric partial contraction on $E \times E$ with contraction constant k. Furthermore, if $\psi_{\mathscr{T}}(r) < r$ for $r > 0$, then $\mathscr{T}$ is called a nonlinear symmetric partial $\mathscr{D}$- contraction on $E \times E$.

REMARK 3.1. Note that linear partial contraction coupled operators are considered by Bhaskar and Lakshmikantham [4] whereas symmetric linear partial contraction coupled operators are considered by Berinde [3] in the study of coupled fixed point theorems in the partially ordered metric spaces with applications. It is clear that every partial $\mathscr{D}$-Lipschitz is symmetric partial $\mathscr{D}$-Lipschitz, but the converse may not be true.

A slight generalization of Corollary 2.6 may be stated as follows.

THEOREM 3.1. *Let $(E, \preceq, \|\cdot\|_E)$ be a regular partially ordered Banach space and let α and $\|\cdot\|$ be the Kasu partial order and Kasu norm defined in E^2 respectively. Suppose that $\mathscr{Q}: E^2 \to E^2$ is a monotone nondecreasing operator satisfying the nonlinear partial contraction condition viz., there exists a $\mathscr{D}$-function $\psi \in \mathfrak{D}$ such that*

$$\|\mathscr{Q}Z - \mathscr{Q}W\| \leqslant \psi\big(\|Z-W\|\big) \tag{3.8}$$

for all comparable elements $Z, W \in E^2$, where $\psi(r) < r$, $r > 0$. If there exists an element $Z_0 = (x_0, y_0) \in E \times E$ such that $(x_0, y_0)\, \alpha\, \mathscr{Q}(x_0, y_0)$ or $\mathscr{Q}(x_0, y_0)\, \alpha\, (x_0, y_0)$, then $\mathscr{Q}$ has a unique comparable fixed point $Z^* = (x^*, y^*) \in E \times E$ and the sequence $\{\mathscr{Q}^n(x_0, y_0)\}$ of successive iterations converges monotonically to (x^*, y^*). Moreover, the fixed point is unique if every pair of elements in E have a lower bound or an upper bound.

Proof. The proof is similar to Corollary 2.6 with obvious modifications and now the conclusion follws from the arguments given in the papers of Dhage [14, 15, 16, 17]. We omit the details. □

As a consequence of Theorem 3.1 we obtain the following coupled hybrid fixed point results studied earlier in the literature.

COROLLARY 3.1. (Berinde [3] and Dhage [19]) *Let $(E, \preceq, \|\cdot\|_E)$ be a regular partially ordered Banach space. Suppose that $\mathscr{Q}: E^2 \to E$ is a mixed monotone nondecreasing coupled operator satisfying the condition of nonlinear symmetric partial contraction. If there exists an element $Z_0 = (x_0, y_0) \in E \times E$ such that $x_0 \preceq \mathscr{Q}(x_0, y_0)$ and $\mathscr{Q}(y_0, x_0) \preceq y_0$, then $\mathscr{Q}$ has a unique comparable coupled fixed point $Z^* = (x^*, y^*) \in E \times E$ and the sequences $\{\mathscr{Q}^n(x_0, y_0)\}$ and $\{\mathscr{Q}^n(y_0, x_0)\}$ of successive iterations converge monotonically to x^* and y^* respectively. Moreover, the coupled fixed point is unique if every pair of elements in E have a lower bound or an upper bound.*

Proof. Define a Kasu norm $\|\cdot\|_{E^2}$ and a Kasu partial order $\preceq_m$ in E^2 by the relations

$$\|(x,y)\|_{E^2} = \|x\|_E + \|y\|_E$$

and

$$(x,y) \preceq_m (u,v) \iff x \preceq u \wedge y \succeq v,$$

for all $(x,y), (u,v) \in E^2$. Define an operator $\mathscr{T}: E^2 \to E^2$ by

$$\mathscr{T}(x,y) = \big(\mathscr{Q}(x,y)\,,\, \mathscr{Q}(y,x)\big).$$

We show that the operator $\mathscr{T}$ is a monotone nondecreasing operator and satisfies the condition of nonlinear partial $\mathscr{D}$-contraction on E^2. Let $Z = (x,y)$ and $W = (u,v)$

be any two elements in E^2 such that $Z \succeq_m W$. Then, by mixed monotonicity of operator $\mathscr{Q}$, we obtain $\mathscr{T}Z \succeq_m \mathscr{T}W$. Again, we have

$$\begin{aligned}\|\mathscr{T}Z-\mathscr{T}W\|_{E^2} &= \Big\|\big(\mathscr{Q}(x,y)\,,\,\mathscr{Q}(y,x)\big)-\big(\mathscr{Q}(u,v)\,,\,\mathscr{Q}(v,u)\big)\Big\|_{E^2}\\ &= \Big\|\big(\mathscr{Q}(x,y)-\mathscr{Q}(u,v)\,,\,\mathscr{Q}(y,x)-\mathscr{Q}(v,u)\big)\Big\|_{E^2}\\ &= \|\mathscr{Q}(x,y)-\mathscr{Q}(u,v)\|_E+\|\mathscr{Q}(y,x)-\mathscr{Q}(v,u)\|_E\\ &\leqslant \psi(\|x-u\|_E+\|y-v\|_E)\\ &= \psi(\|Z-W\|_{E^2}).\end{aligned}$$

Now the desired conclusion follows by an application of Theorem 3.1. □

COROLLARY 3.2. (Bhaskar and Lakshmikantham [4]) *Let $\big(E,\preceq,\|\cdot\|_E\big)$ be a regular partially ordered Banach space. Suppose that $\mathscr{Q}:E^2\to E$ is a mixed monotone nondecreasing coupled nonlinear partial contraction. If there exists an element $Z_0=(x_0,y_0)\in E\times E$ such that $x_0\preceq\mathscr{Q}(x_0,y_0)$ and $\mathscr{Q}(y_0,x_0)\preceq y_0$, then $\mathscr{Q}$ has a unique comparable coupled fixed point $(x^*,y^*)\in E\times E$ and the sequences $\{\mathscr{Q}^n(x_0,y_0)\}$ and $\{\mathscr{Q}^n(y_0,x_0)\}$ of successive iterations converge monotonically to x^* and ,y^* respectively. Moreover, the coupled fixed point is unique if every pair of elements in E have a lower bound or an upper bound.*

The following two lemmas are fundamental in the study of coupled hybrid fixed point theorems for the sum and product of three coupled operators on a partially ordered Banach algebra.

LEMMA 3.2. *Let $(E,\leqslant,\|\cdot\|)$ be a partially ordered Banach space and let $\mathscr{T}: E\times E\to E$ be a mixed monotone, partially bounded and partial $\mathscr{D}$-Lipschitz coupled operator with $\mathscr{D}$-function $\psi_{\mathscr{T}}$. Then,*

$$\alpha_p\big(\mathscr{T}(C\times D)\big)+\alpha_p\big(\mathscr{T}(D\times C)\big)\leqslant\psi_{\mathscr{T}}\big(\alpha_p(C)+\alpha_p(D)\big) \tag{3.9}$$

for all bounded chains C and D in E, where α_p is a partial Kuratowskii measure of noncompactness in E.

Proof. Let $\varepsilon>0$ be given and let C and D be two bounded chains in the partially ordered Banach space E. Then, by partial Kuratowskii measure of noncompactness, there exist subchains $C_1,C_2,\cdots,C_m$ of C such that

$$C=\bigcup_{i=1}^{m}C_i \quad \text{and} \quad \operatorname{diam}C_i<\alpha_p(C)+\frac{\varepsilon}{2}. \tag{3.10}$$

Similarly, there exist subchains $D_1,D_2,\cdots,D_n$ of D such that

$$D=\bigcup_{j=1}^{n}D_j \quad \text{and} \quad \operatorname{diam}D_j<\alpha_p(D)+\frac{\varepsilon}{2}. \tag{3.11}$$

Now,

$$\begin{aligned}\operatorname{diam}\Big(\mathscr{T}(C\times D)\Big) &= \sup_{(x,y),(u,v)\in C\times D} \|\mathscr{T}(x,y)-\mathscr{T}(u,v)\|\\ &\leqslant \frac{1}{2}\cdot \sup_{(x,y),(u,v)\in C\times D} \psi_{\mathscr{T}}\Big(\|x-u\|+\|v-y\|\Big)\\ &\leqslant \frac{1}{2}\cdot \psi_{\mathscr{T}}\Big(\operatorname{diam}C+\operatorname{diam}D\Big). \qquad (3.12)\end{aligned}$$

Similarly, we have

$$\operatorname{diam}\Big(\mathscr{T}(D\times C)\Big) \leqslant \frac{1}{2}\cdot \psi_{\mathscr{T}}\Big(\operatorname{diam}D+\operatorname{diam}C\Big). \qquad (3.13)$$

Therefore, from (3.10) and (3.12) we obtain,

$$\begin{aligned}\alpha_p\,\big(\,\mathscr{T}(C\times D)\,\big) &\leqslant \operatorname{diam}\,\big(\,\mathscr{T}(C_i\times D_j)\,\big)\\ &\leqslant \frac{1}{2}\cdot \psi_{\mathscr{T}}\Big(\operatorname{diam}C_i+\operatorname{diam}D_j\Big)\\ &\leqslant \frac{1}{2}\cdot \psi_{\mathscr{T}}\Big(\alpha_p(C)+\alpha_p(D)+\varepsilon\Big). \qquad (3.14)\end{aligned}$$

Similarly, from (3.11) and (3.13) we obtain

$$\alpha_p\,\big(\,\mathscr{T}(D\times C)\,\big)\leqslant \frac{1}{2}\cdot \psi_{\mathscr{T}}\Big(\alpha_p(D)+\alpha_p(C)+\varepsilon\Big). \qquad (3.15)$$

Now adding (3.14) and (3.15) together implies that

$$\alpha_p\,\big(\,\mathscr{T}(C\times D)\,\big)+\alpha_p\,\big(\,\mathscr{T}(D\times C)\,\big)\leqslant \psi_{\mathscr{T}}\Big(\alpha_p(C)+\alpha_p(D)+\varepsilon\Big).$$

Since ε is arbitrary, one has

$$\alpha_p\,\big(\,\mathscr{T}(C\times D)\,\big)+\alpha_p\,\big(\,\mathscr{T}(D\times C)\,\big)\leqslant \psi_{\mathscr{T}}\Big(\alpha_p(C)+\alpha_p(D)\Big).$$

This completes the proof. □

LEMMA 3.3. *Let* $(E,\leqslant,\|\cdot\|)$ *be a partially ordered Banach algebra and let* $\mathscr{F},\mathscr{G} : E\times E\to K$ *be two mixed monotone coupled operators satisfying the following conditions.*

(a) $\mathscr{F}$ *is a partially bounded and partial* $\mathscr{D}$*-Lipschitz coupled operator with bound* $M_{\mathscr{F}}$ *and* $\mathscr{D}$*-function* $\psi_{\mathscr{F}}$*, and*

(b) $\mathscr{G}$ *is a uniformly partially compact with uniform bound* $M_{\mathscr{G}}$*.*

If the coupled operator $\mathscr{T}: E\times E \to E$ *be defined by* $\mathscr{T}(x,y) = \mathscr{F}(x,y)\mathscr{G}(x,y)$*, then* $\mathscr{T}$ *is partially continuous and partially bounded mixed monotone coupled operator satisfying*

$$\alpha_p\big(\mathscr{T}(C\times D)\big)+\alpha_p\big(\mathscr{T}(D\times C)\big) \leqslant M_{\mathscr{G}}\,\psi_{\mathscr{F}}\big(\alpha_p(C)+\alpha_p(D)\big) \tag{3.16}$$

for all bounded chains C *and* D *in* E*.*

Proof. Define an operator $\mathscr{T}: E\times E \to E$ by

$$\mathscr{T}(x,y) = \mathscr{F}(x,y)\mathscr{G}(x,y) \tag{3.17}$$

so that we have

$$\mathscr{T}(y,x) = \mathscr{F}(y,x)\mathscr{G}(y,x). \tag{3.18}$$

Since $\mathscr{F}$ is a partial $\mathscr{D}$-Lipschitz, it is partially continuous on $E\times E$. As a result, $\mathscr{T} = \mathscr{F}\mathscr{G}$ is well defined and partially continuous coupled operator on $E\times E$ into E. As the coupled operators $\mathscr{F}$ and $\mathscr{G}$ are mixed monotone and partially bounded, the coupled operator $\mathscr{T}$ is also a mixed monotone and partially bounded on $E\times E$ into K in view of Lemma 3.1. We show that $\mathscr{T}$ satisfies the condition (3.16) on $E\times E$ into E.

Let $(x,y),(u,v)\in E\times E$ be two comparable elements. Then, by definition of the coupled operator $\mathscr{T}$, we obtain

$$\begin{aligned}
&\|\mathscr{T}(x,y)-\mathscr{T}(u,v)\|\\
&\qquad = \|\mathscr{F}(x,y)\mathscr{G}(x,y)-\mathscr{F}(u,v)\mathscr{G}(u,v)\|\\
&\qquad \leqslant \|\mathscr{G}(x,y)\|\,\|\mathscr{F}(x,y)-\mathscr{F}(u,v)\| + \|\mathscr{F}(u,v)\|\,\|\mathscr{G}(x,y)-\mathscr{G}(u,v)\|\\
&\qquad \leqslant M_{\mathscr{G}}\,\|\mathscr{F}(x,y)-\mathscr{F}(u,v)\| + M_{\mathscr{F}}\,\|\mathscr{G}(x,y)-\mathscr{G}(u,v)\|\\
&\qquad \leqslant \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\|x-u\|+\|v-y\|\Big) + M_{\mathscr{F}}\,\|\mathscr{G}(x,y)-\mathscr{G}(u,v)\|.
\end{aligned} \tag{3.19}$$

Similarly, we have

$$\begin{aligned}
\|\mathscr{T}(y,x)-\mathscr{T}(v,u)\| \leqslant{}& \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\|x-u\|+\|v-y\|\Big)\\
&+ M_{\mathscr{F}}\,\|\mathscr{G}(y,x)-\mathscr{G}(v,u)\|.
\end{aligned} \tag{3.20}$$

Let $\varepsilon > 0$ be given and let C and D be two bounded chains in the Banach algebra E satisfying the conditions (3.10) and (3.11) respectively. Then,

$$\begin{aligned}
\operatorname{diam}\Big(\mathscr{T}(C\times D)\Big) &= \sup_{(x,y),(u,v)\in C\times D} \|\mathscr{T}(x,y)-\mathscr{T}(u,v)\|\\
&\leqslant \sup_{(x,y),(u,v)\in C\times D} \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\|x-u\|+\|v-y\|\Big)\\
&\quad + \sup_{(x,y),(u,v)\in C\times D} M_{\mathscr{F}}\,\|\mathscr{G}(x,y)-\mathscr{G}(u,v)\|\\
&\leqslant \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\operatorname{diam} C+\operatorname{diam} D\Big) + M_{\mathscr{F}}\operatorname{diam}\mathscr{G}(C\times D).
\end{aligned} \tag{3.21}$$

Similarly, we have

$$\operatorname{diam}\Big(\mathscr{T}(D\times C)\Big) \leqslant \frac{1}{2}M_{\mathscr{G}}\psi_{\mathscr{F}}\Big(\operatorname{diam}C+\operatorname{diam}D\Big)+M_{\mathscr{F}}\operatorname{diam}\mathscr{G}(D\times C). \tag{3.22}$$

Next, since $\mathscr{G}$ is partially compact, $\mathscr{G}(C\times D)=G$ is relatively compact subset of E. Therefore, for above $\varepsilon>0$, by partial Kuratowskii measure of noncompactness, there exist subchains G_1, $G_2,\dots,G_{m_1}$ of G such that

$$\mathscr{G}(C\times D)=\bigcup_{\ell=1}^{m_1}G_\ell=G \quad\Longrightarrow\quad \bigcup_{\ell=1}^{m_1}\mathscr{G}^{-1}(G_\ell)=C\times D$$

and

$$\operatorname{diam}(G_\ell)<\frac{\varepsilon}{2M_{\mathscr{F}}} \text{ for each } \ell. \tag{3.23}$$

Similarly, $\mathscr{G}(D\times C)=G'$ is a relatively compact subset of E and, for above $\varepsilon>0$, by partial Kuratowskii measure of noncompactness, there exist subchains G'_1, $G'_2,\dots,G'_{n_1}$ of G' such that

$$\mathscr{G}(D\times C)=\bigcup_{k=1}^{n_1}G'_k=G' \quad\Longrightarrow\quad \bigcup_{k=1}^{n_1}\mathscr{G}^{-1}(G'_k)=D\times C$$

and

$$\operatorname{diam}(G'_k)<\frac{\varepsilon}{2M_{\mathscr{F}}} \text{ for each } k. \tag{3.24}$$

Denote

$$\mathtt{C}_{\ell,i,j}=\mathscr{G}^{-1}(G_\ell)\bigcap\big(C_i\times D_j\big).$$

Then we have

$$\bigcup_{\ell,i,j}\mathtt{C}_{\ell,i,j}=C\times D.$$

Similarly, let

$$\mathtt{C}'_{k,j,i}=\mathscr{G}^{-1}(G'_k)\bigcap\big(D_j\times C_i\big).$$

Therefore, we have

$$\bigcup_{k,i,j}\mathtt{C}_{k,j,i}=D\times C.$$

Now, from (3.21) and (3.23), it follows that

$$\begin{aligned}\alpha_p\ \big(\ \mathscr{T}(C\times D)\ \big) &\leqslant \operatorname{diam}\ \big(\ \mathscr{T}\big(\mathtt{C}_{\ell,i,j}\big)\ \big)\\ &\leqslant \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\operatorname{diam}C_i+\operatorname{diam}D_j\Big)+M_{\mathscr{F}}\operatorname{diam}\mathscr{G}\big(\mathtt{C}_{\ell,i,j}\big)\\ &\leqslant \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\operatorname{diam}C_i+\operatorname{diam}D_j\Big)+M_{\mathscr{F}}\operatorname{diam}G_\ell\\ &< \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\alpha_p(C)+\alpha_p(D)+\varepsilon\Big)+\frac{\varepsilon}{2}.\end{aligned} \tag{3.25}$$

Similarly, from (3.22) and (3.24) it follows that

$$\begin{aligned}\alpha_p\,\big(\,\mathscr{T}(D\times C)\,\big) &\leqslant \operatorname{diam}\,\big(\,\mathscr{T}(\mathtt{C}'_{k,i,j})\,\big)\\ &\leqslant \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\operatorname{diam}C_i+\operatorname{diam}D_j\Big)+M_{\mathscr{F}}\operatorname{diam}\mathscr{G}\big(\mathtt{C}'_{k,i,j}\big)\\ &\leqslant \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\operatorname{diam}C_i+\operatorname{diam}D_j\Big)+M_{\mathscr{F}}\operatorname{diam}G'_k\\ &< \frac{1}{2}M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\alpha_p(C)+\alpha_p(D)+\varepsilon\Big)+\frac{\varepsilon}{2}. \end{aligned}\tag{3.26}$$

Adding expressions (3.25) and (3.26) together implies that

$$\alpha_p\,\big(\,\mathscr{T}(C\times D)\,\big)+\alpha_p\,\big(\,\mathscr{T}(D\times C)\,\big)< M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\alpha_p(C)+\alpha_p(D)+\varepsilon\Big)+\varepsilon.$$

Since ε is arbitrary, we have

$$\alpha_p\,\big(\,\mathscr{T}(C\times D)\,\big)+\alpha_p\,\big(\,\mathscr{T}(D\times C)\,\big)\leqslant M_{\mathscr{G}}\,\psi_{\mathscr{F}}\Big(\alpha_p(C)+\alpha_p(D)\Big)$$

for all bounded chains C and D in E and the proof of the lemma is complete. □

THEOREM 3.2. *Let $(E,\leqslant,\|\cdot\|)$ be a partially ordered Banach algebra and let every compact chain C in E be Janhavi. Suppose that $\mathscr{F},\mathscr{G}:E\times E\to K$ and $\mathscr{H}:E\times E\to E$ are three mixed monotone coupled operators satisfying the following conditions.*

- *(a) $\mathscr{F}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{F}}$,*
- *(b) $\mathscr{G}$ is partially continuous and uniformly partially compact with uniform bound $M_{\mathscr{G}}=\sup\big\{\|\mathscr{G}(C\times D)\|:C,D\in\mathscr{P}_{bd,cn}(E)\big\}$,*
- *(c) $\mathscr{H}$ is partially bounded and partially $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{H}}$,*
- *(d) $M_{\mathscr{G}}\,\psi_{\mathscr{F}}(r)+\psi_{\mathscr{H}}(r)<r$ for $r>0$, and*
- *(e) there exists an element $(x_0,y_0)\in E\times E$ such that $x_0\leqslant\mathscr{F}(x_0,y_0)\mathscr{G}(x_0,y_0)+\mathscr{H}(x_0,y_0)$ and $y_0\geqslant\mathscr{F}(y_0,x_0)\mathscr{G}(y_0,x_0)+\mathscr{H}(y_0,x_0)$ or $x_0\geqslant\mathscr{F}(x_0,y_0)\mathscr{G}(x_0,y_0)+\mathscr{H}(x_0,y_0)$ and $y_0\leqslant\mathscr{F}(y_0,x_0)\mathscr{G}(y_0,x_0)+\mathscr{H}(y_0,x_0)$.*

Then the coupled operator equations (3.1) and (3.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by

$$x_{n+1}=\mathscr{F}(x_n,y_n)\mathscr{G}(x_n,y_n)+\mathscr{H}(x_n,y_n)\tag{3.27}$$

and

$$y_{n+1}=\mathscr{F}(y_n,x_n)\mathscr{G}(y_n,x_n)+\mathscr{H}(y_n,x_n)\tag{3.28}$$

for $n\geqslant 0$, converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.

Proof. Now, the coupled operator equations (3.1) and (3.2) can be written as

$$x = \mathscr{T}(x,y) + \mathscr{H}(x,y) \tag{3.29}$$

and

$$y = \mathscr{T}(y,x) + \mathscr{H}(y,x) \tag{3.30}$$

for all $(x,y),(y,x) \in E \times E$, where $\mathscr{T}$ is a coupled operator defined by

$$\mathscr{T}(x,y) = \mathscr{F}(x,y)\mathscr{G}(x,y),\ (x,y) \in E \times E.$$

Define the coupled operator $\mathfrak{T} : E \times E \to E$ by

$$\mathfrak{T}(x,y) = \mathscr{T}(x,y) + \mathscr{H}(x,y) \tag{3.31}$$

so that we have

$$\mathfrak{T}(y,x) = \mathscr{T}(y,x) + \mathscr{H}(y,x). \tag{3.32}$$

We show that the coupled operator $\mathfrak{T}$ satisfies all the conditions of Theorem 2.1 on $E \times E$. Since $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ are mixed monotone coupled operators on $E \times E$, the coupled operator $\mathfrak{T}$ is mixed monotone on $E \times E$. As $\mathscr{F}$ and $\mathscr{H}$ are partial $\mathscr{D}$-Lipschitz, they are partially continuous coupled operators on $E \times E$. From the continuity of the multiplicative composition, it follows that the coupled operator $\mathfrak{T}$ is a partially continuous on $E \times E$. Next we show that $\mathfrak{T}$ is a partially condensing on $E \times E$, that is, $\mathfrak{T}$ satisfies the set-contractive condition (2.25) of Theorem 2.8 with respect to the partial Kuratowskii measure α_p of noncompactness in E.

