Differential Equations & Applications
Volume 11, Number 1 (2019), 129–142
doi:10.7153/dea-2019-11-04

L^2–CONCENTRATION FOR A COUPLED NONLINEAR SCHRÖDINGER SYSTEM

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(*Communicated by Shangbin Cui*)

Abstract. In this work we adapt Bourgain's ideas in [2] to a coupled system and we prove the L^2-concentration of blow-up solutions for two-coupled nonlinear Schrödinger equations at critical dimension.

1. Introduction

In this work we consider the following nonlinear Schrödinger system

$$\begin{cases} iu_t+\Delta u+(\alpha|u|^{2p}+\beta|u|^{p-1}|v|^{p+1})u=0, & (x,t)\in\mathbb{R}^n\times\mathbb{R}, \\ iv_t+\Delta v+(\alpha|v|^{2p}+\beta|v|^{p-1}|u|^{p+1})v=0, & \\ u(x,0)=u_0(x), \qquad v(x,0)=v_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where u and v are complex-valued functions and α and β are real constants and p is a constant not less than 1. This system is a model for propagation of polarized laser beams in birefringent Kerr medium in nonlinear optics (see, [1, 8, 9, 13] and the references therein for a complete discussion of the physics of the problem). The system (1.1) with $p=1$ is known as Kerr nonlinearity in the physical literature.

In the case $np<2$, it has been proven by Fanelli and Montefusco [7] that the Cauchy problem to (1.1) is globally well posed in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ and in the case $np=2$ they showed that there exists a constant c_0 such that the Cauchy problem (1.1) is globally well posed in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ with the condition $\|u_0\|_{L^2}+\|v_0\|_{L^2}<c_0$ and moreover they also showed that there exists a pair (u_0,v_0) such that $\|u_0\|_{L^2}+\|v_0\|_{L^2}=c_0$ and the corresponding solution blows up in a finite time (see also [6, 7, 10, 15]). On the other hand, the solution of the Cauchy problem (1.1) exists globally for other initial data, especially for a class of sufficiently small data (see [4, 7, 11]).

Well-posedness issues, the blow-up phenomenon and a sharp threshold of blow-up solution for the IVP (1.1) has been studied in the literature, see for example in [4, 6, 7, 10, 11, 15, 18] and references therein.

Mathematics subject classification (2010): 35A01, 35Q55.
Keywords and phrases: Nonlinear Schrödinger system, well-posedness, blow-up phenomenon, L^2-concentration.

Paper DEA-11-04

The Hamiltonian associated with the system (1.1) is given by

$$E(t) := E(u,v) = \frac{1}{2}\|(\nabla u, \nabla v)\|_2^2 - \frac{\alpha}{2(p+1)}\|(u,v)\|_{2p+2}^{2p+2} - \frac{\beta}{(p+1)}\|uv\|_{p+1}^{p+1} = E(0)$$

where

$$\|(f,g)\|_r = \left(\int_{\mathbb{R}^2} |f|^r + |g|^r\, dx\right)^{1/r} \quad \text{and} \quad \|f\|_r = \left(\int_{\mathbb{R}^2} |f|^r\, dx\right)^{1/r}.$$

In particular if $p = 1$, the Hamiltonian associated with (1.1) is of the form

$$E(t) = \frac{1}{2}\|(\nabla u, \nabla v)\|_2^2 - \frac{\alpha}{4}\|(u,v)\|_4^4 - \frac{\beta}{2}\|uv\|_2^2 = E(0). \tag{1.2}$$

In this paper, we analyze the L^2-concentration on small balls for two-coupled nonlinear Schrödinger equations (1.1) at critical dimension ($n = 2,\ p = 1$) with data in H^1 and L^2 i.e. to the following system:

$$\begin{cases} iu_t + \Delta u + (\alpha|u|^2 + \beta|v|^2)u = 0, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \\ iv_t + \Delta v + (\alpha|v|^2 + \beta|u|^2)v = 0, & \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.3}$$

when t approaches $T^* > 0$, where T^* is the maximal time of existence of a solution $(u(t), v(t))$ in $X \times X$ where $X = H^1$ or $X = L^2$. More precisely we will prove