Let C and D be any two bounded chains in E. Then from partial boundedness of $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ it follows that $\mathscr{T}(C \times D)$ and $\mathscr{H}(C \times D)$ are bounded chains in E. Now, by definition of the coupled operator $\mathfrak{T}$, we obtain

$$\mathfrak{T}(C \times D) \subset \mathscr{T}(C \times D) + \mathscr{H}(C \times D)$$

and

$$\mathfrak{T}(D \times C) \subset \mathscr{T}(D \times C) + \mathscr{H}(D \times C).$$

Again, from monotonicity and subadditivity of the partial measure α_p of noncompactness it follows that

$$\alpha_p\left(\mathfrak{T}(C \times D)\right) \leqslant \alpha_p\left(\mathscr{T}(C \times D)\right) + \alpha_p\left(\mathscr{H}(C \times D)\right) \tag{3.33}$$

and

$$\alpha_p\left(\mathfrak{T}(D \times C)\right) \leqslant \alpha_p\left(\mathscr{T}(D \times C)\right) + \alpha_p\left(\mathscr{H}(D \times C)\right). \tag{3.34}$$

Adding the above two inequalities together implies that

$$\begin{aligned}
\alpha_p\left(\mathfrak{T}(C \times D)\right) + \alpha_p\left(\mathfrak{T}(D \times C)\right)& \\
&\leqslant \alpha_p\left(\mathscr{T}(C \times D)\right) + \alpha_p\left(\mathscr{H}(C \times D)\right) \\
&\quad + \alpha_p\left(\mathscr{T}(D \times C)\right) + \alpha_p\left(\mathscr{H}(D \times C)\right) \\
&\leqslant M_G\, \psi_{\mathscr{F}}\left(\alpha_p(C) + \alpha_p(D)\right) + \psi_{\mathscr{H}}\left(\alpha_p(C) + \alpha_p(D)\right) \\
&= \psi_{\mathfrak{T}}\left(\alpha_p(C) + \alpha_p(D)\right)
\end{aligned} \tag{3.35}$$

where, $\psi_{\mathfrak{T}}(r) = M_G \psi_{\mathscr{F}}(r) + \psi_{\mathscr{H}}(r) < r$ for $r > 0$. Thus the coupled operator $\mathfrak{T}$ satisfies the set-contractive condition (2.25) of partially condensing coupled operator with respect to the partial Kuratowskii measure α_p of noncompactness on E.

Next, there exists an element $(x_0, y_0) \in E \times E$ such that

$$x_0 \leqslant \mathscr{F}(x_0,y_0)\mathscr{G}(x_0,y_0) + \mathscr{H}(x_0,y_0) = \mathfrak{T}(x_0,y_0)$$

and

$$y_0 \geqslant \mathscr{F}(y_0,x_0)\mathscr{G}(y_0,x_0) + \mathscr{H}(y_0,x_0) = \mathfrak{T}(y_0,x_0)$$

or

$$x_0 \geqslant \mathscr{F}(x_0,y_0)\mathscr{G}(x_0,y_0) + \mathscr{H}(x_0,y_0) = \mathfrak{T}(x_0,y_0)$$

and

$$y_0 \leqslant \mathscr{F}(y_0,x_0)\mathscr{G}(y_0,x_0) + \mathscr{H}(y_0,x_0) = \mathfrak{T}(y_0,x_0).$$

Hence, the element (x_0, y_0) is a lower coupled or an upper coupled solution of the coupled operator equations

$$x = \mathfrak{T}(x,y) \quad \text{and} \quad y = \mathfrak{T}(y,x).$$

Thus the coupled operator $\mathfrak{T}$ satisfies all the conditions of Theorem 2.1 and so it has a coupled solution, that is there exists a point $(x^*, y^*) \in E \times E$ such that $x^* = \mathfrak{T}(x^*, y^*)$ and $y^* = \mathfrak{T}(y^*, x^*)$ which further by definition of the coupled operator $\mathfrak{T}$ implies that

$$x^* = \mathscr{T}(x^*,y^*) + \mathscr{H}(x^*,y^*)$$

and

$$y^* = \mathscr{T}(y^*,x^*) + \mathscr{H}(y^*,x^*)$$

or

$$x^* = \mathscr{F}(x^*,y^*)\mathscr{G}(x^*,y^*) + \mathscr{H}(x^*,y^*)$$

and

$$y^* = \mathscr{F}(y^*,x^*)\mathscr{G}(y^*,x^*) + \mathscr{H}(y^*,x^*)$$

and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.27) and (3.28) converge monotonically to x^* and y^* respectively. Moreover, the set of all such comparable coupled solutions is compact. This completes the proof. □

THEOREM 3.3. *Let $(E, \leqslant, \|\cdot\|)$ be a partially ordered Banach algebra and let every compact chain C in E be Janhavi. Suppose that $\mathscr{F}, \mathscr{G} : E \times E \to K$ and $\mathscr{H} : E \times E \to E$ are three mixed monotone coupled operators satisfying the following conditions.*

(a) $\mathscr{F}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{F}}$,

(b) $\mathscr{G}$ is partially continuous and uniformly partially compact with uniform bound $M_{\mathscr{G}} = \sup\{\|\mathscr{G}(C \times D)\| : C, D \in \mathscr{P}_{bd,cn}(E)\}$,

(c) $\mathscr{H}$ is partially continuous and partially compact,

(d) $M_{\mathscr{G}}\psi_{\mathscr{F}}(r) < r$ *for* $r > 0$*, and*

(e) *there exists an element* $(x_0,y_0) \in E\times E$ *such that* $x_0 \leqslant \mathscr{F}(x_0,y_0)\mathscr{G}(x_0,y_0) + \mathscr{H}(x_0,y_0)$ *and* $y_0 \geqslant \mathscr{F}(y_0,x_0)\mathscr{G}(y_0,x_0)+\mathscr{H}(y_0,x_0)$ *or* $x_0 \geqslant \mathscr{F}(x_0,y_0)\mathscr{G}(x_0,y_0) + \mathscr{H}(x_0,y_0)$ *and* $y_0 \leqslant \mathscr{F}(y_0,x_0)\mathscr{G}(y_0,x_0)+\mathscr{H}(y_0,x_0)$.

Then the coupled operator equations (3.1) *and* (3.2) *have a coupled solution* (x^*,y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by* (3.27) *and* (3.28) *converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

Proof. The proof of the theorem is similar to Theorem 3.2, but for the sake of completeness we give the details of it. Define a coupled operator $\mathfrak{T} : E\times E \to E$ by

$$\mathfrak{T}(x,y) = \mathscr{T}(x,y)+\mathscr{H}(x,y)$$

where $\mathscr{T}$ is a coupled operator defined by

$$\mathscr{T}(x,y) = \mathscr{F}(x,y)\mathscr{G}(x,y),\ (x,y) \in E\times E.$$

We show that the coupled operator $\mathfrak{T}$ satisfies all the conditions of Theorem 2.1 on $E\times E$. Since $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ are mixed monotone coupled operators on $E\times E$, the coupled operator $\mathfrak{T}$ is mixed monotone on $E\times E$. As $\mathscr{F}$ is partial $\mathscr{D}$-Lipschitz, it is partially continuous coupled operator on $E\times E$. From the continuity of the multiplicative composition in E it follows that the coupled operator $\mathfrak{T}$ is a partially continuous on $E\times E$. Next we show that $\mathfrak{T}$ satisfies the set-contractive condition (2.25) of Theorem 2.8 with respect to the partial Kuratowskii measure α_p of noncompactness in E.

Let C and D be any two bounded chains in E. Then from partial boundedness of $\mathscr{F}$ and the partial compactness of $\mathscr{G}$ and $\mathscr{H}$ it follows that $\mathscr{T}(C\times D)$ and $\mathscr{H}(C\times D)$ are bounded chains in E. Now, by definition of the coupled operator $\mathfrak{T}$, we obtain

$$\mathfrak{T}(C\times D) \subset \mathscr{T}(C\times D)+\mathscr{H}(C\times D)$$

and

$$\mathfrak{T}(D\times C) \subset \mathscr{T}(D\times C)+\mathscr{H}(D\times C).$$

Next, from monotonicity and subadditivity of the partial measure α_p of noncompactness it follows that

$$\alpha_p\left(\mathfrak{T}(C\times D)\right) \leqslant \alpha_p\left(\mathscr{T}(C\times D)\right) + \alpha_p\left(\mathscr{H}(C\times D)\right) \tag{3.36}$$

and

$$\alpha_p\left(\mathfrak{T}(D\times C)\right) \leqslant \alpha_p\left(\mathscr{T}(D\times C)\right) + \alpha_p\left(\mathscr{H}(D\times C)\right). \tag{3.37}$$

Since $\mathscr{H}(C\times D)$ and $\mathscr{H}(D\times C)$ are compact sets, it can be shown as in the proof of Theorem 3.2 that

$$\alpha_p\left(\mathscr{H}(C\times D)\right) < \frac{\varepsilon}{2} \quad \text{and} \quad \alpha_p\left(\mathscr{H}(D\times C)\right) < \frac{\varepsilon}{2}$$

for some arbitrary $\varepsilon > 0$. Making use of these inequalities and adding the two inequalities (3.36) and (3.37) together implies that

$$\begin{aligned}\alpha_p\left(\mathfrak{T}(C\times D)\right)+\alpha_p\left(\mathfrak{T}(D\times C)\right) &< \alpha_p\left(\mathscr{T}(C\times D)\right)+\alpha_p\left(\mathscr{T}(D\times C)\right)+\varepsilon\\ &\leqslant M_{\mathscr{G}}\,\psi_{\mathscr{F}}\left(\alpha_p(C)+\alpha_p(D)\right)\\ &= \psi_{\mathfrak{T}}\left(\alpha_p(C)+\alpha_p(D)\right)\end{aligned}\tag{3.38}$$

where $\psi_{\mathfrak{T}}(r)=M_{\mathscr{G}}\,\psi_{\mathscr{F}}(r)<r$ for $r>0$. The rest of the proof is obtained by closely observing the proof of Theorem 3.2 and so we omit the details. Hence, the proof of the theorem is complete. □

On taking $\mathscr{H}(x,y)\equiv 0$, we obtain the following new coupled hybrid fixed point theorem to the subject of nonlinear analysis and applications.

COROLLARY 3.3. *Let $(E,\leqslant,\|\cdot\|)$ be a partially ordered Banach algebra and let every compact chain C in E be Janhavi. Suppose that $\mathscr{F},\mathscr{G}:E\times E\to K$ are two mixed monotone coupled operators satisfying the following conditions.*

(a) $\mathscr{F}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{F}}$,

(b) $\mathscr{G}$ is partially continuous and uniformly partially compact with uniform bound $M_{\mathscr{G}}=\sup\{\|\mathscr{G}(C\times D)\|:C,D\in\mathscr{P}_{bd,cn}(E)\}$,

(c) $M_{\mathscr{G}}\,\psi_{\mathscr{F}}(r)<r$ for $r>0$, and

(d) there exists an element $(x_0,y_0)\in E\times E$ such that $x_0\leqslant\mathscr{F}(x_0,y_0)\mathscr{G}(x_0,y_0)$ and $y_0\geqslant\mathscr{F}(y_0,x_0)\mathscr{G}(y_0,x_0)$ or $x_0\geqslant\mathscr{F}(x_0,y_0)\mathscr{G}(x_0,y_0)$ and $y_0\leqslant\mathscr{F}(y_0,x_0)\mathscr{G}(y_0,x_0)$.

Then the coupled operator equations

$$x=\mathscr{F}(x,y)\mathscr{G}(x,y)\tag{3.39}$$

and

$$y=\mathscr{F}(y,x)\mathscr{G}(y,x)\tag{3.40}$$

have a positive coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by

$$x_{n+1}=\mathscr{F}(x_n,y_n)\mathscr{G}(x_n,y_n)\tag{3.41}$$

and

$$y_{n+1}=\mathscr{F}(y_n,x_n)\mathscr{G}(y_n,x_n)\tag{3.42}$$

for $n\geqslant 0$, converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable positive coupled solutions is compact.

REMARK 3.2. Note that a couple of coupled hybrid fixed point theorems similar to Theorems 3.2 and 3.3 are also obtained in Dhage [21] using the partial Kuratowskii measure of noncompactness under the condition that one of the coupled operators $\mathscr{F}$ and $\mathscr{H}$ are symmetric partial $\mathscr{D}$-Lipschitz on $E \times E$. However, we remark that though the theorems are correct, there is a flaw in the proofs of the theorems with that method. Therefore, Theorems 3.2 and 3.3 are new coupled hybrid fixed point theorems different from those presented in Dhage [19] containing the sum and product of three coupled operators on $E \times E$ under a little stronger partial $\mathscr{D}$-Lipschitz condition of one of the coupled operators $\mathscr{F}$ and $\mathscr{H}$. Again, the regularity of the partially ordered Banach algebra E in above coupled hybrid fixed point theorems, Theorems 3.2 and 3.3 may be relaxed and compensated with the continuity of the coupled operators $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ on $E \times E$. See Dhage [9, 17, 18, 19, 20] and the references therein.

REMARK 3.3. If $x = y$ in the coupled operator equations (3.1) and (3.2), then they reduce to the operator equation $\mathscr{A}x\mathscr{B}x + \mathscr{C}x = x$, where $\mathscr{A}x = \mathscr{F}(x,x)$, $\mathscr{B}x = \mathscr{G}(x,x)$ and $\mathscr{C}x = \mathscr{H}(x,x)$, and consequently Theorems 3.2 and 3.3 reduce to the hybrid fixed point theorems for the sum and product of three nonlinear nondecreasing operators in a partially ordered Banach algebra E proved in Dhage [15, 16, 17].

In view of the Lemmas 2.1 and 2.2 we obtain the following interesting applicable coupled hybrid fixed point results in an ordered Banach algebra (E,K).

COROLLARY 3.4. *Let (E,K) be an ordered Banach algebra and let $\mathscr{F},\mathscr{G} : E \times E \to K$ and $\mathscr{H} : E \times E \to E$ are three mixed monotone coupled operators satisfying the conditions (a) through (e) of Theorem 3.2. Then the coupled operator equations (3.1) and (3.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by (3.27) and (3.28) converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.*

COROLLARY 3.5. *Let (E,K) be an ordered Banach algebra and let $\mathscr{F},\mathscr{G} : E \times E \to K$ and $\mathscr{H} : E \times E \to E$ are three mixed monotone coupled operators satisfying the conditions (a) through (e) of Theorem 3.3. Then the coupled operator equations (3.1) and (3.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by (3.27) and (3.28) converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.*

COROLLARY 3.6. *Let (E,K) be an ordered Banach algebra and let $\mathscr{F},\mathscr{G} : E \times E \to K$ be two mixed monotone coupled operators satisfying the conditions (a) through (d) of Corollary 3.3. Then the coupled operator equations (3.39) and (3.40) have a positive coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by (3.41) and (3.42) converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable positive coupled solutions is compact.*

Similarly, if $x = y$ and $\mathscr{A}x = \mathscr{F}(x,x)$, $\mathscr{B}x = \mathscr{G}(x,x)$ and $\mathscr{C}x = \mathscr{H}(x,x)$, then Corollary 3.5 reduces to the following hybrid fixed point theorem involving the sum and product of three operators in an ordered Banach space (E,K).

COROLLARY 3.7. *Let (E,K) be an ordered Banach algebra and let $\mathscr{A},\mathscr{B}:E\to K$ and $\mathscr{C}:E\to E$ be three nondecreasing operators satisfying the following conditions.*

(a) $\mathscr{A}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with $\mathscr{D}$-function $\psi_{\mathscr{A}}$,

(b) $\mathscr{B}$ is partially continuous and uniformly partially compact,

(c) $\mathscr{C}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with $\mathscr{D}$-function $\psi_{\mathscr{C}}$, and

(d) $M_{\mathscr{B}}\psi_{\mathscr{A}}(r)+\psi_{\mathscr{C}}(r)<r$ for each $r>0$, where $M_{\mathscr{B}}=\sup\{\|\mathscr{B}(C)\|:C\in\mathscr{P}_{bd,cn}(E)\}$.

If there exists an element $x_0\in E$ such that $x_0\leqslant\mathscr{A}x_0\mathscr{B}x_0+\mathscr{C}x_0$ or $x_0\geqslant\mathscr{A}x_0\mathscr{B}x_0+\mathscr{C}x_0$, then the operator equation $\mathscr{A}x\mathscr{B}x+\mathscr{C}x=x$ has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1}=\mathscr{A}x_n\mathscr{B}x_n+\mathscr{C}x_n$ converges monotonically to x^*. Moreover, the set of all comparable solutions is compact.

COROLLARY 3.8. *Let (E,K) be an ordered Banach algebra and let $\mathscr{A},\mathscr{B}:E\to K$ be two nondecreasing operators satisfying the following conditions.*

(a) $\mathscr{A}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with $\mathscr{D}$-function $\psi_{\mathscr{A}}$,

(b) $\mathscr{B}$ is partially continuous and uniformly partially compact, and

(c) $M_{\mathscr{B}}\psi_{\mathscr{A}}(r)<r$ for each $r>0$, where $M_{\mathscr{B}}=\sup\{\|\mathscr{B}(C)\|:C\in\mathscr{P}_{bd,cn}(E)\}$.

If there exists an element $x_0\in E$ such that $x_0\leqslant\mathscr{A}x_0\mathscr{B}x_0$ or $x_0\geqslant\mathscr{A}x_0\mathscr{B}x_0$, then the operator equation $\mathscr{A}x\mathscr{B}x=x$ has a positive solution x^* and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1}=\mathscr{A}x_n\mathscr{B}x_n$ converges monotonically to x^*. Moreover, the set of all comparable solutions is compact.

REMARK 3.4. The hybrid fixed point corollaries, Corollaries 3.3, 3.4, 3.5 and 3.6 are new to the literature on the theory of coupled fixed point theorems in the ordered Banach spaces and applications. Note that the above mentioned coupled hybrid fixed point results are very much useful in the subject of nonlinear analysis for proving the existence and approximation theorems for nonlinear coupled differential and integral equations in finite and infinite dimensional Banach spaces.

4. Mixed coupled hybrid fixed point theorems

Next, we consider the case in which the operators and the coupled operators appear in the coupled equations simultaneously at the same time. There are ample examples of such coupled equations in nonlinear analysis and applications. Therefore, it is of interest to obtain the coupled solutions to such mixed coupled equations under certain suitable mixed hybrid conditions of algebra, analysis and topology.

4.1. Mixed coupled equations of type I

Now, consider the coupled operator equations

$$x = \mathscr{A}x\mathscr{G}(x,y) + \mathscr{C}x \tag{4.1}$$

and

$$y = \mathscr{A}y\mathscr{G}(y,x) + \mathscr{C}y, \tag{4.2}$$

where $\mathscr{A},\mathscr{C} : E \to E$ are nonlinear operators and $\mathscr{G} : E \times E \to E$ is a coupled operator which are not necessarily continuous.

A pair of elements $(x^*,y^*) \in E \times E$ is called a *coupled fixed point* of the coupled operator equations (4.1) and (4.2) if

$$x^* = \mathscr{A}x^*\mathscr{G}(x^*,y^*) + \mathscr{C}x^* \tag{4.3}$$

and

$$y^* = \mathscr{A}y^*\mathscr{G}(y^*,x^*) + \mathscr{C}y^*. \tag{4.4}$$

THEOREM 4.1. *Let $(E,\leqslant,\|\cdot\|)$ be a partially ordered Banach algebra and let every compact chain C in E be Janhavi. Suppose that $\mathscr{A} : E \to K$ and $\mathscr{C} : E \to E$ are nondecreasing operators and $\mathscr{G} : E \times E \to K$ is a mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{A}}$,

(b) $\mathscr{G}$ is partially continuous and uniformly partially compact with uniform bound $M_{\mathscr{G}} = \sup\big\{\|\mathscr{G}(C \times D)\| : C,D \in \mathscr{P}_{bd,cn}(E)\big\}$,

(c) $\mathscr{C}$ is partially bounded and partially $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{C}}$,

(d) $M_{\mathscr{G}}\,\psi_{\mathscr{A}}(r) + \psi_{\mathscr{C}}(r) < r$ for each $r > 0$, and

(e) there exists an element $(x_0,y_0) \in E \times E$ such that $x_0 \leqslant \mathscr{A}x_0\mathscr{G}(x_0,y_0) + \mathscr{C}x_0$ and $y_0 \geqslant \mathscr{A}y_0\mathscr{G}(y_0,x_0) + \mathscr{C}y_0$ or $x_0 \geqslant \mathscr{A}x_0\mathscr{G}(x_0,y_0) + \mathscr{C}x_0$ and $y_0 \leqslant \mathscr{A}y_0\mathscr{G}(y_0,x_0) + \mathscr{C}y_0$.

Then the coupled operator equations (4.1) and (4.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by

$$x_{n+1} = \mathscr{A}x_n\mathscr{G}(x_n,y_n) + \mathscr{C}x_n \tag{4.5}$$

and

$$y_{n+1} = \mathscr{A}y_n\mathscr{G}(y_n,x_n) + \mathscr{C}y_n \tag{4.6}$$

for $n \geqslant 0$, converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.

Proof. Define a coupled operator $\mathscr{T} : E \times E \to E$ by

$$\mathscr{T}(x,y) = \mathscr{A}x\mathscr{G}(x,y) + \mathscr{C}x \tag{4.7}$$

so that

$$\mathscr{T}(y,x) = \mathscr{A}y\mathscr{G}(y,x) + \mathscr{C}y. \tag{4.8}$$

Since the operator $\mathscr{A}$ and $\mathscr{C}$ are nondecreasing on E and the coupled operator $\mathscr{G}(x,y)$ is nondecreasing in x for each $y \in E$, the coupled operator $\mathscr{T}(x,y)$ is nondecreasing in x for each $y \in E$. Similarly, since the operator $\mathscr{G}(x,y)$ is nonincreasing in y for each $x \in E$, the coupled operator $\mathscr{T}(x,y)$ is nonincreasing in y for each $x \in E$. Thus the coupled operator $\mathscr{T}$ is mixed monotone on $E \times E$. Again, as $\mathscr{A}$ and $\mathscr{C}$ are partially bounded on E and $\mathscr{G}$ is partially bounded on $E \times E$, the coupled operator $\mathscr{T}$ is partially bounded on $E \times E$. We show that $\mathscr{T}$ satisfies the measure theoretic set-contractive condition (2.25) of Theorem 2.8 on $E \times E$.

Now, for any comparable elements $(x,y), (u,v) \in E \times E$, by definition of the coupled operator $\mathscr{T}$, we obtain

$$\begin{aligned}
\|\mathscr{T}(x,y) - \mathscr{T}(u,v)\| &\leqslant \|\mathscr{A}x\mathscr{G}(x,y) - \mathscr{A}u\mathscr{G}(u,v)\| + \|\mathscr{C}x - \mathscr{C}u\| \\
&\leqslant \|\mathscr{A}x - \mathscr{A}u\|\,\|\mathscr{G}(x,y)\| + \|\mathscr{A}u\|\,\|G(x,y) - G(u,v)\| \\
&\quad + \|\mathscr{C}x - \mathscr{C}u\| \\
&\leqslant M_{\mathscr{G}}\,\psi_{\mathscr{A}}\big(\|x-u\|\big) + M_{\mathscr{A}}\|\mathscr{G}(x,y) - \mathscr{G}(u,v)\| \\
&\quad + \psi_{\mathscr{C}}\big(\|x-u\|\big).
\end{aligned} \tag{4.9}$$

Similarly,

$$\begin{aligned}
\|\mathscr{T}(y,x) - \mathscr{T}(v,u)\| \leqslant M_{\mathscr{G}}\,\psi_{\mathscr{A}}\big(\|v-y\|\big) + M_{\mathscr{A}}\|\mathscr{G}(y,x) - \mathscr{G}(v,u)\| \\
+ \psi_{\mathscr{C}}\big(\|v-y\|\big).
\end{aligned} \tag{4.10}$$

Next, let $\varepsilon > 0$ be given and let C and D be two bounded chains in the Banach algebra E satisfying the conditions (3.10) and (3.11) respectively. As $\mathscr{T}$ is partially bounded, we have that $\mathscr{T}(C \times D)$ is a bounded chain of E. Therefore, by definition of the diameter of a set in E, we obtain

$$\begin{aligned}
\operatorname{diam}\Big(\mathscr{T}\big(C \times D\big)\Big) &= \sup_{(x,y),(u,v)\in C\times D} \|\mathscr{T}(x,y) - \mathscr{T}(u,v)\| \\
&\leqslant \sup_{(x,y),(u,v)\in C\times D} M_{\mathscr{G}}\,\psi_{\mathscr{A}}\big(\|x-u\|\big) \\
&\quad + M_{\mathscr{A}} \sup_{(x,y),(u,v)\in C\times D} \|\mathscr{G}(y,x) - \mathscr{G}(v,u)\| \\
&\quad + \sup_{(x,y),(u,v)\in C\times D} \psi_{\mathscr{C}}\big(\|x-u\|\big) \\
&\leqslant M_{\mathscr{G}}\,\psi_{\mathscr{A}}\big(\operatorname{diam}(C)\big) + M_{\mathscr{A}} \operatorname{diam}\Big(\mathscr{G}(C \times D)\Big) \\
&\quad + \psi_{\mathscr{C}}\big(\operatorname{diam}(C)\big).
\end{aligned} \tag{4.11}$$

Similarly, we obtain

$$\operatorname{diam}\Big(\mathscr{T}\big(D\times C\big)\Big) \leqslant M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\,\operatorname{diam}(D)\,\big) + M_{\mathscr{A}}\operatorname{diam}\Big(\mathscr{G}(D\times C)\Big) + \psi_{\mathscr{C}}\,\big(\,\operatorname{diam}(D)\,\big)\,. \tag{4.12}$$

Next, since $\mathscr{G}$ is partially compact, $\mathscr{G}(C\times D) = G$ is relatively compact subset of E. Therefore, for above $\varepsilon > 0$, by partial Kuratowskii measure of noncompactness, there exist subchains G_1, $G_2, \dots, G_{m_2}$ of G such that

$$\mathscr{G}(C\times D) = \bigcup_{\lambda=1}^{m_2} G_\lambda = G \quad \Rightarrow \quad \bigcup_{\lambda=1}^{m_2} \mathscr{G}^{-1}(G_\lambda) = C\times D$$

and

$$\operatorname{diam}(G_\lambda) < \frac{\varepsilon}{2M_{\mathscr{A}}} \text{ for each } \lambda. \tag{4.13}$$

Similarly, $\mathscr{G}(D\times C) = G'$ is a relatively compact subset of E and for above $\varepsilon > 0$, by partial Kuratowskii measure of noncompactness, there exist subchains G'_1, $G'_2, \dots, G'_{n_2}$ of G' such that

$$\mathscr{G}(D\times C) = \bigcup_{\gamma=1}^{n_2} G'_\gamma = G' \quad \Rightarrow \quad \bigcup_{\gamma=1}^{n_2} \mathscr{G}^{-1}(G'_\gamma) = D\times C$$

and

$$\operatorname{diam}(G'_\gamma) < \frac{\varepsilon}{2M_{\mathscr{A}}} \text{ for each } \gamma. \tag{4.14}$$

Denote

$$\mathtt{C}_{\lambda,i,j} = \mathscr{G}^{-1}(G_\lambda)\bigcap\big(C_i\times D_j\big).$$

Then we have

$$\bigcup_{\lambda,i,j}\mathtt{C}_{\lambda,i,j} = C\times D.$$

Similarly, let

$$\mathtt{C}'_{\gamma,j,i} = \mathscr{G}^{-1}(G'_\gamma)\bigcap\big(D_j\times C_i\big).$$

Therefore, we have

$$\bigcup_{\gamma,j,i}\mathtt{C}'_{\gamma,j,i} = D\times C.$$

Now, from the inequalities (4.11) and (4.13), we obtain

$$\begin{aligned}
\alpha_p\Big(\mathscr{T}(C\times D)\Big) &\leqslant \operatorname{diam}\Big(\mathscr{T}\big(\mathtt{C}_{\lambda,i,j}\big)\Big) \\
&\leqslant M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\,\operatorname{diam}(C_i)\,\big) + M_{\mathscr{A}}\operatorname{diam}\Big(\mathscr{G}\big(\mathtt{C}_{\lambda,i,j}\big)\Big) \\
&\quad + \psi_{\mathscr{C}}\,\big(\,\operatorname{diam}(C_i)\,\big) \\
&\leqslant M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\,\alpha_p(C)+\varepsilon\,\big) + M_{\mathscr{A}}\operatorname{diam} G_\lambda \\
&\quad + \psi_{\mathscr{C}}\,\big(\,\alpha_p(C)+\varepsilon\,\big) \\
&< M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\,\alpha_p(C)+\varepsilon\,\big) + \psi_{\mathscr{C}}\,\big(\,\alpha_p(C)+\varepsilon\,\big) + \frac{\varepsilon}{2}.
\end{aligned} \tag{4.15}$$

Similarly, from the inequalities (4.12) and (4.14), we obtain

$$\begin{aligned}\alpha_p\Big(\mathscr{T}(D\times C)\Big) &\leqslant \operatorname{diam}\Big(\mathscr{T}(\mathtt{C}'_{\gamma,j,i})\Big)\\ &\leqslant M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\operatorname{diam}(D_j)\big) + M_{\mathscr{A}}\operatorname{diam}\Big(\mathscr{G}(\mathtt{C}_{\gamma,j,i})\Big)\\ &\quad + \psi_{\mathscr{C}}\,\big(\operatorname{diam}(D_i)\big)\\ &\leqslant M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\alpha_p(D)+\varepsilon\big) + M_{\mathscr{A}}\operatorname{diam} G'_\gamma\\ &\quad + \psi_{\mathscr{C}}\,\big(\alpha_p(D)+\varepsilon\big)\\ &< M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\alpha_p(D)+\varepsilon\big) + \psi_{\mathscr{C}}\,\big(\alpha_p(D)+\varepsilon\big) + \frac{\varepsilon}{2}. \end{aligned} \tag{4.16}$$

Adding (4.15) and (4.16) together implies that

$$\begin{aligned}&\alpha_p\Big(\mathscr{T}(C\times D)\Big) + \alpha_p\Big(\mathscr{T}(D\times C)\Big)\\ &\qquad < M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\alpha_p(C)+\varepsilon\big) + \psi_{\mathscr{C}}\,\big(\alpha_p(C)+\varepsilon\big)\\ &\qquad\quad + M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\alpha_p(D)+\varepsilon\big) + \psi_{\mathscr{C}}\,\big(\alpha_p(D)+\varepsilon\big) + \varepsilon.\end{aligned}$$

Since ε is arbitrary, one has

$$\begin{aligned}&\alpha_p\Big(\mathscr{T}(C\times D)\Big) + \alpha_p\Big(\mathscr{T}(D\times C)\Big)\\ &\qquad \leqslant M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\alpha_p(C)\big) + \psi_{\mathscr{C}}\,\big(\alpha_p(C)\big)\\ &\qquad\quad + M_{\mathscr{G}}\,\psi_{\mathscr{A}}\,\big(\alpha_p(D)\big) + \psi_{\mathscr{C}}\,\big(\alpha_p(D)\big)\\ &\qquad < \alpha_p(C) + \alpha_p(D)\end{aligned}$$

for all bounded chains C and D in E for which $\alpha_p(C)+\alpha_p(D) > 0$. As a result, the coupled operator $\mathscr{T}$ is a partially condensing with respect to the partial Kuratowskii measure α_p of noncompactness in E. Thus, $\mathscr{T}$ satisfies all the conditions of Theorem 2.8 and so the coupled operator equations $x = \mathscr{T}(x,y)$ and $y = \mathscr{T}(y,x)$ have a coupled solution (x^*,y^*). Consequently the coupled operator equations (4.1) and (4.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by (4.5) and (4.6) converge monotonically to x^* and y^* respectively. This completes the proof. □

THEOREM 4.2. *Let $(E,\leqslant,\|\cdot\|)$ be a partially ordered Banach algebra and let every compact chain C in E be Janhavi. Suppose that $\mathscr{A}: E\to K$ and $\mathscr{C}: E\to E$ are nondecreasing operators and $\mathscr{G}: E\times E\to K$ is a mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{A}}$,

(b) $\mathscr{G}$ is partially continuous and uniformly partially compact with uniform bound $M_{\mathscr{G}} = \sup\big\{\|\mathscr{G}(C\times D)\| : C,D\in\mathscr{P}_{bd,cn}(E)\big\}$,

(c) $\mathscr{C}$ is partially continuous and partially compact,

(d) $M_{\mathscr{G}}\,\psi_{\mathscr{A}}(r) < r$ *for each* $r > 0$*, and*

(e) there exists an element $(x_0, y_0) \in E \times E$ *such that* $x_0 \leqslant \mathscr{A}x_0\mathscr{G}(x_0, y_0) + \mathscr{C}x_0$ *and* $y_0 \geqslant \mathscr{A}y_0\mathscr{G}(y_0, x_0) + \mathscr{C}y_0$ *or* $x_0 \geqslant \mathscr{A}x_0\mathscr{G}(x_0, y_0) + \mathscr{C}x_0$ *and* $y_0 \leqslant \mathscr{A}y_0\mathscr{G}(y_0, x_0) + \mathscr{C}y_0$.

Then the coupled operator equations (4.1) and (4.2) have a coupled solution (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by (4.5) and (4.6) converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

The proof of Theorem 4.2 is simple and can be obtained by giving similar arguments and closely observing the proof of Theorem 4.1. We omit the details. As a consequence of above Theorems 4.1 and 4.2 we obtain the following corollary.

COROLLARY 4.1. *Let* $(E, \leqslant, \|\cdot\|)$ *be a partially ordered Banach algebra and let every compact chain* C *in* E *be Janhavi. Suppose that* $\mathscr{A} : E \to K$ *is nondecreasing operator and* $\mathscr{G} : E \times E \to K$ *is a mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ *is partially bounded and partial* $\mathscr{D}$*-Lipschitz with a* $\mathscr{D}$*-function* $\psi_{\mathscr{A}}$*,*

(b) $\mathscr{G}$ *is partially continuous and uniformly partially compact with uniform bound* $M_{\mathscr{G}} = \sup\{\|\mathscr{G}(C \times D)\| : C, D \in \mathscr{P}_{bd,cn}(E)\}$*,*

(c) $M_{\mathscr{G}}\,\psi_{\mathscr{A}}(r) < r$ *for each* $r > 0$*, and*

(d) there exists an element $(x_0, y_0) \in E \times E$ *such that* $x_0 \leqslant \mathscr{A}x_0\mathscr{G}(x_0, y_0)$ *and* $y_0 \geqslant \mathscr{A}y_0\mathscr{G}(y_0, x_0)$ *or* $x_0 \geqslant \mathscr{A}x_0\mathscr{G}(x_0, y_0)$ *and* $y_0 \leqslant \mathscr{A}y_0\mathscr{G}(y_0, x_0)$.

Then the coupled operator equations

$$x = \mathscr{A}x\mathscr{G}(x, y) \tag{4.17}$$

and

$$y = \mathscr{A}y\mathscr{G}(y, x) \tag{4.18}$$

have a positive coupled solution (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by*

$$x_{n+1} = \mathscr{A}x_n\mathscr{G}(x_n, y_n) \tag{4.19}$$

and

$$y_{n+1} = \mathscr{A}y_n\mathscr{G}(y_n, x_n) \tag{4.20}$$

for $n \geqslant 0$*, converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

Now, in view of the Lemmas 2.1 and 2.2, we obtain

COROLLARY 4.2. *Let* (E,K) *be an ordered Banach algebra. Suppose that* $\mathscr{A}:E\to K$ *and* $\mathscr{C}:E\to E$ *are nondecreasing operators and* $\mathscr{G}:E\times E\to K$ *is a mixed monotone coupled operator satisfying the conditions (a) through (e) of Theorem 4.1 or 4.2. Then the coupled operator equations (4.1) and (4.2) have a positive coupled solution* (x^*,y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by (4.5) and (4.6) converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

4.2. Mixed coupled equations of type II

Next, we consider the mixed coupled operator equations

$$x=\mathscr{A}x\,\mathscr{F}(x,y)+\mathscr{C}x \tag{4.21}$$

and

$$y=\mathscr{A}y\,\mathscr{F}(y,x)+\mathscr{C}y, \tag{4.22}$$

where $\mathscr{A},\mathscr{C}:E\to E$ are nonlinear operators and $\mathscr{F}:E\times E\to E$ is a coupled operator which are not necessarily continuous.

A pair of elements $(x^*,y^*)\in E\times E$ is called a *coupled fixed point* of the coupled operator equations (4.21) and (4.22) if

$$x^*=\mathscr{A}x^*\,\mathscr{F}(x^*,y^*)+\mathscr{C}x^* \tag{4.23}$$

and

$$y^*=\mathscr{A}y^*\,\mathscr{F}(y^*,x^*)+\mathscr{C}y^*. \tag{4.24}$$

THEOREM 4.3. *Let* $(E,\leqslant,\|\cdot\|)$ *be a partially ordered Banach algebra and let every compact chain* C *in* E *be Janhavi. Suppose that* $\mathscr{A}:E\to K$ *and* $\mathscr{C}:E\to E$ *are nondecreasing operators and* $\mathscr{F}:E\times E\to K$ *is a mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ *is partially continuous and uniformly partially compact with uniform bound* $M_{\mathscr{A}}=\sup\{\|\mathscr{A}(C)\|:C\in\mathscr{P}_{bd,cn}(E)\}$,

(b) $\mathscr{F}$ *is partially bounded and partial* $\mathscr{D}$*-Lipschitz with a* $\mathscr{D}$*-function* $\psi_{\mathscr{F}}$,

(c) $\mathscr{C}$ *is partially bounded and partial* $\mathscr{D}$*-Lipschitz with a* $\mathscr{D}$*-function* $\psi_{\mathscr{C}}$,

(d) $M_{\mathscr{A}}\,\psi_{\mathscr{F}}(r_1+r_2)+\psi_{\mathscr{C}}(r_1)+\psi_{\mathscr{C}}(r_2)<r_1+r_2$ *for each* $r_1>0$, $r_2>0$, *and*

(e) there exists an element $(x_0,y_0)\in E\times E$ *such that* $x_0\leqslant\mathscr{A}x_0\,\mathscr{F}(x_0,y_0)+\mathscr{C}x_0$ *and* $y_0\geqslant\mathscr{A}y_0\,\mathscr{F}(y_0,x_0)+\mathscr{C}y_0$ *or* $x_0\geqslant\mathscr{A}x_0\,\mathscr{F}(x_0,y_0)+\mathscr{C}x_0$ *and* $y_0\leqslant\mathscr{A}y_0\,\mathscr{F}(y_0,x_0)+\mathscr{C}y_0$.

Then the coupled operator equations (4.21) and (4.22) have a coupled solution (x^*,y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by*

$$x_{n+1}=\mathscr{A}x_n\,\mathscr{F}(x_n,y_n)+\mathscr{C}x_n \tag{4.25}$$

and

$$y_{n+1} = \mathscr{A}y_n\mathscr{F}(y_n, x_n) + \mathscr{C}y_n \tag{4.26}$$

for $n \geqslant 0$, converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.

Proof. Define a coupled operator $\mathscr{T} : E \times E \to E$ by

$$\mathscr{T}(x,y) = \mathscr{A}x\mathscr{F}(x,y) + \mathscr{C}x \tag{4.27}$$

so that we have

$$\mathscr{T}(y,x) = \mathscr{A}y\mathscr{F}(y,x) + \mathscr{C}y. \tag{4.28}$$

Since the operator $\mathscr{A}$ and $\mathscr{C}$ are nondecreasing on E and the coupled operator $\mathscr{F}(x,y)$ is nondecreasing in x for each $y \in E$, the coupled operator $\mathscr{T}(x,y)$ is nondecreasing in x for each $y \in E$. Similarly, since the operator $\mathscr{F}(x,y)$ is nonincreasing in y for each $x \in E$, the coupled operator $\mathscr{T}(x,y)$ is nonincreasing in y for each $x \in E$. Thus the coupled operator $\mathscr{T}$ is mixed monotone on $E \times E$. Again, as $\mathscr{A}$ and $\mathscr{C}$ are partially bounded on E and $\mathscr{F}$ is partially bounded on $E \times E$, the coupled operator $\mathscr{T}$ is partially bounded on $E \times E$. We show that $\mathscr{T}$ satisfies the measure theoretic set-contractive condition (2.25) of Theorem 2.8 with respect to the partial Kuratowskii measure of noncompactness on $E \times E$.

Now for any two comparable elements $(x,y),(u,v) \in E \times E$, by definition of the coupled operator $\mathscr{T}$, we obtain

$$\begin{aligned}
\|\mathscr{T}(x,y) - \mathscr{T}(u,v)\| &\leqslant \|\big(\mathscr{A}x\mathscr{F}(x,y)\big) - \big(\mathscr{A}u\mathscr{F}(u,v)\big)\| + \|\mathscr{C}x - \mathscr{C}u\| \\
&\leqslant \|\mathscr{A}x - \mathscr{A}u\|\,\|\mathscr{F}(x,y)\| + \|\mathscr{A}u\|\,\|\mathscr{F}(x,y) - \mathscr{F}(u,v)\| \\
&\quad + \|\mathscr{C}x - \mathscr{C}u\| \\
&\leqslant M_{\mathscr{F}}\|\mathscr{A}x - \mathscr{A}u\| + \frac{1}{2}M_{\mathscr{A}}\,\psi_{\mathscr{F}}\left(\|x-u\| + \|v-y\|\right) \\
&\quad + \psi_{\mathscr{C}}\left(\|x-u\|\right).
\end{aligned} \tag{4.29}$$

Similarly,

$$\begin{aligned}
\|\mathscr{T}(y,x) - \mathscr{T}(v,u)\| \leqslant M_{\mathscr{F}}\|\mathscr{A}y - \mathscr{A}v\| &+ \frac{1}{2}M_{\mathscr{A}}\,\psi_{\mathscr{F}}\left(\|x-u\| + \|v-y\|\right) \\
&+ \psi_{\mathscr{C}}\left(\|y-v\|\right).
\end{aligned} \tag{4.30}$$

Next, let $\varepsilon > 0$ be given and let C and D be two bounded chains in the Banach algebra E satisfying the inequalities (3.10) and (3.11) respectively. As $\mathscr{T}$ is partially bounded, we have that $\mathscr{T}(C \times D)$ is a bounded chain of E. Therefore, by definition of the diameter of a set in E, we obtain

$$\begin{aligned} \operatorname{diam}\Big(\mathscr{T}\big(C\times D\big)\Big) &= \sup_{(x,y),(u,v)\in C\times D} \|\mathscr{T}(x,y)-\mathscr{T}(u,v)\| \\ &\leqslant \sup_{(x,y),(u,v)\in C\times D} M_{\mathscr{F}}\,\|\mathscr{A}x-\mathscr{A}u\| \\ &\quad + \frac{1}{2}\cdot \sup_{(x,y),(u,v)\in C\times D} M_{\mathscr{A}}\,\psi_{\mathscr{F}}\,\big(\,\|x-u\|+\|v-y\|\,\big) \\ &\quad + \sup_{(x,y),(u,v)\in C\times D} \psi_{\mathscr{C}}\,\big(\,\|x-u\|\,\big) \\ &\leqslant M_{\mathscr{F}}\operatorname{diam}\mathscr{A}(C)+\frac{1}{2}\cdot M_{\mathscr{A}}\,\psi_{\mathscr{F}}\,\big(\operatorname{diam}C+\operatorname{diam}D\big) \\ &\quad + \psi_{\mathscr{C}}\,\big(\operatorname{diam}C\,\big)\,. \end{aligned} \tag{4.31}$$

Similarly, we obtain

$$\begin{aligned} \operatorname{diam}\Big(\mathscr{T}\big(D\times C\big)\Big) &\leqslant M_{\mathscr{F}}\operatorname{diam}\mathscr{A}(D)+\frac{1}{2}\cdot M_{\mathscr{A}}\,\psi_{\mathscr{F}}\,\big(\operatorname{diam}D+\operatorname{diam}C\big) \\ &\quad +\psi_{\mathscr{C}}\,\big(\operatorname{diam}D\,\big)\,. \end{aligned} \tag{4.32}$$

Next, since $\mathscr{A}$ is partially compact, $\mathscr{A}(C)=A$ is relatively compact subset of E. Therefore, for above $\varepsilon>0$, by partial Kuratowskii measure of noncompactness, there exist subchains A_1, $A_2,\ldots,A_{m_3}$ of A such that

$$\mathscr{A}(C)=\bigcup_{\mu=1}^{m_3} A_\mu = A \quad\Longrightarrow\quad \bigcup_{\mu=1}^{m_3}\mathscr{A}^{-1}(A_\mu)=C$$

and

$$\operatorname{diam}(A_\mu)<\frac{\varepsilon}{2M_{\mathscr{F}}} \text{ for each } \mu. \tag{4.33}$$

Similarly, $\mathscr{A}(D)=A'$ is a relatively compact subset of E and therefore, for above $\varepsilon>0$, by partial Kuratowskii measure of noncompactness, there exist subchains A'_1, $A'_2,\ldots,A'_{n_3}$ of A' such that

$$\mathscr{A}(D)=\bigcup_{\nu=1}^{n_3} A'_\nu = A' \quad\Longrightarrow\quad \bigcup_{\nu=1}^{n_3}\mathscr{A}^{-1}(A'_\nu)=D$$

and

$$\operatorname{diam}(A'_\nu)<\frac{\varepsilon}{2M_{\mathscr{F}}} \text{ for each } \nu. \tag{4.34}$$