THEOREM 1. *If $u(t), v(t) \in H^1,\ t \in [0, T^*)$ are solutions of the IVP (1.3) with $\alpha > 0,\ \beta > 0$ and $(u(t), v(t))$ blows up at finite time T^*, then there exists $x_0 \in \mathbb{R}^2$ such that*

$$\limsup_{t \nearrow T^*} \sup_{x_0 \in \mathbb{R}^2} \int_{|x - x_0| \lesssim (T^* - t)^{1/2}} |u(x,t)|^2\, dx \geqslant c, \tag{1.4}$$

and

$$\limsup_{t \nearrow T^*} \sup_{x_0 \in \mathbb{R}^2} \int_{|x - x_0| \lesssim (T^* - t)^{1/2}} |v(x,t)|^2\, dx \geqslant c, \tag{1.5}$$

where $c = c(\|u_0\|_2 + \|v_0\|_2) > 0$.

REMARK 1. i) There exists symmetry in the nonlinearity, i.e., when interchanging u with v in the system (1.1), it remains the same.

ii) Observe also that if $t_n \nearrow T^*$, then $u(t_n)$ and $v(t_n)$ do not have a strong limit in L^2. This result is proved by contradiction using the conservation of the Hamiltonian and the Gagliardo-Nirenberg inequality (see [5]).

Next we have also the same result with data in L^2.

THEOREM 2. *If $u(t), v(t) \in L^2,\ t \in [0, T^*)$ are solutions of the IVP (1.3) with $\alpha > 0,\ \beta > 0$ and $(u(t), v(t))$ blows up at finite time T^*, then there exists $x_0 \in \mathbb{R}^2$ such that (1.4) and (1.5) hold.*

Initially the rate of the L^2-norm concentration was obtained by Tsutsumi and Merle (see [14, 17]) for radially symmetric solutions to the critical nonlinear Schrödinger

$$iu_t + \Delta u + |u|^{2p}u = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \quad np = 2. \tag{1.6}$$

Recently Martel and Raphael [12], gave the first example of solution blowing up in finite time with a rate strictly above the pseudo-conformal one. Such solution concentrates K bubbles at a point.

Adapting ideas of Tsutsumi and Merle to a coupled system and considering radially symmetric blow-up solutions of (1.3), the rate of L^2-concentration was obtained recently by Z. Lü and Z. Liu in [19] with initial data in $H^1 \times H^1$ and the condition $0 < \beta < \alpha$. See also [20], for the L^2 concentration for radially symmetric blow-up solutions of two-coupled nonlinear Schrödinger equations with harmonic potential.

Adapting an argument of Bourgain [2] to a coupled system in the bidimensional case, we obtain the L^2-norm concentration to the system (1.3) without the use of radially symmetric solutions and without the condition $0 < \beta < \alpha$. In the following three sections we give in details the proofs of Theorems 1 and 2 by using this idea (in [2] there are some parts that are true but that are not proven, see for example the estimate of the term I_2 in Section 2).

We denote by C a general constant, that may vary from line to line. For $x, y \in \mathbb{R}$, $x \lesssim y$ means that there exist $C > 0$ such that $x \leqslant Cy$, $x \sim y$ means that $x \lesssim y$ and $y \lesssim x$.

2. Proof of Theorem 1

Proof of Theorem 1. Let $\psi := u(t)$, $\phi := v(t)$, $0 \leqslant t < T^*$ with t really close to T^*. In Section 4, (see (4.14)) we will prove the following inequality:

$$\lambda := \|\nabla(\psi,\phi)\|_2 \gtrsim \frac{1}{(T^* - t)^{1/2}} \gg 1, \quad 0 \leqslant t < T^*. \tag{2.1}$$