Denote

$$\mathsf{C}_{\mu,i,j}=\Big(\mathscr{G}^{-1}(G_\mu)\bigcap C_i\Big)\times D_j,$$

then we have

$$\bigcup_{\mu,i,j}\mathsf{C}_{\mu,i,j}=C\times D.$$

Similarly, let

$$\mathtt{C}'_{\nu,j,i} = \Big(\mathscr{G}^{-1}(A'_\nu)\bigcap D_j\Big)\times C_i,$$

so, we have

$$\bigcup_{\nu,j,i}\mathtt{C}'_{\nu,j,i} = D\times C.$$

Now, from the expressions (4.31) and (4.33) we obtain

$$\begin{aligned}\alpha_p\,\big(\,\mathscr{T}(C\times D)\,\big) &\leqslant \operatorname{diam}\,\big(\,\mathscr{T}(\mathtt{C}_{\mu,i,j})\,\big)\\ &\leqslant M_{\mathscr{F}}\operatorname{diam}A_\mu + \frac{1}{2}\cdot M_{\mathscr{A}}\,\psi_{\mathscr{F}}\,\big(\operatorname{diam}C_i + \operatorname{diam}D_j\big)\\ &\quad + \psi_{\mathscr{C}}\,\big(\operatorname{diam}C_i\,\big)\\ &< \frac{1}{2}\cdot M_{\mathscr{A}}\,\psi_{\mathscr{F}}\,\big(\,\alpha(C)+\alpha(D)+\varepsilon\,\big)\\ &\quad + \psi_{\mathscr{C}}\,\big(\,\alpha(C)+\varepsilon\,\big) + \frac{\varepsilon}{2}. \end{aligned}\tag{4.35}$$

Similarly, we have

$$\begin{aligned}\alpha_p\,\big(\,\mathscr{T}(D\times C)\,\big) &\leqslant \operatorname{diam}\,\big(\,\mathscr{T}(\mathtt{C}'_{\nu,i,j})\,\big)\\ &\leqslant M_{\mathscr{F}}\operatorname{diam}A'_\nu + \frac{1}{2}\cdot M_{\mathscr{A}}\,\psi_{\mathscr{F}}\,\big(\operatorname{diam}C_i + \operatorname{diam}D_j\big)\\ &\quad + \psi_{\mathscr{C}}\,\big(\operatorname{diam}D_i\,\big)\\ &< \frac{1}{2}\cdot M_{\mathscr{A}}\,\psi_{\mathscr{F}}\,\big(\,\alpha(C)+\alpha(D)+\varepsilon\,\big)\\ &\quad + \psi_{\mathscr{C}}\,\big(\,\alpha(D)+\varepsilon\,\big) + \frac{\varepsilon}{2}. \end{aligned}\tag{4.36}$$

Adding (4.35) and (4.36) together implies that

$$\begin{aligned}\alpha_p\Big(\mathscr{T}(C\times D)\Big)+\alpha_p\Big(\mathscr{T}(D\times C)\Big) &< M_{\mathscr{A}}\,\psi_{\mathscr{F}}\,\big(\,\alpha_p(C)+\alpha_p(D)+\varepsilon\,\big)\\ &\quad + \psi_{\mathscr{C}}\,\big(\,\alpha_p(C)+\varepsilon\,\big) + \psi_{\mathscr{C}}\,\big(\,\alpha_p(D)+\varepsilon\,\big) + \varepsilon.\end{aligned}$$

Since ε is arbitrary, one has

$$\begin{aligned}\alpha_p\Big(\mathscr{T}(C\times D)\Big)+\alpha_p\Big(\mathscr{T}(D\times C)\Big) &\leqslant M_{\mathscr{A}}\,\psi_{\mathscr{F}}\,\big(\,\alpha_p(C)+\alpha_p(D)\,\big)\\ &\quad + \psi_{\mathscr{C}}\,\big(\,\alpha_p(C)\,\big) + \psi_{\mathscr{C}}\,\big(\,\alpha_p(D)\,\big)\\ &< \alpha_p(C)+\alpha_p(D)\end{aligned}$$

for all bounded chains C and D in E for which $\alpha_p(C)+\alpha_p(D) > 0$. As a result, the coupled operator $\mathscr{T}$ is a partially condensing with respect to the partial Kuratowskii measure α_p in E. Thus, $\mathscr{T}$ satisfies all the conditions of Theorem 2.8 and so the coupled operator equations $x = \mathscr{T}(x,y)$ and $y = \mathscr{T}(y,x)$ have a coupled solution (x^*,y^*). Consequently the coupled operator equations (4.21) and (4.22) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by (4.25) and (4.26) converge monotonically to x^* and y^* respectively. This completes the proof. □

THEOREM 4.4. *Let* $(E, \leqslant, \|\cdot\|)$ *be a partially ordered Banach algebra and let every compact chain* C *in* E *be Janhavi. Suppose that* $\mathscr{A}: E \to K$ *and* $\mathscr{C}: E \to E$ *are nondecreasing operators and* $\mathscr{G}: E \times E \to K$ *is a mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ *is partially continuous and uniformly partially compact with uniform bound* $M_{\mathscr{A}} = \sup\{\|\mathscr{A}(C)\| : C \in \mathscr{P}_{bd,cn}(E)\}$,

(b) $\mathscr{F}$ *is partially bounded and partial* $\mathscr{D}$*-Lipschitz with a* $\mathscr{D}$*-function* $\psi_{\mathscr{F}}$,

(c) $\mathscr{C}$ *is partially continuous and partially compact,*

(d) $M_{\mathscr{A}}\, \psi_{\mathscr{F}}(r) < r$ *for each* $r > 0$, *and*

(e) there exists an element $(x_0, y_0) \in E \times E$ *such that* $x_0 \leqslant \mathscr{A}x_0\,\mathscr{F}(x_0, y_0) + \mathscr{C}x_0$ *and* $y_0 \geqslant \mathscr{A}y_0\,\mathscr{F}(y_0, x_0) + \mathscr{C}y_0$ *or* $x_0 \geqslant \mathscr{A}x_0\,\mathscr{F}(x_0, y_0) + \mathscr{C}x_0$ *and* $y_0 \leqslant \mathscr{A}y_0\,\mathscr{F}(y_0, x_0) + \mathscr{C}y_0$.

Then the coupled operator equations (4.21) *and* (4.22) *have a coupled solution* (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by* (4.25) *and* (4.26) *converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

The proof of Theorem 4.4 is obtained by closely observing the proof of Theorem 4.3. We omit the details. As a consequence of above Theorems 4.3 and 4.4 we obtain the following corollary.

COROLLARY 4.3. *Let* (E, K) *be an ordered Banach algebra. Suppose that* $\mathscr{A}: E \to K$ *and* $\mathscr{C}: E \to E$ *are nondecreasing operators and* $\mathscr{F}: E \times E \to K$ *is a mixed monotone coupled operator satisfying the conditions (a) through (e) of Theorem 4.3 or 4.4. Then the coupled operator equations* (4.21) *and* (4.22) *have a coupled solution* (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by* (4.25) *and* (4.26) *converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

4.3. Mixed coupled equations of type III

Now, consider the coupled operator equations

$$x = \mathscr{A}x\mathscr{G}(x, y) + \mathscr{C}y \tag{4.37}$$

and

$$y = \mathscr{A}y\mathscr{G}(y, x) + \mathscr{C}x, \tag{4.38}$$

where $\mathscr{A}, \mathscr{C}: E \to E$ are nonlinear operators and $\mathscr{F}: E \times E \to E$ is a coupled operator which are not necessarily continuous.

A pair of elements $(x^*, y^*) \in E \times E$ is called a *coupled fixed point* of the coupled operator equations (4.37) and (4.38) if

$$x^* = \mathscr{A}x^*\mathscr{G}(x^*, y^*) + \mathscr{C}y^* \tag{4.39}$$

and

$$y^* = \mathscr{A} y^* \mathscr{G}(y^*, x^*) + \mathscr{C} x^*. \tag{4.40}$$

THEOREM 4.5. *Let $(E, \leqslant, \|\cdot\|)$ be a partially ordered Banach algebra and let every compact chain C in E be Janhavi. Suppose that $\mathscr{A} : E \to K$ is nondecreasing operator, $\mathscr{C} : E \to E$ is nonincreasing operator and $\mathscr{G} : E \times E \to K$ is mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{A}}$,

(b) $\mathscr{G}$ is partially continuous and uniformly partially compact with uniform bound $M_{\mathscr{G}} = \sup\{\|\mathscr{G}(C \times D)\| : C, D \in \mathscr{P}_{bd,cn}(E)\}$,

(c) $\mathscr{C}$ is partially bounded and partially $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{C}}$,

(d) $M_{\mathscr{G}}\, \psi_{\mathscr{A}}(r) + \psi_{\mathscr{C}}(r) < r$ for each $r > 0$, and

(e) there exists an element $(x_0, y_0) \in E \times E$ such that $x_0 \leqslant \mathscr{A} x_0 \mathscr{G}(x_0, y_0) + \mathscr{C} y_0$ and $y_0 \geqslant \mathscr{A} y_0 \mathscr{G}(y_0, x_0) + \mathscr{C} x_0$ or $x_0 \geqslant \mathscr{A} x_0 \mathscr{G}(x_0, y_0) + \mathscr{C} y_0$ and $y_0 \leqslant \mathscr{A} y_0 \mathscr{G}(y_0, x_0) + \mathscr{C} x_0$.

Then the coupled operator equations (4.37) and (4.38) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by

$$x_{n+1} = \mathscr{A} x_n \mathscr{G}(x_n, y_n) + \mathscr{C} y_n \tag{4.41}$$

and

$$y_{n+1} = \mathscr{A} y_n \mathscr{G}(y_n, x_n) + \mathscr{C} x_n \tag{4.42}$$

for $n \geqslant 0$, converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.

Proof. Define a coupled operator $\mathscr{T} : E \times E \to E$ by

$$\mathscr{T}(x, y) = \mathscr{A} x \mathscr{G}(x, y) + \mathscr{C} y \tag{4.43}$$

so that we have

$$\mathscr{T}(y, x) = \mathscr{A} y \mathscr{G}(y, x) + \mathscr{C} x. \tag{4.44}$$

Since the operator $\mathscr{A}$ is nondecreasing and the coupled operator $\mathscr{G}(x, y)$ is nondecreasing in x for each $y \in E$, the coupled operator $\mathscr{T}(x, y)$ is nondecreasing in x for each $y \in E$. Similarly, since the operator $\mathscr{C}$ is nonincreasing and the coupled operator $\mathscr{G}(x, y)$ is nonincreasing in y for each $x \in E$, the coupled operator $\mathscr{T}(x, y)$ is nonincreasing in y for each $x \in E$. Thus the coupled operator $\mathscr{T}$ is mixed monotone on $E \times E$. Again, as $\mathscr{A}$ and $\mathscr{C}$ are partially bounded on E and $\mathscr{G}$ is partially bounded on $E \times E$, the coupled operator $\mathscr{T}$ is partially bounded on $E \times E$. The rest of the proof is similar to Theorem 4.1 and hence we omit the details. □

THEOREM 4.6. *Let* $(E,\leqslant,\|\cdot\|)$ *be a partially ordered Banach algebra and let every compact chain* C *in* E *be Janhavi. Suppose that* $\mathscr{A}:E\to K$ *is nondecreasing operator,* $\mathscr{C}:E\to E$ *is nonincreasing operator and* $\mathscr{G}:E\times E\to K$ *is mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ *is partially bounded and partial* $\mathscr{D}$*-Lipschitz with a* $\mathscr{D}$*-function* $\psi_{\mathscr{A}}$,

(b) $\mathscr{G}$ *is partially continuous and uniformly partially compact with uniform bound* $M_{\mathscr{G}}=\sup\big\{\|\mathscr{G}(C\times D)\|:C,D\in\mathscr{P}_{bd,cn}(E)\big\}$,

(c) $\mathscr{C}$ *is partially continuous and partially compact,*

(d) $M_{\mathscr{G}}\,\psi_{\mathscr{A}}(r)<r$ *for each* $r>0$, *and*

(e) there exists an element $(x_0,y_0)\in E\times E$ *such that* $x_0\leqslant\mathscr{A}x_0\mathscr{G}(x_0,y_0)+\mathscr{C}y_0$ *and* $y_0\geqslant\mathscr{A}y_0\mathscr{G}(y_0,x_0)+\mathscr{C}x_0$ *or* $x_0\geqslant\mathscr{A}x_0\mathscr{G}(x_0,y_0)+\mathscr{C}y_0$ *and* $y_0\leqslant\mathscr{A}y_0\mathscr{G}(y_0,x_0)+\mathscr{C}x_0$.

Then the coupled operator equations (4.37) *and* (4.38) *have a coupled solution* (x^*,y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by* (4.41) *and* (4.42) *converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

COROLLARY 4.4. *Let* (E,K) *be an ordered Banach algebra. Suppose that* $\mathscr{A}:E\to K$ *and* $\mathscr{C}:E\to E$ *are nondecreasing operators and* $\mathscr{F}:E\times E\to K$ *is a mixed monotone coupled operator satisfying the conditions (a) through (e) of Theorem 4.5 or 4.6. Then the coupled operator equations* (4.37) *and* (4.38) *have a positive coupled solution* (x^*,y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by* (4.41) *and* (4.42) *converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

4.4. Mixed coupled equations of type IV

Now, consider the coupled operator equations

$$x=\mathscr{A}x\,\mathscr{F}(x,y)+\mathscr{C}y \tag{4.45}$$

and

$$y=\mathscr{A}y\,\mathscr{F}(y,x)+\mathscr{C}x, \tag{4.46}$$

where $\mathscr{A},\mathscr{C}:E\to E$ are nonlinear operators and $\mathscr{F}:E\times E\to E$ is a coupled operator which are not necessarily continuous.

A pair of elements $(x^*,y^*)\in E\times E$ is called a *coupled fixed point* of the coupled operator equations (4.45) and (4.46) if

$$x^*=\mathscr{A}x^*\,\mathscr{F}(x^*,y^*)+\mathscr{C}y^* \tag{4.47}$$

and

$$y^*=\mathscr{A}y^*\,\mathscr{F}(y^*,x^*)+\mathscr{C}x^*. \tag{4.48}$$

THEOREM 4.7. *Let $(E, \leqslant, \|\cdot\|)$ be a partially ordered Banach algebra and let every compact chain C in E be Janhavi. Suppose that $\mathscr{A} : E \to K$ is nondecreasing operator, $\mathscr{C} : E \to E$ is nonincreasing operator and $\mathscr{F} : E \times E \to K$ is mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ is partially continuous and uniformly partially compact with uniform bound $M_{\mathscr{A}} = \sup\{\|\mathscr{A}(C)\| : C \in \mathscr{P}_{bd,cn}(E)\}$,

(b) $\mathscr{F}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{F}}$,

(c) $\mathscr{C}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{C}}$,

(d) $M_{\mathscr{A}}\, \psi_{\mathscr{F}}(r_1 + r_2) + \psi_{\mathscr{C}}(r_1) + \psi_{\mathscr{C}}(r_2) < r_1 + r_2$ for each $r_1 > 0$, $r_2 > 0$, and

(e) there exists an element $(x_0, y_0) \in E \times E$ such that $x_0 \leqslant \mathscr{A}x_0\, \mathscr{F}(x_0, y_0) + \mathscr{C}y_0$ and $y_0 \geqslant \mathscr{A}y_0\, \mathscr{F}(y_0, x_0) + \mathscr{C}x_0$ or $x_0 \geqslant \mathscr{A}x_0\, \mathscr{F}(x_0, y_0) + \mathscr{C}y_0$ and $y_0 \leqslant \mathscr{A}y_0\, \mathscr{F}(y_0, x_0) + \mathscr{C}x_0$.

Then the coupled operator equations (4.45) and (4.46) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by

$$x_{n+1} = \mathscr{A}x_n\, \mathscr{F}(x_n, y_n) + \mathscr{C}y_n \tag{4.49}$$

and

$$y_{n+1} = \mathscr{A}y_n\, \mathscr{F}(y_n, x_n) + \mathscr{C}x_n \tag{4.50}$$

for $n \geqslant 0$, converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.

Proof. The proof of the theorem is obtained by closely observing the proof of Theorems 4.3 and 4.5 and hence we omit the details. □

THEOREM 4.8. *Let $(E, \leqslant, \|\cdot\|)$ be a partially ordered Banach algebra and let every compact chain C in E be Janhavi. Suppose that $\mathscr{A} : E \to K$ is nondecreasing operator, $\mathscr{C} : E \to E$ is nonincreasing operator and $\mathscr{F} : E \times E \to K$ is mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ is partially continuous and uniformly partially compact with uniform bound $M_{\mathscr{A}} = \sup\{\|\mathscr{A}(C)\| : C \in \mathscr{P}_{bd,cn}(E)\}$,

(b) $\mathscr{F}$ is partially bounded and partial $\mathscr{D}$-Lipschitz with a $\mathscr{D}$-function $\psi_{\mathscr{F}}$,

(c) $\mathscr{C}$ is partially continuous and partially compact,

(d) $M_{\mathscr{A}}\, \psi_{\mathscr{F}}(r) < r$ for each $r > 0$, and

(e) there exists an element $(x_0, y_0) \in E \times E$ such that $x_0 \leqslant \mathscr{A}x_0\, \mathscr{F}(x_0, y_0) + \mathscr{C}y_0$ and $y_0 \geqslant \mathscr{A}y_0\, \mathscr{F}(y_0, x_0) + \mathscr{C}x_0$ or $x_0 \geqslant \mathscr{A}x_0\, \mathscr{F}(x_0, y_0) + \mathscr{C}y_0$ and $y_0 \leqslant \mathscr{A}y_0\, \mathscr{F}(y_0, x_0) + \mathscr{C}x_0$.

Then the coupled operator equations (4.45) *and* (4.46) *have a coupled solution* (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by* (4.49) *and* (4.50) *converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

Proof. The proof of the theorem is obtained by closely observing the proof of Theorems 4.4 and 4.6 and hence we omit the details. □

COROLLARY 4.5. *Let* (E, K) *be an ordered Banach algebra. Suppose that* $\mathscr{A} : E \to K$ *and* $\mathscr{C} : E \to E$ *are nonincreasing operators and* $\mathscr{F} : E \times E \to K$ *is a mixed monotone coupled operator satisfying the conditions (a) through (e) of Theorem* 4.7 *or* 4.8. *Then the coupled operator equations* (4.45) *and* (4.46) *have a coupled solution* (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by* (4.49) *and* (4.50) *converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

4.5. Mixed coupled equations of type V

Now, consider the coupled operator equations

$$x = \mathscr{A}x + \mathscr{G}(x, y) \tag{4.51}$$

and

$$y = \mathscr{A}y + \mathscr{G}(y, x), \tag{4.52}$$

where $\mathscr{A} : E \to E$ is a nonlinear operators and $\mathscr{G} : E \times E \to E$ is a coupled operator which are not necessarily continuous.

A pair of elements $(x^*, y^*) \in E \times E$ is called a *coupled fixed point* of the coupled operator equations (4.51) and (4.52) if

$$x^* = \mathscr{A}x^* + \mathscr{G}(x^*, y^*) \tag{4.53}$$

and

$$y^* = \mathscr{A}y^* + \mathscr{G}(y^*, x^*) \tag{4.54}$$

THEOREM 4.9. *Let* $(E, \leqslant, \|\cdot\|)$ *be a partially ordered Banach space and let every compact chain* C *in* E *be Janhavi. Suppose that* $\mathscr{A} : E \to E$ *is a nondecreasing operator and* $\mathscr{G} : E \times E \to K$ *is a mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ *is partially bounded and nonlinear partial* $\mathscr{D}$*-contraction with a* $\mathscr{D}$*-function* $\psi_{\mathscr{A}}$*,*

(b) $\mathscr{G}$ *is partially continuous and partially compact, and*

(c) *there exists an element* $(x_0, y_0) \in E \times E$ *such that* $x_0 \leqslant \mathscr{A}x_0 + \mathscr{G}(x_0, y_0)$ *and* $y_0 \geqslant \mathscr{A}y_0 + \mathscr{G}(y_0, x_0)$ *or* $x_0 \geqslant \mathscr{A}x_0 + \mathscr{G}(x_0, y_0)$ *and* $y_0 \leqslant \mathscr{A}y_0 + \mathscr{G}(y_0, x_0)$.

Then the coupled operator equations (4.51) *and* (4.52) *have a coupled solution* (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by*

$$x_{n+1} = \mathscr{A}x_n + \mathscr{G}(x_n, y_n) \tag{4.55}$$

and

$$y_{n+1} = \mathscr{A}y_n + \mathscr{G}(y_n, x_n) \tag{4.56}$$

for $n \geqslant 0$*, converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

Proof. Define a coupled operator $\mathscr{T} : E \times E \to E$ by

$$\mathscr{T}(x,y) = \mathscr{A}x + \mathscr{G}(x,y) \tag{4.57}$$

so that

$$\mathscr{T}(y,x) = \mathscr{A}y + \mathscr{G}(y,x). \tag{4.58}$$

Since the operator $\mathscr{A}$ is nondecreasing on E and the coupled operator $\mathscr{G}(x,y)$ is nondecreasing in x for each $y \in E$, the coupled operator $\mathscr{T}(x,y)$ is nondecreasing in x for each $y \in E$. Similarly, since the operator $\mathscr{G}(x,y)$ is nonincreasing in y for each $x \in E$, the coupled operator $\mathscr{T}(x,y)$ is nonincreasing in y for each $x \in E$. Thus the coupled operator $\mathscr{T}$ is mixed monotone on $E \times E$. Again, as $\mathscr{A}$ is partially bounded on E and $\mathscr{G}$ is partially bounded on $E \times E$, the coupled operator $\mathscr{T}$ is partially bounded on $E \times E$. We show that $\mathscr{T}$ satisfies the measure theoretic set-contractive condition (2.25) of Theorem 2.8 on $E \times E$.

Let C and D be two bounded chains in the Banach space E. As $\mathscr{T}$ is partially bounded, we have that $\mathscr{T}(C \times D)$ is a bounded chain of E. Then we have

$$\mathscr{T}(C \times D) \subseteq \mathscr{A}(C) + \mathscr{G}(C \times D)$$

and

$$\mathscr{T}(D \times C) \subseteq \mathscr{A}(D) + \mathscr{G}(D \times C).$$

Now, by property (P6) of partial Kuratowski measure α_p of noncompactness in E, we obtain

$$\alpha_p\big(\mathscr{T}(C \times D)\big) \leqslant \alpha_p\big(\mathscr{A}(C)\big) + \alpha_p\big(\mathscr{G}(C \times D)\big) \leqslant \psi_{\mathscr{A}}\big(\alpha_p(C)\big) \tag{4.59}$$

and

$$\alpha_p\big(\mathscr{T}(D \times C)\big) \leqslant \alpha_p\big(\mathscr{A}(D)\big) + \alpha_p\big(\mathscr{G}(D \times C)\big) \leqslant \psi_{\mathscr{A}}\big(\alpha_p(D)\big) \tag{4.60}$$

Adding (4.59) and (4.60) together implies that

$$\begin{aligned}\alpha_p\Big(\mathscr{T}(C \times D)\Big) + \alpha_p\Big(\mathscr{T}(D \times C)\Big) \\ &\leqslant \psi_{\mathscr{A}}\big(\alpha_p(C)\big) + \psi_{\mathscr{A}}\big(\alpha_p(D)\big) \\ &< \alpha_p(C) + \alpha_p(D)\end{aligned}$$

for all bounded chains C and D in E for which $\alpha_p(C)+\alpha_p(D)>0$. As a result, the coupled operator $\mathscr{T}$ is a partially condensing with respect to the partial Kuratowskii measure α_p of noncompactness in E. Thus, $\mathscr{T}$ satisfies all the conditions of Theorem 2.1 and so the coupled operator equations $x=\mathscr{T}(x,y)$ and $y=\mathscr{T}(y,x)$ have a coupled solution (x^*,y^*). Consequently the coupled operator equations (4.51) and (4.52) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by (4.55) and (4.56) converge monotonically to x^* and y^* respectively. This completes the proof. □

As a consequence of above Theorem 4.9 we obtain the following corollary.

COROLLARY 4.6. *Let (E,K) be an ordered Banach space. Suppose that $\mathscr{A}:E\to E$ is a nondecreasing operator and $\mathscr{G}:E\times E\to E$ is a mixed monotone coupled operator satisfying the conditions (a) through (c) of Theorem 4.9. Then the coupled operator equations (4.51) and (4.52) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive iterations defined by (4.55) and (4.56) converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.*

Next, we consider the mixed coupled operator equations

$$x=\mathscr{A}x+\mathscr{F}(x,y) \tag{4.61}$$

and

$$y=\mathscr{A}y+\mathscr{F}(y,x), \tag{4.62}$$

where $\mathscr{A}:E\to E$ is a nonlinear operator and $\mathscr{F}:E\times E\to E$ is a coupled operator which are not necessarily continuous.

A pair of elements $(x^*,y^*)\in E\times E$ is called a *coupled fixed point* of the coupled operator equations (4.61) and (4.62) if

$$x^*=\mathscr{A}x^*+\mathscr{F}(x^*,y^*) \tag{4.63}$$

and

$$y^*=\mathscr{A}y^*+\mathscr{F}(y^*,x^*). \tag{4.64}$$

THEOREM 4.10. *Let $(E,\leqslant,\|\cdot\|)$ be a partially ordered Banach space and let every compact chain C in E be Janhavi. Suppose that $\mathscr{A}:E\to E$ is a nondecreasing operator and $\mathscr{F}:E\times E\to K$ is a mixed monotone coupled operator satisfying the following conditions.*

(a) $\mathscr{A}$ is partially continuous and partially compact,

(b) $\mathscr{F}$ is partially bounded and nonlinear partial $\mathscr{D}$-contraction with a $\mathscr{D}$-function $\psi_{\mathscr{F}}$, and

(c) there exists an element $(x_0,y_0)\in E\times E$ such that $x_0\leqslant\mathscr{A}x_0+\mathscr{F}(x_0,y_0)$ and $y_0\geqslant\mathscr{A}y_0+\mathscr{F}(y_0,x_0)$ or $x_0\geqslant\mathscr{A}x_0+\mathscr{F}(x_0,y_0)$ and $y_0\leqslant\mathscr{A}y_0+\mathscr{F}(y_0,x_0)$.

Then the coupled operator equations (4.61) *and* (4.62) *have a coupled solution* (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by*

$$x_{n+1} = \mathscr{A}x_n + \mathscr{F}(x_n, y_n) \tag{4.65}$$

and

$$y_{n+1} = \mathscr{A}y_n + \mathscr{F}(y_n, x_n) \tag{4.66}$$

for $n \geqslant 0$, *converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

Proof. The proof is similar to Theorem 4.9 with appropriate modifications and hence we omit the details. □

As a consequence of above Theorem 4.10 we obtain the following corollary.

COROLLARY 4.7. *Let* (E, K) *be an ordered Banach space. Suppose that* $\mathscr{A} : E \to E$ *is a nondecreasing operator and* $\mathscr{F} : E \times E \to K$ *is a mixed monotone coupled operator satisfying the conditions (a) through (c) of Theorem* 4.10*. Then the coupled operator equations* (4.61) *and* (4.62) *have a coupled solution* (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive iterations defined by* (4.65) *and* (4.66) *converge monotonically to* x^* *and* y^* *respectively. Moreover, the set of all comparable coupled solutions is compact.*

The common existence principle running through all of the above coupled and mixed coupled hybrid fixed point theorems is called "*Dhage monotone iteration principle*" in nonlinear analysis and which may be described as "*the monotonic convergence of the sequences of successive iterations or approximations to the coupled solutions of the nonlinear coupled equations beginning with a lower or an upper coupled solution of the related coupled equations as their first or initial approximation*" and the procedure of applying the above iteration principle to nonlinear coupled equations is called the "*Dhage monotone iteration method*". This method is very much useful in the study of nonlinear coupled equations in view of the fact that we obtain the algorithms along with existence of the coupled solutions for a system of nonlinear coupled and mixed coupled equations under consideration.

5. Coupled periodic boundary value problems

The periodic boundary value problems are often times discussed in the literature for different aspects of the solutions via applications of the tools from nonlinear functional analysis. See for example, Dhage [18, 21, 22, 23, 24], Dhage and Dhage [27] and the references therein. In the following we consider a coupled periodic boundary value problem of nonlinear first order quadratic differential equations with linear and quadratic perturbations of second type to be discussed by an application of coupled hybrid fixed point principle embodied in Theorem 4.1.

Given a closed and bounded interval $J = [0,T]$ of the real line $\mathbb{R}$, we consider the coupled hybrid quadratic periodic boundary value problems (in short coupled hybrid QPBVPs) of nonlinear first order ordinary differential equations,

$$\left.\begin{aligned} \left(\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right)' + \lambda\left(\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right) &= g(t,x(t),y(t)),\ t \in J, \\ x(0) &= x(T), \end{aligned}\right\} \tag{5.1}$$

and

$$\left.\begin{aligned} \left(\frac{y(t)-k(t,y(t))}{f(t,y(t))}\right)' + \lambda\left(\frac{y(t)-k(t,y(t))}{f(t,y(t))}\right) &= g(t,y(t),x(t)),\ t \in J, \\ y(0) &= y(T), \end{aligned}\right\} \tag{5.2}$$

for $\lambda \in \mathbb{R}$, $\lambda > 0$, where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a *coupled solution* of the coupled hybrid QPBVPs (5.1) and (5.2) we mean a pair of functions $(x^*, y^*) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ such that the functions $t \mapsto \frac{x(t)-k(t,x(t))}{f(t,x(t))}$ and $t \mapsto \frac{y(t)-k(t,y(t))}{f(t,y(t))}$ are well defined, continuous and differentiable satisfying the equations (5.1) and (5.2) on J, where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on J.

The special case of the coupled hybrid QPBVPs of the form

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= g(t,x(t),y(t)),\ t \in J, \\ x(0) &= x(T), \end{aligned}\right\} \tag{5.3}$$

and

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= g(t,y(t),x(t)),\ t \in J, \\ y(0) &= y(T), \end{aligned}\right\} \tag{5.4}$$

have been discussed by Bhaskar and Lakshmikantham [4] and Berinde [3] for the existence and uniqueness theorem if the nonlinearities f and g satisfy a Lipschitz type condition and when f and g satisfy a compactness type condition it has been discussed in Dhage [16] for the existence and approximation of coupled solutions on J. However, the special cases of the coupled hybrid PBVPs (5.1) and (5.2) in the form

$$\left.\begin{aligned} \left[x(t)-k(t,x(t))\right]' + \lambda\left[x(t)-k(t,x(t))\right] &= g(t,x(t),y(t)),\ t \in J, \\ x(0) &= x(T), \end{aligned}\right\} \tag{5.5}$$

and

$$\left.\begin{aligned} \left[y(t)-k(t,y(t))\right]' + \lambda\left[y(t)-k(t,y(t))\right] &= g(t,y(t),x(t)),\ t \in J, \\ y(0) &= y(T), \end{aligned}\right\} \tag{5.6}$$

as well as the coupled hybrid QPBVPs

$$\left.\begin{aligned} \left(\frac{x(t)}{f(t,x(t))}\right)' + \lambda\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t),y(t)),\ t \in J,\\ x(0) = x(T), \end{aligned}\right\} \tag{5.7}$$

and

$$\left.\begin{aligned} \left(\frac{y(t)}{f(t,y(t))}\right)' + \lambda\left(\frac{y(t)}{f(t,y(t))}\right) = g(t,y(t),x(t)),\ t \in J,\\ y(0) = y(T), \end{aligned}\right\} \tag{5.8}$$

are also new and not discussed in the literature. In view of above presentation, the coupled hybrid QPBVPs (5.1) and (5.2) are more general in the theory of nonlinear coupled differential equations and therefore, it is of interest to discuss them for different aspects of the coupled solutions.

The existence theorems for all the above PBVPs without monotonic property may be proved using the hybrid fixed point theorems given in Dhage [23], however in that case we do not get the algorithms for the coupled solutions. The purpose of the present study is to establish an existence result and develop an algorithm for approximating the coupled solutions of the coupled hybrid QPBVPs (5.1) and (5.2) under certain mixed hybrid conditions on the nonlinearities f, g and k.

The following useful lemmas are obvious and may be found in Nieto [32, 33], Nieto and Lopez [34, 35] and Dhage [18, 20] and the references therein.

LEMMA 5.1. *For any function $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x'(t) + \lambda x(t) = \sigma(t),\ t \in J,\\ x(0) = x(T), \end{aligned}\right\} \tag{5.9}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_\lambda(t,s)\,\sigma(s)\,ds,\ t \in J, \tag{5.10}$$

where, the Green's function $G(t,s)$ is given by

$$G_\lambda(t,s) = \begin{cases} \dfrac{e^{\lambda s-\lambda t+\lambda T}}{e^{\lambda T}-1}, & \text{if} \quad 0 \leqslant s \leqslant t \leqslant T,\\[2ex] \dfrac{e^{\lambda s-\lambda t}}{e^{\lambda T}-1}, & \text{if} \quad 0 \leqslant t < s \leqslant T. \end{cases} \tag{5.11}$$

Notice that the Green's function G_λ is continuous and nonnegative on $J \times J$ and therefore, the number

$$M_{G_\lambda} := \max\{\,|G_\lambda(t,s)| : t,s \in [0,T]\,\}$$

exists for all $\lambda \in \mathbb{R}^+$. For the sake of convenience, we write $G_\lambda(t,s) = G(t,s)$ and $M_{G_\lambda} = M_G$.

Other useful results for establishing the main result are as follows.

LEMMA 5.2. (Dhage [22, 23]) *If there exists a differentiable function $u \in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t) + \lambda u(t) \leqslant \sigma(t),\ t \in J, \\ u(0) \leqslant u(T), \end{aligned}\right\} \tag{5.12}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, then

$$u(t) \leqslant \int_0^T G(t,s)\,\sigma(s)\,ds, \tag{5.13}$$

for all $t \in J$, where $G(t,s)$ is the Green's function given by the expression (5.11) on $J \times J$.

Proof. The proof of the lemma appears in Dhage [15, 16, 17, 20] and Dhage and Dhage [27, 28] and so we omit the details. □

Similarly, we have the following result of differential inequality related to the first order periodic boundary value problems defined on J.

LEMMA 5.3. (Dhage [22, 23]) *If there exists a differentiable function $v \in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} v'(t) + \lambda v(t) \geqslant \sigma(t),\ t \in J, \\ v(0) \geqslant v(T), \end{aligned}\right\} \tag{5.14}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, then

$$v(t) \geqslant \int_0^T G(t,s)\,\sigma(s)\,ds, \tag{5.15}$$

for all $t \in J$, where $G(t,s)$ is the Green's function given by the expression (5.11) on $J \times J$.

Now we are ready to apply our abstract mixed coupled hybrid fixed point theorem to coupled hybrid QPBVPs (5.1) and (5.2) under suitable natural conditions. In the following section we prove our main existence and approximation theorem for coupled solutions of the coupled hybrid QPBVPs (5.1) and (5.2) defined on J.

6. Existence and approximation results

The equivalent integral forms of the coupled hybrid QPBVPs (5.1) and (5.2) are considered in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leqslant$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{6.1}$$

and

$$x \leqslant y \quad \text{if and only if} \quad x(t) \leqslant y(t) \quad \text{for all} \quad t \in J. \tag{6.2}$$

Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm which is also a partially ordered with respect to the above partially order relation $\leqslant$. Moreover, $C(J,\mathbb{R})$ is also a Banach algebra with respect to the multiplication "$\cdot$" defined by

$$(x \cdot y)(t) = x(t) \cdot y(t) \ \text{ for all } \ t \in J. \tag{6.3}$$

It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and has some nice properties concerning the compatibility of norm and order relation in it. The following lemma concerning the Janhavi sets in $C(J,\mathbb{R})$ follows by an application of the Arzelá-Ascoli theorem.

LEMMA 6.1. *Let $\big(C(J,\mathbb{R}), \leqslant, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leqslant$ defined by (6.1) and (6.2) respectively. Then, every partially compact subset of $C(J,\mathbb{R})$ possesses $\mathscr{D}$-compatibility property with respect to $\|\cdot\|$ and $\leqslant$ and so is Janhavi.*

Proof. The proof of the lemma is well-known and appears in the papers of Dhage [14, 16, 17, 18] and Dhage and Dhage [26, 28]. Here we give the proof of the lemma using somewhat different arguments via cones in a Banach space $C(J,\mathbb{R})$. Define a subset K of $C(J,\mathbb{R})$ by

$$K = \{x \in C(J,\mathbb{R}) \mid x(t) \geqslant 0 \ \text{ for all } t \in J\}. \tag{6.4}$$

Clearly K is a non-empty, closed and convex subset of the Banach space $C(J,\mathbb{R})$ satisfying the properties (i)- (iv) of a cone in $C(J,\mathbb{R})$. So K is a positive cone in $C(J,\mathbb{R})$. Now, the order relation $\leqslant$ given by (6.2) is equivalent to the order relation $\leqslant$ defined by the cone K in $C(J,\mathbb{R})$. Therefore, the desired conclusion follows by an application of Lemma 2.2. This completes the proof. □

We need the following definition in what follows.

DEFINITION 6.1. A pair of differentiable functions $(u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is said to be a lower coupled solution of the coupled equations (5.1) and (5.2) if the functions $t \mapsto \dfrac{u(t)-k(t,u(t))}{f(t,u(t))}$ and $t \mapsto \dfrac{v(t)-k(t,v(t))}{f(t,v(t))}$ are continuous and differentiable on J satisfying

$$\left.\begin{aligned} \left(\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right)' + \lambda \left(\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right) \leqslant g(t,u(t),v(t)),\ t \in J,\\ u(0) \leqslant u(T), \end{aligned}\right\} \tag{6.5}$$

and

$$\left.\begin{aligned} \left(\frac{v(t)-k(t,v(t))}{f(t,v(t))}\right)' + \lambda \left(\frac{v(t)-k(t,v(t))}{f(t,v(t))}\right) \geqslant g(t,v(t),u(t)),\ t \in J,\\ v(0) \geqslant v(T). \end{aligned}\right\} \tag{6.6}$$

Similarly, a pair of differentiable functions $(w,z) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is called an upper coupled solution of the coupled hybrid QPBVPs (5.1) and (5.2) if the functions $t \mapsto \dfrac{w(t)-k(t,w(t))}{f(t,w(t))}$ and $t \mapsto \dfrac{z(t)-k(t,z(t))}{f(t,z(t))}$ are continuous and differentiable on J and the above inequalities are satisfied with reverse sign.

The coupled hybrid QPBVPs (5.1) and (5.2) will be considered under the following assumptions:

(A_1) f defines a mapping $f : J \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$.

(A_2) $f(t,x)$ is periodic in t with period T for each $x \in \mathbb{R}$.

(A_3) There exists a $\mathscr{D}$-function $\varphi_f \in \mathscr{D}$ such that

$$0 \leqslant f(t,x_1) - f(t,x_2) \leqslant \varphi_f\big(x_1 - x_2\big)$$

for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \geqslant x_2$.

(A_4) The function f is bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound M_f.

(B_1) g defines a mapping $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$.

(B_2) $g(t,x,y)$ is nondecreasing in x and nonincreasing in y for each $t \in J$.

(B_3) The function g is bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound M_g.

(C_1) $k(t,x)$ is periodic in t with period T for each $x \in \mathbb{R}$.

(C_2) There exists a $\mathscr{D}$-function $\varphi_k \in \mathfrak{D}$ such that

$$0 \leqslant k(t,x_1) - k(t,x_2) \leqslant \varphi_k\big(x_1 - x_2\big)$$

for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \geqslant x_2$.

(C_3) The function k is bounded on $J \times \mathbb{R}$ with bound M_k.

(D_1) The map $x \mapsto \dfrac{x-k(0,x)}{f(0,x)}$ is injection on $\mathbb{R}$.

(D_2) The coupled hybrid QPBVPs (5.1) and (5.2) have a lower coupled solution $(u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

The hypotheses (A_1)–(A_4), (B_1)–(B_3) and (C_1)–(C_3) are standard and have been widely used in the literature on nonlinear differential and integral equations. The special case of the hypothesis (A_3) and (C_2) with $\varphi(r) = \dfrac{Lr}{M+r}$, where $L > 0$ and $M > 0$ satisfy $L \leqslant M$, is considered recently in Dhage [9, 12].

REMARK 6.1. The hypothesis (D_1) holds, in particular if the map $x \mapsto \dfrac{x-k(0,x)}{f(0,x)}$ is increasing in $\mathbb{R}$.

The following useful lemma follows by an application of Lemma 5.1.

LEMMA 6.2. *Assume that the hypotheses* (A_2), (C_1) *and* (D_1) *hold. Then a pair of differentiable functions* $(x,y) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ *is a coupled solution to the coupled hybrid QPBVPs* (5.1) *and* (5.2) *if and only if* x *and* y *are the solutions of the quadratic coupled integral equations*

$$x(t) = k(t,x(t)) + \big[\, f(t,x(t)) \,\big] \left(\int_0^T G(t,s) g(s,x(s),y(s))\, ds \right) \tag{6.7}$$

and

$$y(t) = k(t,y(t)) + \big[\, f(t,y(t)) \,\big] \left(\int_0^T G(t,s) g(s,y(s),x(s))\, ds \right) \tag{6.8}$$

for all $t \in J$.

Proof. Set $z(t) = \dfrac{x(t) - k(t,x(t))}{f(t,x(t))}$ and $w(t) = \dfrac{y(t) - k(t,y(t))}{f(t,y(t))}$ for $t \in J$. Then the functions $t \mapsto z(t)$ and $t \mapsto w(t)$ are continuous and differentiable on J. Moreover, by the periodic boundary conditions $x(0) = x(T)$ and $y(0) = y(T)$, get

$$z(0) = \frac{x(0) - k(0,x(0))}{f(0,x(0))} = \frac{x(T) - k(T,x(T))}{f(T,x(T))} = z(T)$$

and

$$w(0) = \frac{y(0) - k(0,y(0))}{f(0,y(0))} = \frac{y(T) - k(T,y(T))}{f(T,y(T))} = w(T)$$

and conversely in view of hypotheses (A_2), (C_1) and (D_1). Substituting these values in the coupled hybrid QPBVPs (5.1) and (5.2) we obtain the PBVPs

$$\left.\begin{aligned} z'(t) + \lambda z(t) &= g(t,x(t),y(t)) \\ z(0) &= z(T) \end{aligned}\right\} \tag{$*$}$$

and

$$\left.\begin{aligned} w'(t) + \lambda w(t) &= g(t,y(t),x(t)) \\ w(0) &= w(T) \end{aligned}\right\} \tag{$**$}$$

for all $t \in J$ and vice versa. Now, from Lemma 5.1 we infer that the above PBVPs $(*)$ and $(**)$ are equivalent to the integral equations

$$z(t) = \int_0^T G(t,s) g(s,x(s),y(s))\, ds,$$

and

$$w(t) = \int_0^T G(t,s) g(s,y(s),x(s))\, ds,$$

for all $t \in J$. The desired quadratic integral equations (6.7) and (6.8) now follow from definitions of the functions z and w on J and the proof of the lemma is complete. □

Now we formulate the main existence and approximation result for the coupled hybrid QPBVPs (5.1) and (5.2) under previously mentioned natural conditions.

THEOREM 6.1. *Assume that the hypotheses* $(A_1)-(A_4)$*,* $(B_1)-(B_3)$*,* $(C_1)-(C_3)$ *and* $(D_1)-(D_2)$ *hold. Furthermore, if the inequality*

$$M_g M_G T \varphi_f(r) + \varphi_k(r) < r, \; r > 0, \tag{6.9}$$

holds, then the coupled hybrid QPBVPs (5.1)–(5.2) *have a coupled solution* (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive approximations defined by*

$$\begin{aligned} x_0 = u, \quad x_{n+1}(t) &= k(t, x_n(t)) \\ &+ \big[\, f(t, x_n(t)) \,\big] \left(\int_0^T G(t,s) g(s, x_n(s), y_n(s))\, ds \right) \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} y_0 = v, \quad y_{n+1}(t) &= k(t, y_n(t)) \\ &+ \big[\, f(t, y_n(t)) \,\big] \left(\int_0^T G(t,s) g(s, y_n(s), x_n(s))\, ds \right) \end{aligned} \tag{6.11}$$

for $t \in J$*, converge monotonically to* x^* *and* y^* *respectively.*

Proof. Set $E = C(J, \mathbb{R})$. Then, by Lemma 6.1, every compact chain C in E is Janhavi. We introduce a Kasu norm $\|\cdot\|_{E^2}$ and a Kasu partial order $\preceq_m$ in $E^2 = E \times E$ by the relation

$$\|(x,y)\|_{E^2} = \|x\| + \|y\|$$

and

$$(x,y) \preceq_m (u,v) \iff x \leqslant u \wedge y \geqslant v$$

for $(x,y), (u,v) \in E \times E$. Clearly $\left(E^2, \preceq_m, \|\cdot\|_{E^2}\right)$ is a regular partially ordered Banach algebra with respect to above norm and partial order and every compact chain is E^2 is Janhavi in view of Theorem 2.5. Now, by Lemma 6.2, the coupled hybrid QPBVPs (5.1) and (5.2) are equivalent to the nonlinear coupled integral equations of Fredholm type (6.7) and (6.8) respectively.

Next, we define the three mappings $\mathscr{A}$ and $\mathscr{C}$ on E and $\mathscr{G}$ on $E \times E$ by

$$\mathscr{A}x(t) = f(t, x(t)), \; t \in J, \tag{6.12}$$

$$\mathscr{C}x(t) = k(t, x(t)), \; t \in J, \tag{6.13}$$

and

$$\mathscr{G}(x,y)(t) = \int_0^T G(t,s) g(s, x(s), y(s))\, ds, \; t \in J, \tag{6.14}$$

respectively.

Then, the nonlinear coupled hybrid quadratic integral equations (6.7) and (6.8) are equivalent to the coupled operator equations,

$$x(t) = \mathscr{A}x(t)\,\mathscr{G}(x,y)(t) + \mathscr{C}x(t), \; t \in J, \tag{6.15}$$

and

$$y(t) = \mathscr{A}y(t)\mathscr{G}(y,x)(t) + \mathscr{C}y(t),\ t \in J. \tag{6.16}$$

Since the hypotheses (A_1), (B_1) hold, $\mathscr{A}$ and $\mathscr{G}$ define the mappings $\mathscr{A} : E \to K$ and $\mathscr{G} : E \times E \to K$. Also the operator $\mathscr{C}$ maps E into itself. We shall show that the operators $\mathscr{A}$ and $\mathscr{C}$ and the coupled operator $\mathscr{G}$ satisfy all the conditions of Theorem 4.1 on their respective domains of definition into E. This will be done in a series of following steps:

Step I: The operators $\mathscr{A}$, $\mathscr{C}$ and $\mathscr{G}$ are monotone.

Let $x, y \in C(J,\mathbb{R})$ be two elements such that $x \geqslant y$. Then by hypothesis (A_1), we get

$$\mathscr{A}x(t) = f(t,x(t)) \geqslant f(t,y(t)) = \mathscr{A}y(t)$$

for all $t \in J$, and so $\mathscr{A}x \geqslant \mathscr{A}y$. Similarly, by hypothesis (C_2), we obtain

$$\mathscr{C}x(t) = k(t,x(t)) \geqslant k(t,y(t)) = \mathscr{C}y(t)$$

for all $t \in J$, and so $\mathscr{C}x \geqslant \mathscr{C}y$. Therefore the operators $\mathscr{A}$ and $\mathscr{C}$ are nondecreasing on E.

Next, let $(x,y),(u,v) \in E \times E$ be arbitrary elements such that $(x,y) \preceq_m (u,v)$. Then by definition of $\preceq_m$, we get $x \leqslant u$ and $y \geqslant v$. Now, by hypotheses (B_2),

$$\begin{aligned}\mathscr{G}(x,y)(t) &= \int_0^T G(t,s)f(s,x(s),y(s))\,ds \\ &\leqslant \int_0^T G(t,s)g(s,u(s),v(s))\,ds \\ &= \mathscr{G}(u,v)(t)\end{aligned}$$

for all $t \in J$. Hence $\mathscr{G}$ is a mixed monotone coupled operator on $E \times E$ into K.

Step II: $\mathscr{A}$ and $\mathscr{C}$ are partially bounded and partial $\mathscr{D}$-Lipschitz operator on E.

Let $x \in E$ be arbitrary. Then by (A_2),

$$|\mathscr{A}x(t)| \leqslant |f(t,x(t))| \leqslant M_f$$

for all $t \in J$. Taking the supremum over t in the above inequality, we obtain

$$\|\mathscr{A}x\| \leqslant M_f$$

for all $x \in E$. So, $\mathscr{A}$ is bounded and consequently a partially bounded operator on E. Similarly, it can be shown that the operator $\mathscr{C}$ is also bounded with bound M_k.

Next, let $x, y \in E$ be such that $x \geqslant y$. Then, we have

$$\begin{aligned}|\mathscr{A}x(t) - \mathscr{A}y(t)| &= \big|f(t,x(t)) - f(t,y(t))\big| \\ &\leqslant \varphi_f\,\big(\,|x(t)-y(t)|\,\big) \\ &\leqslant \varphi_f\,\big(\,\|x-y\|\,\big)\end{aligned}$$

for all $t \in J$, where $\varphi_f \in \mathfrak{D}$. Taking the supremum over t, we obtain

$$\|\mathscr{A}x - \mathscr{A}y\| \leqslant \varphi_f(\|x-y\|),$$

for all $x, y \in E$ with $x \geqslant y$. Hence, $\mathscr{A}$ is a partially $\mathscr{D}$-Lipschitz on E with a $\mathscr{D}$-function φ_f and which further also implies that $\mathscr{A}$ is a partially continuous operator on E. Similarly, we have

$$\|\mathscr{A}x - \mathscr{A}y\| \leqslant \varphi_k(\|x-y\|),$$

for all $x, y \in E$ with $x \geqslant y$, and so $\mathscr{C}$ is also partially $\mathscr{D}$-Lipschitz on E with a $\mathscr{D}$-function φ_k and which further also implies that $\mathscr{C}$ is a partially continuous operator on E.

Step III: $\mathscr{G}$ is a partially continuous coupled operator on $E \times E$.

Let C and D be any two chains in E and let $\{x_n\}$ and $\{y_n\}$ be two sequences in C and D respectively such that $x_n \to x$ and $y_n \to y$. Then, by continuity of the function g, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{G}(x_n, y_n)(t) &= \lim_{n\to\infty} \int_0^T G(t,s) g(s, x_n(s), y_n(s))\, ds \\
&= \int_0^T G(t,s) \left[\lim_{n\to\infty} g(s, x_n(s), y_n(s))\right] ds \\
&= \int_0^T G(t,s) g(s, x(s), y(s))\, ds \\
&= \mathscr{G}(x,y)(t)
\end{aligned}$$

for all $t \in J$. This shows that the sequence $\{\mathscr{G}(x_n, y_n)\}$ converges to $\mathscr{G}(x,y)$ pointwise on J. We show that the convergence is uniform. To do so, it is enough to show that the sequence $\{\mathscr{G}(x_n, y_n)\}$ is equicontinuous set of functions in E. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}
&\left|\mathscr{G}(x_n, y_n)(t_1) - \mathscr{G}(x_n, y_n)(t_2)\right| \\
&\qquad \leqslant \int_0^T |G(t_1, s) - G(t_2, s)|\, |g(s, x_n(s), y_n(s))|\, ds \\
&\qquad \leqslant M_g \int_0^T |G(t_1, s) - G(t_2, s)|\, ds \\
&\qquad \to 0 \quad \text{as} \quad t_1 \to t_2,
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This proves the equicontinuity of the sequence $\{\mathscr{G}(x_n, y_n)\}$ of functions in E. As a result, $\mathscr{G}(x_n, y_n) \to \mathscr{G}(x,y)$ uniformly. Hence $\mathscr{G}$ is continuous coupled operator on $C \times D$. Consequently, $\mathscr{G}$ is partially continuous on $E \times E$ into K.

Step IV: $\mathscr{G}$ is a partially compact coupled operator on $E \times E$.

Let C and D be any two chains in E. We show that $\mathscr{G}(C \times D)$ is a relatively compact subset of E. First we show that $\mathscr{G}(C \times D)$ is a uniformly bounded subset of E. Let $z \in \mathscr{G}(C \times D)$ be a fixed element. Then there exists a point $(x,y) \in C \times D$ such that $z = \mathscr{G}(x,y)$. Then,

$$\left|z(t)\right| = \left|\mathscr{G}(x,y)(t)\right| \leqslant \int_0^T G(t,s)|g(s,x(s),y(s))|\,ds \leqslant M_g M_G T$$

for all $t \in J$. Taking the supremum over t, $\|z\| \leqslant M_g M_G T$ for all $z \in \mathscr{G}(C \times D)$. Hence $\mathscr{G}(C \times D)$ is a uniformly bounded subset of E.

Next, we show that $\mathscr{G}(C \times D)$ is an equicontinuous subset of E. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}
|z(t_1) - z(t_2)| &= \left|\mathscr{G}(x,y)(t_1) - \mathscr{G}(x,y)(t_2)\right| \\
&\leqslant \int_0^T |G(t_1,s) - G(t_2,s)|\,|g(s,x(s),y(s))|\,ds \\
&\leqslant M_g \int_0^T |G(t_1,s) - G(t_2,s)|\,ds \\
&\to 0 \quad \text{as} \quad t_1 \to t_2,
\end{aligned}$$

uniformly for all $z \in \mathscr{G}(C \times D)$. This proves the equicontinuity of the set $\mathscr{G}(C \times D)$ in E. As a result, $\mathscr{G}(C \times D)$ is compact and hence relatively compact in view of Arzelá-Ascoli theorem. Hence $\mathscr{G}$ is a partially compact coupled operator on $E \times E$ into E.

Step V: The operators $\mathscr{G}$, $\mathscr{A}$ and $\mathscr{C}$ satisfy condition (d) of Theorem 4.1.

Now, since condition (6.9) holds, we have

$$M_{\mathscr{G}}\, \psi_{\mathscr{A}}(r) + \psi_{\mathscr{C}}(r) \leqslant M_g M_G T\, \varphi_f(r) + \varphi_k(r) < r$$

for each $r > 0$ and so, condition (d) of Theorem 4.1 is satisfied.

Step VI: Coupled equations (6.15)–(6.16) *have a lower coupled solution.*

Now, by hypothesis (H_5), there exists an element $(u,v) \in E \times E$ such that the functions $t \mapsto \dfrac{u(t) - k(t,u(t))}{f(t,u(t))}$ and $t \mapsto \dfrac{v(t) - k(t,v(t))}{f(t,v(t))}$ are continuous and differentiable satisfying the inequalities

$$\left.\begin{aligned}
\left(\frac{u(t) - k(t,u(t))}{f(t,u(t))}\right)' + \lambda\left(\frac{u(t) - k(t,u(t))}{f(t,u(t))}\right) \leqslant g(t,u(t),v(t)), \\
u(0) \leqslant u(T),
\end{aligned}\right\}$$

and

$$\left.\begin{aligned}
\left(\frac{v(t) - k(t,v(t))}{f(t,v(t))}\right)' + \lambda\left(\frac{v(t) - k(t,v(t))}{f(t,v(t))}\right) \geqslant g(t,v(t),u(t)), \\
u(0) \geqslant u(T).
\end{aligned}\right\}$$

for all $t \in J$. This further in view of Lemmas 5.1, 5.2 and 5.3 implies that

$$u(t) \leqslant k(t,u(t)) + \left[\, f(t,u(t)) \,\right] \left(\int_0^T G(t,s)g(s,u(s),v(s))\,ds \right)$$

and

$$v(t) \geqslant k(t,v(t)) + \left[\, f(t,v(t)) \,\right] \left(\int_0^T G(t,s)g(s,v(s),u(s))\,ds \right)$$

for all $t \in J$. Again, from definition of the operators $\mathscr{A}$ and $\mathscr{C}$ and the coupled operator $\mathscr{G}$ it follows that

$$u(t) \leqslant \mathscr{A}u(t)\mathscr{G}(u,y)(t) + \mathscr{C}u(t),\ t \in J,$$

and

$$v(t) \geqslant \mathscr{A}v(t)\mathscr{G}(v,u)(t) + \mathscr{C}v(t),\ t \in J.$$

Therefore, the coupled operator equations (6.15)–(6.16) have a lower coupled solution (u,v) in $E \times E$. Thus the coupled operators $\mathscr{F}$ and $\mathscr{G}$ satisfy all the conditions of Theorem 4.1 and hence the coupled operator equations (6.15)–(6.16) and consequently the coupled hybrid QPBVPs (5.1)–(5.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (6.10) and (6.11) converge monotonically to x^* and y^* respectively. □

REMARK 6.2. The conclusion of Theorem 6.1 also remains true if we replace the hypothesis (D_2) by the following one:

(D_3) The coupled hybrid QPBVPs (5.1)–(5.2) have a upper coupled solution $(w,z) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

The proof under this new hypothesis (D_3) is obtained by giving similar arguments as in the proof of Theorem 6.1 with appropriate modifications.

COROLLARY 6.1. *Assume that the hypotheses* (A_1)–(A_4), (B_1)–(B_3) *and* (D_1)–(D_2) *hold with* $k \equiv 0$. *Furthermore, if the inequality*

$$M_g M_G T \varphi_f(r) < r,\ r > 0, \tag{6.17}$$

holds, then the coupled hybrid QPBVPs (5.7) *and* (5.8) *have a positive coupled solution* (x^*,y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive approximations defined by*

$$x_0 = u, \quad x_{n+1}(t) = \left[\, f(t,x_n(t)) \,\right] \left(\int_0^T G(t,s)g(s,x_n(s),y_n(s))\,ds \right) \tag{6.18}$$

and

$$y_0 = v, \quad y_{n+1}(t) = \left[\, f(t,y_n(t)) \,\right] \left(\int_0^T G(t,s)g(s,y_n(s),x_n(s))\,ds \right) \tag{6.19}$$

for $t \in J$, *converge monotonically to* x^* *and* y^* *respectively.*

Proof. The proof is similar to Theorem 6.1 and now the desired conclusion follows by an application of Corollary 4.1. □

COROLLARY 6.2. *Assume that the hypotheses* (B_1)–(B_3), (C_1)–(C_3) *and* (D_1)–(D_2) *hold with* $f \equiv 1$. *Furthermore, if the inequality*

$$\varphi_f(r) < r,\ r > 0, \tag{6.20}$$

holds, then the coupled hybrid QPBVPs (5.5) and (5.6) have a coupled solution (x^*, y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive approximations defined by*

$$x_0 = u, \quad x_{n+1}(t) = f(t, x_n(t)) + \left(\int_0^T G(t,s) g(s, x_n(s), y_n(s))\, ds \right) \tag{6.21}$$

and

$$y_0 = v, \quad y_{n+1}(t) = f(t, y_n(t)) + \left(\int_0^T G(t,s) g(s, y_n(s), x_n(s))\, ds \right) \tag{6.22}$$

for $t \in J$, *converge monotonically to* x^* *and* y^* *respectively.*

Proof. The proof is similar to Theorem 6.1 and now the desired conclusion follows by an application of Theorem 4.9. □

REMARK 6.3. We note that if the coupled hybrid QPBVPs (5.1) and (5.2) have a lower coupled solution (u,v) as well as an upper coupled solution (w,z) such that $(u,v) \preceq_m (w,z)$, then under the given hypotheses of Theorem 6.1, they have a corresponding coupled solution (x_*, y^*) of the coupled hybrid QPBVPs (5.1)–(5.2) which satisfies the inequalities $u \leqslant x_* \leqslant w$ and $v \geqslant y^* \geqslant z$ and therefore, $(x_*, y^*) \in [u,w] \times [z,v]$, where the vector segment $[u,w]$ and $[z,v]$ are the sets in $C(J,\mathbb{R})$ defined by

$$[u,w] = \{x \in C(J,\mathbb{R}) \mid u \leqslant x \leqslant w\}$$

and

$$[z,v] = \{x \in C(J,\mathbb{R}) \mid z \leqslant x \leqslant v\}.$$

Moreover, if (u,v) is a lower coupled solution of the coupled hybrid QPBVPs (5.1) and (5.2) with $u \leqslant v$, then they have a coupled solution (x_*, y^*) satisfying the inequality

$$u = x_0 \leqslant \cdots \leqslant x_n \leqslant x_* \leqslant y^* \leqslant y_n \leqslant \cdots \leqslant y_0 = v, \tag{6.23}$$

where the sequences $\{x_n\}$ and $\{y_n\}$ are defined by (6.10) and (6.11) respectively. Thus, (x_*, y^*) is a maximal coupled solution of the coupled hybrid QPBVPs (5.1) and (5.2) in the vector segment $[u,v]$ of the Banach space $E = C(J,\mathbb{R})$ with respect to the order relation $\preceq_m$. The similar conclusion holds for the first order PBVPs (5.3)–(5.4), (5.5)–(5.6) and (5.7)–(5.8) on J. Again, we note that the present study via Dhage iteration method does not require any property of the cone K in the existence theorems of this paper which is otherwise to the case of nonlinear differential equations for proving the existence of maximal and minimal solutions.

7. Coupled hybrid QPBVPs of second order differential equations

Given a closed and bounded interval $J=[0,T]$ of the real line $\mathbb{R}$, we consider the coupled hybrid quadratic periodic boundary value problems (in short coupled hybrid QPBVPs) of nonlinear second order ordinary differential equations,

$$\left.\begin{aligned} -\left(\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right)'' + \lambda^2\left(\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right) = g(t,x(t),y(t)),\ t\in J,\\ x(0)=x(T),\ x'(0)=x'(T), \end{aligned}\right\} \tag{7.1}$$

and

$$\left.\begin{aligned} -\left(\frac{y(t)-k(t,y(t))}{f(t,y(t))}\right)'' + \lambda^2\left(\frac{y(t)-k(t,y(t))}{f(t,y(t))}\right) = g(t,y(t),x(t)),\ t\in J,\\ y(0)=y(T),\ y'(0)=y'(T), \end{aligned}\right\} \tag{7.2}$$

for $\lambda\in\mathbb{R}$, $\lambda>0$, where the functions $k: J\times\mathbb{R}\to\mathbb{R}$, $f: J\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ and $g: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous and satisfy certain mixed hybrid conditions from algebra, geometry and topology.

By a *coupled solution* of the coupled hybrid QPBVPs (7.1) and (7.2) we mean a pair of functions $(x^*,y^*)\in C^1(J,\mathbb{R})\times C^1(J,\mathbb{R})$ with the property that the functions $t\mapsto \dfrac{x(t)-k(t,x(t))}{f(t,x(t))}$ and $t\mapsto \dfrac{y(t)-k(t,y(t))}{f(t,y(t))}$ are well defined, continuously differentiable satisfying the equations (7.1) and (7.2) on J, where $C^1(J,\mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The QPBVPs (7.1) and (7.2) are the mixed linear and quadratic perturbations of second type of the coupled hybrid PBVPs of nonlinear second order ordinary differential equations,

$$\left.\begin{aligned} -x''(t)+\lambda^2 x(t) = g(t,x(t),y(t)),\ t\in J,\\ x(0)=x(T),\ x'(0)=x'(T), \end{aligned}\right\} \tag{7.3}$$

and

$$\left.\begin{aligned} -y''(t)+\lambda^2 y(t) = g(t,y(t),x(t)),\ t\in J,\\ y(0)=y(T),\ y'(0)=y'(T), \end{aligned}\right\} \tag{7.4}$$

and include the following coupled hybrid PBVPs

$$\left.\begin{aligned} -\big[x(t)-k(t,x(t))\big]'' + \lambda^2\big[x(t)-k(t,x(t))\big] = g(t,x(t),y(t)),\ t\in J,\\ x(0)=x(T),\ x'(0)=x'(T), \end{aligned}\right\} \tag{7.5}$$

and

$$\left.\begin{aligned} -\big[y(t)-k(t,y(t))\big]'' + \lambda^2\big[y(t)-k(t,y(t))\big] = g(t,y(t),x(t)),\ t\in J,\\ y(0)=y(T),\ y'(0)=y'(T), \end{aligned}\right\} \tag{7.6}$$

as well as the coupled hybrid QPBVPs

$$\left.\begin{aligned} -\left(\frac{x(t)}{f(t,x(t))}\right)'' + \lambda^2\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t),y(t)),\ t\in J,\\ x(0)=x(T),\ \ x'(0)=x'(T), \end{aligned}\right\} \tag{7.7}$$

and

$$\left.\begin{aligned} -\left(\frac{y(t)}{f(t,y(t))}\right)'' + \lambda^2\left(\frac{y(t)}{f(t,y(t))}\right) = g(t,y(t),x(t)),\ t\in J,\\ y(0)=y(T),\ \ y'(0)=y'(T), \end{aligned}\right\} \tag{7.8}$$

as special cases (see Dhage [12, 13] and the references therein). Notice that while applying the newly developed mixed coupled hybrid fixed point theorem to the QPBVPs (5.1) and (5.2) we made use of two features of the problem, namely, the nonnegativity of the Green's function and two differential inequalities established in Lemmas 5.2 and 5.3. The following lemma is crucial concerning the Green's function related to the PBVP of second order linear ordinary differential equations (see Nieto [32, 33]).

LEMMA 7.1. *For any function $\sigma\in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} -x''(t)+\lambda^2 x(t) = \sigma(t),\ \ t\in J,\\ x(0)=x(T),\ \ x'(0)=x'(T), \end{aligned}\right\} \tag{7.9}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T \widetilde{G}_\lambda(t,s)\sigma(s)\,ds,\ \ t\in J, \tag{7.10}$$

where $\widetilde{G}_\lambda(t,s)$ is the Green's function associated with the homogeneous PBVP

$$\left.\begin{aligned} -x''(t)+\lambda^2 x(t) = 0,\ \ t\in J,\\ x(0)=x(T),\ \ x'(0)=x'(T), \end{aligned}\right\} \tag{7.11}$$

given by

$$\widetilde{G}_\lambda(t,s) = \begin{cases} \dfrac{1}{2\lambda(e^{\lambda T}-1)}\left[e^{\lambda(t-s)}+e^{\lambda(T-t+s)}\right], & 0\leqslant s\leqslant t\leqslant T,\\ \dfrac{1}{2\lambda(e^{\lambda T}-1)}\left[e^{\lambda(s-t)}+e^{\lambda(T-s+t)}\right], & 0\leqslant t< s\leqslant T. \end{cases} \tag{7.12}$$

Notice that the Green's function $\widetilde{G}_\lambda$ is continuous and nonnegative on $J\times J$ and the numbers

$$\alpha = \min\{\,|\widetilde{G}_\lambda(t,s)| : t,s\in[0,T]\,\} = \frac{e^{\lambda T}}{\lambda(e^{\lambda T}-1)}$$

and

$$\beta = \max\{\,|\widetilde{G}_\lambda(t,s)| : t,s \in [0,T]\,\} = \frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}$$

exist for every positive real number λ.

Again, the linear differential inequalities similar to Lemmas 5.2 and 5.3 related to the PBVP of ordinary linear second order differential equation (7.9) may be stated as follows.

LEMMA 7.2. *If there exists a continuously differentiable function $u : J \to \mathbb{R}$ such that*

$$\left.\begin{aligned} -u''(t)+\lambda^2 u(t) \leqslant \sigma(t),\ t \in J,\\ u(0) \leqslant u(T),\ \ u'(0) \leqslant u'(T), \end{aligned}\right\} \tag{7.13}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, then

$$u(t) \leqslant \int_0^T \widetilde{G}_\lambda(t,s)\,\sigma(s)\,ds,\ \ t \in J, \tag{7.14}$$

where $\widetilde{G}_\lambda(t,s)$ is the Green's function given by the expression (7.12) on $J \times J$.

Proof. The proof of the lemma is obvious and follows from the maximum principle for BVPs of second order ordinary differential equations (see Protter and Weinberger [37] and Dhage and Heikkilä [29]). We omit the details. □

Similarly, we have the following differential inequality related to the second order periodic boundary value problems defined on J.

LEMMA 7.3. *If there exists a continuously differentiable function $v : J \to \mathbb{R}$ such that*

$$\left.\begin{aligned} -v''(t)+\lambda^2 v(t) \geqslant \sigma(t),\ \ t \in J,\\ v(0) \geqslant v(T),\ \ v'(0) \geqslant v'(T), \end{aligned}\right\} \tag{7.15}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, then

$$v(t) \geqslant \int_0^T \widetilde{G}_\lambda(t,s)\,\sigma(s)\,ds,\ \ t \in J, \tag{7.16}$$

where $\widetilde{G}_\lambda(t,s)$ is the Green's function given by the expression (7.12) on $J \times J$.

The following useful result follows by an application of Lemma 7.1.

LEMMA 7.4. *Assume that the hypotheses (A_2), (C_1) and (D_1) hold. Then a pair of continuously differentiable functions $(x,y) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is a solution to the coupled hybrid QPBVPs (7.1) and (7.2) if and only if x and y are the solutions of the nonlinear quadratic coupled integral equations*

$$x(t) = k(t,x(t)) + \big[\,f(t,x(t))\,\big]\left(\int_0^T \widetilde{G}_\lambda(t,s) g(s,x(s),y(s))\,ds\right) \tag{7.17}$$

and

$$y(t)=k(t,y(t))+\left[\,f(t,y(t))\,\right]\left(\int_0^T \widetilde{G}_\lambda(t,s)g(s,y(s),x(s))\,ds\right) \tag{7.18}$$

for all $t\in J$.

We need the following definition in what follows.

DEFINITION 7.1. A pair (u,v) of continuously differentiable functions $u,v:J\to\mathbb{R}$ is said to be a lower coupled solution of the coupled equations (7.1) and (7.2) if the functions $t\mapsto \dfrac{u(t)-k(t,u(t))}{f(t,u(t))}$ and $t\mapsto \dfrac{v(t)-k(t,v(t))}{f(t,v(t))}$ are continuously differentiable on J satisfying the inequalities

$$\left.\begin{aligned} -\left(\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right)''+\lambda^2\left(\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right)\leqslant g(t,u(t),v(t)),\ t\in J,\\ u(0)\leqslant u(T),\ u'(0)\leqslant u'(T), \end{aligned}\right\} \tag{7.19}$$

and

$$\left.\begin{aligned} -\left(\frac{v(t)-k(t,v(t))}{f(t,v(t))}\right)''+\lambda^2\left(\frac{v(t)-k(t,v(t))}{f(t,v(t))}\right)\geqslant g(t,v(t),u(t)),\ t\in J,\\ v(0)\geqslant v(T,)\ v'(0)\geqslant v'(T). \end{aligned}\right\} \tag{7.20}$$

Similarly, a pair (w,z) of continuously differentiable functions $w,z:J\to\mathbb{R}$ is called an upper coupled solution of the coupled hybrid QPBVPs (7.1) and (7.2) if the functions $t\mapsto \dfrac{w(t)-k(t,w(t))}{f(t,w(t))}$ and $t\mapsto \dfrac{z(t)-k(t,z(t))}{f(t,z(t))}$ are continuously differentiable on J and the above inequalities are satisfied with reverse sign.

We consider the following hypothesis in what follows:

(D_4) The coupled hybrid QPBVPs (7.1) and (7.2) have a lower coupled solution $(u,v)\in C^1(J,\mathbb{R})\times C^1(J,\mathbb{R})$.

THEOREM 7.1. *Assume that the hypotheses* (A_1)–(A_4), (B_1)–(B_3), (C_1)–(C_3) *and* (D_1), (D_4) *hold. Furthermore, if the inequality*

$$M_g\beta\, T\varphi_f(r)+\varphi_k(r)<r,\ r>0, \tag{7.21}$$

holds, then the coupled hybrid QPBVPs (7.1) *and* (7.2) *have a coupled solution* (x^*,y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive approximations defined by*

$$x_0=u,\quad x_{n+1}(t)=k(t,x_n(t))+\left[\,f(t,x_n(t))\,\right]\left(\int_0^T \widetilde{G}_\lambda(t,s)g(s,x_n(s),y_n(s))\,ds\right) \tag{7.22}$$

and

$$y_0 = v, \quad y_{n+1}(t) = k(t, y_n(t)) + \left[f(t, y_n(t)) \right] \left(\int_0^T \widetilde{G}_\lambda(t,s) g(s, y_n(s), x_n(s))\, ds \right) \tag{7.23}$$

for $t \in J$, converge monotonically to x^* and y^* respectively.

Proof. The proof is similar to Theorem 6.1 with appropriate modifications. Hence we omit the details. □

REMARK 7.1. The conclusion of Theorem 7.1 also remains true if we replace the hypothesis (D_4) by the following one:

(D_5) The coupled hybrid QPBVPs (7.1) and (7.2) have a upper coupled solution $(w,z) \in C^1(J,\mathbb{R}) \times C^1(J,\mathbb{R})$.

The proof under this new hypothesis (D_5) is obtained by giving similar arguments as in the proof of Theorem 7.1 with appropriate modifications.

COROLLARY 7.1. *Assume that the hypotheses (A_1)–(A_4), (B_1)–(B_3), (D_1) and (D_4) hold with $k \equiv 0$. Furthermore, if the inequality*

$$M_g \beta T \varphi_f(r) < r, \; r > 0, \tag{7.24}$$

holds, then the coupled hybrid QPBVPs (7.7) and (7.8) have a positive coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive approximations defined by

$$x_0 = u, \quad x_{n+1}(t) = \left[f(t, x_n(t)) \right] \left(\int_0^T \widetilde{G}_\lambda(t,s) g(s, x_n(s), y_n(s))\, ds \right) \tag{7.25}$$

and

$$y_0 = v, \quad y_{n+1}(t) = \left[f(t, y_n(t)) \right] \left(\int_0^T \widetilde{G}_\lambda(t,s) g(s, y_n(s), x_n(s))\, ds \right) \tag{7.26}$$

for $t \in J$, converge monotonically to x^* and y^* respectively.

COROLLARY 7.2. *Assume that the hypotheses (B_1)–(B_3), (C_1)–(C_3), (D_1), and (D_4) hold with $f \equiv 1$. Furthermore, if the inequality*

$$\varphi_f(r) < r, \; r > 0, \tag{7.27}$$

holds, then the coupled hybrid QPBVPs (7.5) and (7.6) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive approximations defined by

$$x_0 = u, \quad x_{n+1}(t) = f(t, x_n(t)) + \left(\int_0^T \widetilde{G}_\lambda(t,s) g(s, x_n(s), y_n(s))\, ds \right) \tag{7.28}$$

and

$$y_0 = v, \quad y_{n+1}(t) = f(t, y_n(t)) + \left(\int_0^T \widetilde{G}_\lambda(t,s) g(s, y_n(s), x_n(s))\, ds \right) \tag{7.29}$$

for $t \in J$, converge monotonically to x^* and y^* respectively.

Furthermore, we mention that the obtained existence and approximation results of the coupled hybrid QPBVPs (7.1) and (7.2) also include the existence and approximation results for the coupled hybrid PBVPs (7.3)–(7.4), (7.5)–(7.6) and (7.7)–(7.8) on J as special cases.

REMARK 7.2. The conclusion of Remark 6.3 also remains true if we replace the first order coupled hybrid QPBVPs (5.1) and (5.2) with the second order coupled hybrid QPBVPs (7.1) and (7.2) on J. The similar conclusion holds for the second order PBVPs (7.3)–(7.4), (7.5)–(5.6) and (7.7)–(7.8) on J.

REMARK 7.3. A unified generalization of the coupled hybrid PBVPs of first and second order quadratic differential equations (5.1)–(5.2) and (7.1)–(7.2) is the following problem of coupled hybrid nonlinear Fredholm type quadratic integral equations

$$x(t) = k(t,x(t)) + \big[f(t,x(t))\big]\left(\int_0^T v(t,s)g(s,x(s),y(s))\,ds\right) \tag{7.30}$$

and

$$y(t) = k(t,y(t)) + \big[f(t,y(t))\big]\left(\int_0^T v(t,s)g(s,y(s),x(s))\,ds\right) \tag{7.31}$$

for all $t \in J$, where $v : J \times J \to \mathbb{R}$, $k, f : J \times \mathbb{R} \to \mathbb{R}$ and $G : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. When the kernel function $v(t,s)$ takes the special values like (5.11) and (7.12), we obtain the coupled hybrid PBVPs of quadratic first and second order differential equations (5.1)–(5.2) and (7.1)–(7.2) respectively defined on J.

Furthermore, the existence and approximation theorems along with algorithms for the coupled hybrid nonlinear integral equations (7.30) and (7.31) may be obtained with the arguments similar to existence theorem for the coupled hybrid PBVPs (5.1) and (5.2) with appropriate modifications.

8. Examples

Below in the following we give some numerical examples of the nonlinear coupled hybrid periodic boundary problems of first and second order ordinary quadratic differential equations illustrating the hypotheses and the conclusion of Theorems 6.1 and 7.1 and the Corollaries 6.1, 6.2, 7.1 and 7.2.

EXAMPLE 8.1. Given a closed and bounded interval $J = [0,1]$ of the real line $\mathbb{R}$, we consider the coupled hybrid quadratic periodic boundary value problems (in short coupled hybrid QPBVPs) of nonlinear first order ordinary differential equations,

$$\left.\begin{aligned} \left(\frac{x(t) - k_1(t,x(t))}{f_1(t,x(t))}\right)' + \left(\frac{x(t) - k_1(t,x(t))}{f_1(t,x(t))}\right) &= g_1(t,x(t),y(t)),\ t \in J, \\ x(0) &= x(1), \end{aligned}\right\} \tag{8.1}$$

and

$$\left. \begin{aligned} \left(\frac{y(t)-k_1(t,y(t))}{f_1(t,y(t))}\right)' + \left(\frac{y(t)-k_1(t,y(t))}{f_1(t,y(t))}\right) = g_1(t,y(t),x(t)),\ t\in J,\\ y(0)=y(1), \end{aligned} \right\} \tag{8.2}$$

where $f_1:[0,1]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$, $g_1:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $k_1:[0,1]\times\mathbb{R}\to\mathbb{R}$ are three functions defined by

$$f_1(t,x)=\begin{cases} 2 & \text{if } -\infty<x\leqslant 0,\\ \dfrac{1}{2}\cdot\dfrac{x}{1+x}+2 & \text{if } 0<x<\infty, \end{cases}$$

$$g_1(t,x,y)=\frac{1}{21}\left[\coth x+\cot^{-1}y+4\right]$$

and

$$k_1(t,x)=\begin{cases} 1 & \text{if } -\infty<x\leqslant 0,\\ \dfrac{1}{2}\cdot\dfrac{x}{1+x}+1 & \text{if } 0<x<\infty, \end{cases}$$

for all $t\in[0,1]$.

It is easy to verify that the real-valued function f_1 is continuous on $[0,1]\times\mathbb{R}$ into $\mathbb{R}_+\setminus\{0\}$, g_1 is continuous on $[0,1]\times\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}_+$ and k_1 is continuous on $[0,1]\times\mathbb{R}$ into $\mathbb{R}$ and so the hypotheses (A_1) and (B_1) are satisfied. We shall show that the nonlinearities f_1, g_1 and k_1 satisfy the hypotheses (A_1)–(A_4), (B_1)–(B_3) and (C_1)–(C_3) respectively. Clearly the function $f_1(t,x)$ is periodic in t for each $x\in\mathbb{R}$ and is bounded on $[0,1]\times\mathbb{R}$ with bound $M_{f_1}=2$ and so the hypotheses (A_2) and (A_4) are satisfied. Next let $x_1,x_2\in\mathbb{R}$ be such that $x_1\geqslant x_2>0$. Then, we have

$$\begin{aligned} 0 &\leqslant f_1(t,x_1)-f_1(t,x_2,)\\ &=\frac{1}{2}\left[\frac{x_1}{1+x_1}-\frac{x_2}{1+x_2}\right]\\ &=\frac{1}{2}\left[\frac{x_1-x_2}{(1+x_1)(1+x_2)}\right]\\ &=\frac{1}{2}\left[\frac{x_1-x_2}{1+x_1+x_2+x_1x_2}\right]\\ &\leqslant\frac{1}{2}\left[\frac{x_1-x_2}{1+x_1+x_2}\right]\\ &\leqslant\frac{1}{2}\left[\frac{x_1-x_2}{1+x_1-x_2}\right]\\ &=\varphi_{f_1}(x_1-x_2) \end{aligned}$$

where, $\varphi_{f_1}(r)=\frac{1}{2}\cdot\frac{r}{1+r}$ for $r>0$ and that $\varphi_{f_1}\in\mathfrak{D}$. Again, if $x_1\leqslant 0$ and $x_2\leqslant 0$, then the above inequality is satisfied. Similarly, if $x_1>0$ and $x_2\leqslant 0$, then also the above inequality is satisfied. Therefore, in all cases, the function f_1 satisfies the hypothesis (A_3) on $[0,1]\times\mathbb{R}$.

Similarly, the function $k_1(t,x)$ is periodic in t for each $x\in\mathbb{R}$. Also k_1 is bounded on $[0,1]\times\mathbb{R}$ with bound $M_{k_1}=2$. Further, it can be shown as in the case of function f_1 that the function k_1 satisfies the inequality

$$0\leqslant k_1(t,x_1)-k_1(t,x_2,)\leqslant\varphi_{k_1}(x_1-x_2)$$

for all $t\in[0,1]$ and for all elements $x_1,x_2\in\mathbb{R}$ with $x_1\geqslant x_2$, where $\varphi_{k_1}(r)=\frac{1}{2}\cdot\frac{r}{1+r}$ for $r>0$ and that $\varphi_{k_1}\in\mathfrak{D}$.

Next, the function g_1 is bounded on $[0,1]\times\mathbb{R}\times\mathbb{R}$ with bound $M_{g_1}=\frac{1}{3}$. Again, $g_1(t,x,y)$ is nondecreasing in x and nonincreasing in y for each $t\in[0,1]$. Here, the Green's function $G(t,s)$ is given by

$$G(t,s)=\begin{cases}\dfrac{e^{s-t+1}}{e-1}, & \text{if}\quad 0\leqslant s\leqslant t\leqslant 1,\\[2mm] \dfrac{e^{s-t}}{e-1}, & \text{if}\quad 0\leqslant t<s\leqslant 1.\end{cases}\tag{8.3}$$

Therefore, $M_G=\sup\limits_{(t,s)\in[0,1]\times[0,1]}G(t,s)\leqslant 3$. Then, we have

$$\begin{aligned}M_gM_GT\varphi_{f_1}(r)+\varphi_{k_1}(r)&\leqslant\frac{1}{3}\cdot 3\cdot\varphi_{f_1}(r)+\varphi_{k_1}(r)\\&\leqslant\frac{1}{2}\cdot\frac{r}{1+r}+\frac{1}{2}\cdot\frac{r}{1+r}\\&=\frac{r}{1+r}<r\end{aligned}$$

for each $r>0$. So the condition (6.9) of Theorem 6.1 is satisfied.

Next, we show that the functions f_1 and k_1 satisfy the hypothesis (D_1), that is, the map $x\mapsto\frac{x-k_1(0,x)}{f_1(0,x)}$ is injection on $\mathbb{R}$. If $x\leqslant 0$ and $y\leqslant 0$ are any two real numbers, then the expression

$$\frac{x-k_1(0,x)}{f_1(0,x)}=\frac{y-k_1(0,y)}{f_1(0,y)}$$

implies that

$$\frac{x-1}{2}=\frac{y-1}{2}\quad\Longrightarrow\quad x=y.$$

Similarly, if $x>0$ and $y>0$ be any two real numbers, then the expression

$$\frac{x-k_1(0,x)}{f_1(0,x)}=\frac{y-k_1(0,y)}{f_1(0,y)}$$

implies that

$$\frac{x-\left[\frac{1}{2}\left(\frac{x}{1+x}\right)+1\right]}{\frac{1}{2}\left(\frac{x}{1+x}\right)+2}=\frac{y-\left[\frac{1}{2}\left(\frac{y}{1+y}\right)+1\right]}{\frac{1}{2}\left(\frac{y}{1+y}\right)+2}$$

$$\Longrightarrow \qquad \frac{2x^2-x-2}{5x+4}=\frac{2y^2-y-2}{5y+4}$$

$$\Longrightarrow \qquad 10x^2y-10y+8x^2-4x=10y^2x-10x+8y^2-4y$$

$$\Longrightarrow \qquad (x-y)\big[10xy+8(x+y)+6\big]=0$$

$$\Longrightarrow \qquad x=y.$$

This proves that the map $x\mapsto\dfrac{x-k_1(0,x)}{f_1(0,x)}$ is injection on $\mathbb{R}$, and so the hypothesis (D_1) is satisfied.

Finally, the pair of functions (u,v) given by

$$u(t)=\int_0^1 G(t,s)\,ds-1$$

and

$$v(t)=\int_0^1 G(t,s)\,ds+1$$

for all $t\in[0,1]$, are continuous and form the lower coupled solutions of the coupled hybrid QPBVPs (8.1) and (8.2) defined on $J=[0,1]$, where the Green's function G is defined by (8.3) on $[0,1]\times[0,1]$.

Thus the functions f_1, g_1 and k_1 satisfy all the hypotheses (A_1)–(A_4), (B_1)–(B_3), (C_1)–(C_3) and (D_1)–(D_2) of Theorem 6.1 and therefore, the coupled hybrid QPBVPs (8.1) and (8.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_0(t)=\int_0^1 G(t,s)\,ds-1,$$

$$x_{n+1}(t)=k_1(t,x_n(t))+\big[\,f_1(t,x_n(t))\,\big]\left(\int_0^1 G(t,s)g_1(s,x_n(s),y_n(s))\,ds\right)$$

and

$$y_0(t)=\int_0^1 G(t,s)\,ds+1,$$

$$y_{n+1}(t)=k_1(t,y_n(t))+\big[\,f_1(t,y_n(t))\,\big]\left(\int_0^1 G(t,s)g_1(s,y_n(s),x_n(s))\,ds\right)$$

for all $t\in[0,1]$, converge monotonically to x^* and y^* respectively.

EXAMPLE 8.2. Given a closed and bounded interval $J=[0,1]$ in $\mathbb{R}$, we consider the coupled hybrid quadratic periodic boundary value problems (in short QPBVPs) of nonlinear first order ordinary differential equations,

$$\left.\begin{aligned} \left(\frac{x(t)}{\tan^{-1}x(t)+3}\right)' + \left(\frac{x(t)}{\tan^{-1}x(t)+3}\right) &= \frac{1}{15}\left[\,\tanh x(t)-\tanh y(t)+3\,\right], \\ x(0) &= x(1), \end{aligned}\right\} \tag{8.4}$$

and

$$\left.\begin{aligned} \left(\frac{y(t)}{\tan^{-1}y(t)+3}\right)' + \left(\frac{y(t)}{\tan^{-1}y(t)+3}\right) &= \frac{1}{15}\left[\,\tanh y(t)-\tanh x(t)+3\,\right], \\ y(0) &= y(1), \end{aligned}\right\} \tag{8.5}$$

for all $t\in[0,1]$.

Here, f and g are the functions on $[0,1]\times\mathbb{R}$ and $[0,1]\times\mathbb{R}\times\mathbb{R}$ defined by

$$f(t,x)=\tan^{-1}x+3$$

and

$$g(t,x,y)=\frac{1}{15}\left[\,\tanh x-\tanh y+3\,\right].$$

It is easy to verify that the real-valued functions f and g are continuous on $[0,1]\times\mathbb{R}$ and $[0,1]\times\mathbb{R}\times\mathbb{R}$ respectively. We shall show that the nonlinearities f and g satisfy the hypotheses (A_1)–(A_4) and (B_1)–(B_3) respectively. Obviously f and g define the functions $f:[0,1]\times\mathbb{R}\to\mathbb{R}_+\setminus\{0\}$ and $g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}_+$. So the hypotheses ($A_1$) and ($B_1$) are satisfied. Clearly the function f is bounded on $[0,1]\times\mathbb{R}$ with bound $M_f=5$. Next let $x,y\in\mathbb{R}$ be such that $x\geqslant y$. Then, we have

$$0\leqslant f(t,x)-f(t,y)=\tan^{-1}x-\tan^{-1}y=\frac{1}{1+\xi^2}(x-y)=\varphi_f(x-y)$$

where, $\varphi_f(r)=\dfrac{r}{1+\xi^2}$ for $0<\xi<r$ and that $\varphi_f\in\mathfrak{D}$. Thus, the function f satisfies the hypothesis (A_2).

Next, the function g is bounded on $[0,1]\times\mathbb{R}\times\mathbb{R}$ with bound $M_g=\dfrac{1}{3}$. Again, $g(t,x,y)$ is nondecreasing in x and nonincreasing in y for each $t\in[0,1]$. Here, the Green's function $G(t,s)$ is given by (8.3) on $[0,1]\times[0,1]$. Therefore, we have

$$M_G=\sup_{(t,s)\in[0,1]\times[0,1]} G(t,s)\leqslant 3.$$

Then, we have

$$M_gM_GT\varphi_f(r)\leqslant\frac{1}{3}\cdot 3\cdot\varphi_f(r)\leqslant\frac{r}{1+\xi^2}$$

for each $r>0$, where $0<\xi<r$. So the condition (6.17) of Corollary 6.1 holds.

Next, we show that the function f satisfies the hypothesis (D_1) with $k\equiv 0$, that is, the map $x\mapsto \dfrac{x}{f(0,x)}$ is injection on $\mathbb{R}$. If x and y are any two real numbers, then the expression

$$\frac{x}{f(0,x)}=\frac{x}{\tan^{-1}x+3}$$

implies that

$$\frac{d}{dx}\left[\frac{x}{f(0,x)}\right]=\frac{d}{dx}\left[\frac{x}{\tan^{-1}x+3}\right]=\frac{\tan^{-1}x+3-\dfrac{x}{1+x^2}}{(\tan^{-1}x+3)^2}>0$$

for all $x\in\mathbb{R}$. Therefore, the map $x\mapsto \dfrac{x}{f(0,x)}$ is increasing on $\mathbb{R}$ and consequently the map $x\mapsto \dfrac{x}{f(0,x)}$ is injection on $\mathbb{R}$ in view of Remark 6.1.

Finally, the pair of functions (u,v) given by

$$u(t)=\frac{1}{15}\int_0^1 G(t,s)\,ds$$

and

$$v(t)=\frac{5}{3}\int_0^1 G(t,s)\,ds$$

for all $t\in[0,1]$, are continuous and form the lower coupled solution of the coupled hybrid QPBVPs (8.4) and (8.5) defined on $J=[0,1]$, where the Green's function G is defined by (8.3) on $[0,1]\times[0,1]$.

Thus, the functions f and g satisfy all the hypotheses (A_1)–(A_4), (B_1)–(B_3) and (D_1)–(D_2) of Corollary 6.1 and therefore, the coupled hybrid QPBVPs (8.4) and (8.5) have a positive coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_0(t)=\frac{1}{15}\int_0^1 G(t,s)\,ds,$$
$$x_{n+1}(t)=\frac{1}{15}\left[\tan^{-1}x_n(t)+3\right]\left(\int_0^1 G(t,s)\left[\tanh x_n(s)-\tanh y_n(s)+3\right]\,ds\right)$$

and

$$y_0(t)=\frac{5}{3}\int_0^1 G(t,s)\,ds,$$
$$y_{n+1}(t)=\frac{1}{15}\left[\tan^{-1}y_n(t)+3\right]\left(\int_0^1 G(t,s)\left[\tanh y_n(s)-\tanh x_n(s)+3\right]\,ds\right)$$

for all $t\in[0,1]$, converge monotonically to x^* and y^* respectively.

EXAMPLE 8.3. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, we consider the coupled hybrid quadratic periodic boundary value problems (in short PBVPs) of nonlinear first order ordinary differential equations,

$$\left.\begin{aligned} \left[x(t) - \frac{1}{2}\tan^{-1}x(t)\right]' + \left[x(t) - \frac{1}{2}\tan^{-1}x(t)\right] = \tanh x(t) - \tanh y(t), \\ x(0) = x(1), \end{aligned}\right\} \tag{8.6}$$

and

$$\left.\begin{aligned} \left[y(t) - \frac{1}{2}\tan^{-1}y(t)\right]' + \left[y(t) - \frac{1}{2}\tan^{-1}y(t)\right] = \tanh y(t) - \tanh x(t), \\ y(0) = y(1), \end{aligned}\right\} \tag{8.7}$$

for all $t \in [0,1]$.

Here, k and g are the functions respectively on $[0,1] \times \mathbb{R}$ and $[0,1] \times \mathbb{R} \times \mathbb{R}$ defined by

$$k(t,x) = \frac{1}{2}\tan^{-1}x$$

and

$$g(t,x,y) = \tanh x - \tanh y.$$

It is easy to verify that the real-valued functions k and g are continuous on $[0,1] \times \mathbb{R}$ and $[0,1] \times \mathbb{R} \times \mathbb{R}$ respectively. We shall show that the nonlinearities k and g satisfy the hypotheses (C_1)–(C_4) and (B_1)–(B_3) respectively. Obviously k and g define the functions $k : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. So the hypotheses (C_1) and (B_1) are satisfied. Clearly the function k is bounded on $[0,1] \times \mathbb{R}$ with bound $M_k = \frac{\pi}{2}$ and so, the hypothesis (C_3) is satisfied. Next let $x, y \in \mathbb{R}$ be such that $x \geqslant y$. Then, we have

$$0 \leqslant k(t,x) - k(t,y) = \frac{1}{2}\tan^{-1}x - \frac{1}{2}\tan^{-1}y = \frac{1}{2} \cdot \frac{x-y}{1+\xi^2} = \varphi_f(x-y)$$

where, $\varphi_k(r) = \frac{1}{2} \cdot \frac{r}{1+\xi^2} < r$ for $x < \xi < y$ and that $\varphi_k \in \mathfrak{D}$. Thus, the function k satisfies the hypothesis (C_2).

Next, the function g is bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$ with bound $M_g = 2$. Again, $g(t,x,y)$ is nondecreasing in x and nonincreasing in y for each $t \in [0,1]$. Here, the Green's function $G(t,s)$ is given by (8.3) on $[0,1] \times [0,1]$. Therefore, we have

$$M_G = \sup_{(t,s) \in [0,1] \times [0,1]} G(t,s) \leqslant 3.$$

Then, we have

$$\varphi_k(r) \leqslant \frac{r}{1+\xi^2} < r$$

for each $r > 0$. So the condition (6.20) of Corollary 6.2 holds.

Next, we show that the function k satisfies the hypothesis (D$_1$) with $f \equiv 1$, that is, the map $x \mapsto [x - k(0,x)]$ is injection on $\mathbb{R}$. If x and y are any two real numbers, then the expression

$$x - k(0,x) = x - \frac{1}{2} \cdot \tan^{-1} x$$

implies that

$$\frac{d}{dx}[x - k(0,x)] = \frac{d}{dx}\left[x - \frac{1}{2} \cdot \tan^{-1} x\right] = 1 - \frac{1}{2} \cdot \frac{1}{1+x^2} > 0$$

for all $x \in \mathbb{R}$. Therefore, the map $x \mapsto [x - k(0,x)]$ is increasing on $\mathbb{R}$ and consequently the map $x \mapsto [x - k(0,x)]$ is injection on $\mathbb{R}$ in view of Remark 6.1.

Finally, the pair of functions (u,v) given by

$$u(t) = -\int_0^1 G(t,s)\, ds - 1$$

and

$$v(t) = 2\int_0^1 G(t,s)\, ds + 1$$

for all $t \in [0,1]$, are continuous and form the lower coupled solution of the coupled hybrid PBVPs (8.6) and (8.7) defined on $J = [0,1]$, where the Green's function G is defined by (8.3) on $[0,1] \times [0,1]$.

Thus, the functions k and g satisfy all the hypotheses (C$_1$)–(C$_4$), (B$_1$)–(B$_3$) and (D$_1$)–(D$_2$) of Corollary 6.2 and therefore, the coupled hybrid linearly perturbed PBVPs (8.6) and (8.7) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$\begin{aligned} x_0(t) &= -\int_0^1 G(t,s)\, ds - 1, \\ x_{n+1}(t) &= \frac{1}{2} \cdot \tan^{-1} x_n(t) + \int_0^1 G(t,s) \left[\, \tanh x_n(s) - \tanh y_n(s) \,\right] ds \end{aligned}$$

and

$$\begin{aligned} y_0(t) &= 2\int_0^1 G(t,s)\, ds + 1, \\ y_{n+1}(t) &= \frac{1}{2} \cdot \tan^{-1} y_n(t) + \int_0^1 G(t,s) \left[\, \tanh y_n(s) - \tanh x_n(s) \,\right] ds \end{aligned}$$

for all $t \in [0,1]$, converge monotonically to x^* and y^* respectively.

In the following we give some numerical examples of PBVPs of nonlinear second order ordinary differential equations in order to illustrate the abstract results of Section 7.

EXAMPLE 8.4. Given a closed and bounded interval $J = [0,1]$ of the real line $\mathbb{R}$, we consider the coupled hybrid QPBVPs of nonlinear second order ordinary differential equations,

$$\left.\begin{aligned} -\left(\frac{x(t)-k_1(t,x(t))}{f_1(t,x(t))}\right)'' + \left(\frac{x(t)-k_1(t,x(t))}{f_1(t,x(t))}\right) = g_1(t,x(t),y(t)),\ t \in J, \\ x(0) = x(1),\ x'(0) = x'(1), \end{aligned}\right\} \tag{8.8}$$

and

$$\left.\begin{aligned} -\left(\frac{y(t)-k_1(t,y(t))}{f_1(t,y(t))}\right)'' + \left(\frac{y(t)-k_1(t,y(t))}{f_1(t,y(t))}\right) = g_1(t,y(t),x(t)),\ t \in J, \\ y(0) = y(1),\ y'(0) = y'(1), \end{aligned}\right\} \tag{8.9}$$

where $f_1 : [0,1] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g_1 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$ are three functions defined as in Example 8.1.

Here, $\lambda = 1$ and $T = 1$, so the Green's function $\widetilde{G}_1(t,s)$ associated with the QPBVP (8.8) or (8.9) is given by

$$\widetilde{G}_1(t,s) = \begin{cases} \dfrac{1}{2(e-1)}\left[e^{(t-s)} + e^{(1-t+s)}\right], & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{1}{2(e-1)}\left[e^{(s-t)} + e^{(1-s+t)}\right], & 0 \leqslant t < s \leqslant 1. \end{cases} \tag{8.10}$$

Therefore, $\beta = \sup_{t,s \in J} \widetilde{G}_1(t,s) = \dfrac{e+1}{2(e-1)} \leqslant 2 < 3$. Now following the arguments similar to that given in Example 8.1, it is proved that the functions f, g and k satisfy all the conditions of Theorem 7.1 and a lower coupled solution (u,v) of the QPBVPs (8.8) and (8.9) is given by

$$u(t) = \int_0^1 \widetilde{G}_1(t,s)\,ds - 1$$

and

$$v(t) = \int_0^1 \widetilde{G}_1(t,s)\,ds + 1$$

for all $t \in [0,1]$. Therefore, the coupled hybrid QPBVPs (8.8) and (8.9) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_0(t) = \int_0^1 \widetilde{G}_1(t,s)\,ds - 1,$$

$$x_{n+1}(t) = k_1(t,x_n(t)) + \left[\, f_1(t,x_n(t))\,\right] \left(\int_0^1 \widetilde{G}_1(t,s) g_1(s,x_n(s),y_n(s))\,ds\right)$$

and

$$y_0(t) = \int_0^1 \widetilde{G}_1(t,s)\,ds + 1,$$

$$y_{n+1}(t) = k_1(t,y_n(t)) + \big[\, f_1(t,y_n(t)) \,\big] \left(\int_0^1 \widetilde{G}_1(t,s) g_1(s,y_n(s),x_n(s))\,ds \right)$$

for all $t \in [0,1]$, converge monotonically to x^* and y^* respectively.

EXAMPLE 8.5. Given a closed and bounded interval $J = [0,1]$ of the real line $\mathbb{R}$, we consider the coupled hybrid quadratic periodic boundary value problems (in short QPBVPs) of nonlinear second order ordinary differential equations,

$$\left.\begin{aligned} -\left(\frac{x(t)}{\tan^{-1}x(t)+3}\right)'' + \left(\frac{x(t)}{\tan^{-1}x(t)+3}\right) &= \frac{1}{15}\,\big[\tanh x(t) - \tanh y(t) + 3\,\big], \\ x(0) = x(1),\ x'(0) &= x'(1), \end{aligned}\right\} \tag{8.11}$$

and

$$\left.\begin{aligned} -\left(\frac{y(t)}{\tan^{-1}y(t)+3}\right)'' + \left(\frac{y(t)}{\tan^{-1}y(t)+3}\right) &= \frac{1}{15}\,\big[\tanh y(t) - \tanh x(t) + 3\,\big], \\ y(0) = y(1),\ y'(0) &= y'(1), \end{aligned}\right\} \tag{8.12}$$

for all $t \in [0,1]$.

Here, again the Green's function $\widetilde{G}_1(t,s)$ is given by (8.10). Next, following the arguments similar to that given in Example 8.2, it is shown that the functions involved in (8.11) and (8.12) satisfy all the conditions of Corollary 7.1 and a lower coupled solution (u,v) of the QPBVPs (8.11) and (8.12) is given by

$$u(t) = \frac{1}{15} \int_0^1 \widetilde{G}_1(t,s)\,ds$$

and

$$v(t) = \frac{5}{3} \int_0^1 \widetilde{G}_1(t,s)\,ds$$

for all $t \in [0,1]$. Therefore, the coupled hybrid QPBVPs (8.11) and (8.12) have a positive coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_0(t) = \frac{1}{15} \int_0^1 \widetilde{G}_1(t,s)\,ds,$$

$$x_{n+1}(t) = \frac{1}{15}\,\big[\tan^{-1} x_n(t) + 3\,\big] \left(\int_0^1 \widetilde{G}_1(t,s)\,\big[\tanh x_n(s) - \tanh y_n(s) + 3\,\big]\,ds \right)$$

and

$$y_0(t) = \frac{5}{3}\int_0^1 \widetilde{G}_1(t,s)\,ds,$$
$$y_{n+1}(t) = \frac{1}{15}\left[\,\tan^{-1} y_n(t) + 3\,\right]\left(\int_0^1 \widetilde{G}_1(t,s)\left[\,\tanh y_n(s) - \tanh x_n(s) + 3\,\right]\,ds\right)$$

for all $t \in [0,1]$, converge monotonically to x^* and y^* respectively.

EXAMPLE 8.6. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, we consider the coupled hybrid linear perturbed periodic boundary value problems (in short PBVPs) of second type of nonlinear second order ordinary differential equations,

$$\left.\begin{aligned} -\left[x(t) - \frac{1}{2}\cdot\tan^{-1} x(t)\right]'' + \left[x(t) - \frac{1}{2}\cdot\tan^{-1} x(t)\right] = \tanh x(t) - \tanh y(t),\\ x(0) = x(1)\,,\; x'(0) = x'(1) \end{aligned}\right\} \tag{8.13}$$

and

$$\left.\begin{aligned} -\left[y(t) - \frac{1}{2}\cdot\tan^{-1} y(t)\right]'' + \left[y(t) - \frac{1}{2}\cdot\tan^{-1} y(t)\right] = \tanh y(t) - \tanh x(t),\\ y(0) = y(1)\,,\; x'(0) = x'(1), \end{aligned}\right\} \tag{8.14}$$

for all $t \in [0,1]$.

Here, again the Green's function $\widetilde{G}_1(t,s)$ is given by (8.10). Next, following the arguments similar to that given in Example 8.3, it is shown that the functions involved in (8.13) and (8.14) satisfy all the conditions of Corollary 7.2 and a lower coupled solution (u,v) of the PBVPs (8.13) and (8.14) is given by

$$u(t) = -\int_0^1 \widetilde{G}_1(t,s)\,ds - 1$$

and

$$v(t) = 2\int_0^1 \widetilde{G}_1(t,s)\,ds + 1$$

for all $t \in [0,1]$. Therefore, the coupled hybrid PBVPs (8.13) and (8.14) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_0(t) = -\int_0^1 \widetilde{G}_1(t,s)\,ds - 1,$$
$$x_{n+1}(t) = \frac{1}{2}\cdot\tan^{-1} x_n(t) + \int_0^1 \widetilde{G}_1(t,s)\left[\,\tanh x_n(s) - \tanh y_n(s)\,\right]\,ds$$

and

$$y_0(t) = 2\int_0^1 \widetilde{G}_1(t,s)\,ds + 1,$$

$$y_{n+1}(t) = \frac{1}{2}\cdot \tan^{-1} y_n(t) + \int_0^1 \widetilde{G}_1(t,s)\,\big[\,\tanh y_n(s) - \tanh x_n(s)\,\big]\,ds$$

for all $t \in [0,1]$, converge monotonically to x^* and y^* respectively.

REMARK 8.1. Note that the functions $u,v \in C([0,1],\mathbb{R})$ in the lower coupled solution (u,v) for all coupled hybrid PBVPs given in the Examples 8.1 through 8.6 satisfy the inequality $u \preceq v$ on $J=[0,1]$. Therefore, in view of Remarks 6.3 and 7.2, the functions $x^*,y^* \in C([0,1],\mathbb{R})$ in the coupled solution (x^*,y^*) for each of the coupled hybrid PBVPs considered in above examples satisfy the inequality (6.23) on $[0,1]$. It is known that the nonlinear coupled differential and integral equations occur in many fields of applied sciences and they are studied in the literature for existence and other qualitative aspects of the coupled solutions. The numerical aspects such as approximation and algorithms for the coupled solutions for such equations may be obtained by using the abstract results developed in this paper.

9. Remarks and conclusion

The nonlinear analysis is a multistoryed building and the three basic fixed point theorems of Schauder, Banach and Tarski from topology, analysis and algebra respectively form the three basic pillars and the hybrid fixed point theory initiated by Krasnoselskii for the sum of two operators in a Banach space and by Dhage for the product of two operators in a Banach algebra form the forth basic pillar for this monument (see Dhage [12, 13, 24] and references therein). The hybrid fixed point theorems for the sum of two operators in an ordered Banach space as well as the product of two operators in an ordered Banach algebra is initiated in Dhage [9] and Dhage [10] respectively. It is now well recognized that the hybrid fixed point theory finds numerous applications to diverse areas of nonlinear analysis in mathematics and mathematical sciences. The topic of tupled operator equations and tupled solutions is of common interest and useful in the study of systems of simultaneous nonlinear tupled differential and tupled integral equations involving two or more unknown variables for proving different qualitative aspects of the tupled solutions. The classical coupled fixed point theorems (in short coupled FPTs) deals with the coupled operator equations involving two unknown variables and is a subject of interest for a long time and they are used for proving the existence of coupled solutions to the systems of a couple of simultaneous nonlinear equations with two unknown variables. But the subject of coupled hybrid fixed point theorems (in short HFPTs) and applications is relatively recent in the literature. The advantage of coupled HFPTs over the coupled FPTs is that we obtain additionally the algorithms along with the existence of the coupled solutions for nonlinear coupled hybrid equations. This is mainly because of the monotonic characterization of the coupled operators in the coupled equations on the domains of their definitions. Here, we assumed the mixed monotonicity of the coupled operators, however, we conjecture that other monotonic features of the coupled operators could also be considered for obtaining the coupled hybrid fixed point theorems and their applications. The coupled hybrid fixed point theorems (in short HFPTs) similar to Theorems 3.1 and 3.2 involving the sum and product of three coupled operators are also obtained in Dhage [19] with partial $\mathscr{D}$-Lipschitz condition is replaced by symmetric partial $\mathscr{D}$-Lipschitz condition. However, as for applications to nonlinear quadratic coupled hybrid integral equations of Volterra type, the partial $\mathscr{D}$-Lipschitz condition is employed in the discussion. Therefore, Theorems 3.1 and 3.2 of the present study are new interesting applicable special cases of HFPTs developed in Dhage [18] under a little stronger $\mathscr{D}$-Lipschitz condition in view of Remark 3.1. However, our approach is somewhat different from that of Dhage [17] while establishing the coupled and mixed coupled hybrid fixed point results in an ordered and partially ordered Banach algebra. The mixed coupled hybrid fixed point theorems of Section 4 are completely new to the literature and other similar mixed coupled hybrid fixed point theorems for different algebraic combinations of two or three operators and coupled operators may be obtained on the similar lines with appropriate modifications. We specifically mention that these mixed coupled hybrid fixed point theorems have some nice applications to other areas of mathematics including nonlinear coupled hybrid differential and integral equations. In the present study, our newly developed mixed coupled HFPT, Theorem 4.1 is applied to a very simple but new coupled hybrid periodic boundary problems of first order ordinary quadratic differential equations for proving the existence as well as approximation of the coupled solutions. However, the technique can also be extended and applied to other nonlinear second or higher order coupled differential equations under suitable boundary conditions for proving different aspects and developing algorithms for the coupled solutions. As mentioned in Dhage [19] that these operator theoretic coupled HFPTs have some advantages over the measure theoretic coupled hybrid fixed point theorems in which one needs to construct a handy tool for the partial measures of noncompactness suitable for the prevailing situation and which requires a sophisticated knowledge of advanced functional analysis. Furthermore, a clever mathematician makes the cleverer selection of a coupled hybrid fixed point theorem for dealing with the nonlinear coupled hybrid differential or integral equations and avoid the complexities in the arguments and calculations. This is a fundamental and preliminary work in the direction of nonlinear tupled hybrid operator equations and tupled solutions and many other questions such as attractivity, positivity, stability and monotonicity etc. form the further scope for the research work in the study of simultaneous nonlinear equations of several variables. Finally, while concluding this paper we mention that a few more applications of the abstract coupled and mixed coupled hybrid fixed point theorems of this paper will be considered in the forthcoming papers in future.

Acknowledgement. The author gratefully acknowledges the indebtedness and expresses sincere thanks to his TeX teacher, best friend and co-worker Prof. Sotiris K. Ntouyas (Greece) who taught him the LaTeX and related art-work for the first time in his research career as a mathematician without which this and many other research papers would not have been possible. He is also thankful to the Prof. J. N. Salunke (India) for carefully reading and pointing out some typos in the earlier version of this paper.

REFERENCES

[1] J. APPELL, *Measures of noncompactness, condensing operators and fixed points: An application-oriented survey*, Fixed Point Theory **6** (2005), 157–229.

[2] J. BANAS, K. GOEBEL, *Measures of Noncompactness in Banach Space*, in: Lecture Notes in Pure and Applied Mathematics, Vol. **60**, Dekker, New York, 1980.

[3] V. BERINDE, *Generalized coupled fixed point theorem for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal. **74** (2011), 7347–7355.

[4] T. G. BHASKAR, V. LAKSHMIKANTHAM, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis: TMA **65** (2006), 1379–1393.

[5] S. CARL, S. HEKKILÄ, *Fixed Point Theory in Ordered Sets and Applications*, Springer, 2011.

[6] S. S. CHANG, Y. H. MA, *Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations arising in dynamic programming*, J. Math. Anal. Appl. **160** (1991), 468–479.

[7] K. DEIMLING, *Nonlinear Functional Analysis*, Springer Verlag, 1985.

[8] B. C. DHAGE, *Some fixed point theorems in ordered Banach spaces and applications*, Math. Student **61** (1992), 81–88.

[9] B. C. DHAGE, *Fixed point theorems in ordered Banach algebras and applications*, Pan Amer. Math. J. **9** (4) (1999), 93–102.

[10] B. C. DHAGE, *Remarks on two fixed points theorems involving the sum and the products of two operators*, Computers and Math. Appl. **46** (2003), 1779–1785.

[11] B. C. DHAGE, *Nonlinear alternatives with applications to nonlinear perturbed differential equations*, Nonlinear Studies **13** (2006), 343–354.

[12] B. C. DHAGE, *Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations*, Differ. Equ. Appl. **2** (2010), 465–486.

[13] B. C. DHAGE, *Basic results in the theory of hybrid differential equations with mixed perturbation of second type*, Funct. Diff. Equ. **19** (1–2) (2012), 11–29.

[14] B. C. DHAGE, *Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations*, Differ. Equ. Appl. **5** (2013), 155–184.

[15] B. C. DHAGE, *Partially condensing mappings in ordered normed linear spaces and applications to functional integral equations*, Tamkang J. Math. **45** (2014), 397–426.

[16] B. C. DHAGE, *Nonlinear $\mathscr{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations*, Malaya J. Matematik **3** (2015), 62–86.

[17] B. C. DHAGE, *Some generalizations of a hybrid fixed point theorem in a partially ordered metric space and nonlinear functional integral equations*, Differ. Equ. Appl. **8** (2016), 77–97.

[18] B. C. DHAGE, *Approximating coupled solutions of coupled HPBVPs of nonlinear ordinary differential equations*, Kyungpook Math. J. **56** (1) (2016), 221–231.

[19] B. C. DHAGE, *Coupled hybrid fixed point theory in a partially ordered metric space and attractivity of nonlinear hybrid fractional integral equations*, J. Fixed Point Theory Appl. **19** (2017), 2541-2575.

[20] B. C. DHAGE, *Coupled hybrid fixed point theory involving the sum and product of three coupled operators in a partially ordered Banach algebra with applications*, J. Fixed Point Theory Appl. **19** (2017), 3231-3264.

[21] B. C. DHAGE, *Dhage iteration method for PBVPs of nonlinear first order hybrid integro-differential equations*, Int. J. Nonlinear Anal. Appl. **8** (2017), 95–112.

[22] B. C. DHAGE, *Dhage iteration method in the theory of ordinary nonlinear PBVPs of first order functional differential equations*, Commun. Optim. Theory **2017** (2017), Article ID 32, pp. 22.

[23] B. C. DHAGE, *A coupled hybrid fixed point theorem involving the sum of two coupled operators in a partially ordered Banach space with applications*, Differ. Equ. Appl. **9** (4) (2017), 453–477.

[24] B. C. DHAGE, *Some variants of two basic hybrid fixed point theorems of Krasnoselskii and Dhage with applications*, Nonlinear Studies **25** (3) (2018), 559–573.

[25] B. C. DHAGE, *A coupled hybrid fixed point theorem for sum of two mixed monotone coupled operators in a partially ordered Banach space with applications*, Tamkang J. Math. **50** (1) (2019), 1–36.

[26] B. C. Dhage, S. B. Dhage, *Approximating solutions of nonlinear first order ordinary differential equations*, GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications 13, GJMS Vol. **2**, No. 2, (2014), 25–35.

[27] B. C. Dhage, S. B. Dhage, *Coupled hybrid fixed point theorem in partially ordered metric spaces with applications*, Nonlinear Studies **21** (4) (2014), 675–656.

[28] B. C. Dhage, S. B. Dhage, *Approximating positive solutions of PBVPs of nonlinear first order ordinary hybrid differential equations*, Appl. Math. Lett. **46** (2015), 133–142.

[29] B. C. Dhage, S. Heikkilä, *On nonlinear boundary value problems with deviating arguments and discontinuous right hand side*, J. Appl. Math. Stochastic Anal. **6** (1) (1993), 83–92.

[30] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, London 1988.

[31] S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.

[32] J. J. Nieto, *Basic theory for nonresonance impulsive periodic problems of first order*, J. Math. Anal. Appl. **205** (1997), 423–433.

[33] J. J. Nieto, *Nonlinear second order periodic value problems with Carathéodory functions*, Appl. Anal. **34** (1989), 111–128.

[34] J. J. Nieto, R. Rodriguez-Lopez, *Existence and approximation of solution for nonlinear differential equations with periodic boundary conditions*, Comput. Math. Appl. **40** (2000), 435–442.

[35] J. J. Nieto and R. Rodriguez-Lopez, *Contractive mappings theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239.

[36] P. Nistri, V. V. Obukhovski, P. Zecca, *On the solvability of systems of inclusions involving noncompact operators*, Trans. Amer. Math. Soc. **342** (1994), no. 2, 543–562.

[37] M. H. Protter, H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N. J. 1967.

[38] Y. Sun, *A fixed point theorem for mixed monotone operator with applications*, J. Math. Anal. Appl. **156** (1991), 240–252.

[39] H. Yang, R. P. Agarwal, H. K. Nashine, Y. Liang, *Fixed point theorems in partially ordered Banach spaces with applications to nonlinear fractional evolution equations*, J. Fixed Point Theory Appl. **19** (2017), 1661–1678.

[40] E. Zeidler, *Nonlinear Functional Analysis and Its Applications: Part.I*, Springer-Verlag, New York, 1985.

(Received May 31, 2018)

Bapurao C. Dhage
"Kasubai", Gurukul Colony
Thodga Road, Ahmedpur-413 515
Dist: Latur, Maharashtra, India
e-mail: bcdhage@gmail.com