The L^2 conservation and the conservation of the Hamiltonian (1.2), imply

$$\|(\psi,\phi)\|_2 = \|(u_0,v_0)\|_2 = c_0, \quad \|(\psi,\phi)\|_4^4 \geqslant \frac{2}{\alpha+\beta}\lambda^2 - \frac{4}{\alpha+\beta}E(u_0,v_0) \gtrsim \lambda^2. \tag{2.2}$$

We define

$$\widehat{\psi_j}(\xi) := \widehat{\psi}(\xi)\chi_{\{2^j < |\xi| \leqslant 2^{j+1}\}} \quad \text{and} \quad S(\psi) := \left(\sum_{j\in\mathbb{Z}} |\psi_j|^2\right)^{1/2},$$

and similarly we define ϕ_j and $S(\phi)$. Using the Litlewood-Paley theorem we get

$$\|(\psi,\phi)\|_4 \sim \|(S(\psi),S(\phi))\|_4,$$

then by (2.2) we have

$$\|S(\psi)\|_4^4 + \|S(\phi)\|_4^4 \gtrsim \lambda^2.$$

In order to simplify the calculations, in the next we will consider only $\|S(\psi)\|_4^4$, the same estimates we obtain to the other term $\|S(\phi)\|_4^4$. Therefore we will consider that

$$\int_{\mathbb{R}^2} \sum_j |\psi_j|^2 \sum_{j\geqslant i} |\psi_i|^2 + \int_{\mathbb{R}^2} \sum_j |\psi_j|^2 \sum_{i\geqslant j} |\psi_i|^2 := I_1 + I_2 \gtrsim \lambda^2. \tag{2.3}$$

Following the notation in [2] we denote the diadyc numbers by $N = 2^j$, $N' = 2^i$, $\psi_N := \psi_j$, $\psi_{N'} := \psi_i$,

$$\sum_{j\geqslant i} |\psi_i|^2 := \sum_{N\geqslant N'} |\psi_{N'}|^2,$$

etc., we set

$$N_0 := \lambda k_0, \tag{2.4}$$

where k_0 is a constant which will be chosen after, and we consider

$$\rho_\psi = \sup_{N>N_0} \frac{\|\psi_N\|_\infty}{N}, \quad \rho_\phi = \sup_{N>N_0} \frac{\|\phi_N\|_\infty}{N} \tag{2.5}$$

for all diadyc number N, we have

$$\|\psi_N\|_\infty \lesssim N\|\psi_N\|_2, \quad \|\phi_N\|_\infty \lesssim N\|\phi_N\|_2 \tag{2.6}$$

then

$$\rho_\psi, \rho_\phi \lesssim 1. \tag{2.7}$$

In the I_1 and I_2 estimates, the goal is to try to get N_0 in all the estimates.

Estimate of I_2: We will consider two cases

I) If $N \leqslant N_0$.
In this case we have

$$I_2 = \sum_{N\leqslant N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N\leqslant N'\leqslant N_0} |\psi_{N'}|^2 + \sum_{N\leqslant N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N'>N_0} |\psi_{N'}|^2 := J_1 + J_2,$$

and using (2.6) we get

$$\begin{aligned} J_1 &\lesssim \sum_{N\leqslant N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N\leqslant N'\leqslant N_0} (N')^2 \|\psi_{N'}\|_2^2 \\ &\lesssim \|\psi\|_2^2 \sum_{N\leqslant N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N\leqslant N'\leqslant N_0} (N')^2 \\ &\lesssim \|\psi\|_2^2 \sum_{N\leqslant N_0} \int_{\mathbb{R}^2} |\psi_N|^2 N_0^2 \\ &\lesssim \|\psi\|_2^4 N_0^2. \end{aligned}$$

Using Cauchy-Schwartz inequality three times, Bernstein inequality in $\mathbb{R}^2$: $\|\psi_N\|_q \lesssim N^{2/p-2/q}\|\psi_N\|_p$, where $1 \leqslant p \leqslant q \leqslant \infty$ with $q = 4$, $p = 2$ (see Appendix in [16]) (2.5) and (2.6), give

$$\begin{aligned}
J_2 &\leqslant \sum_{N\leqslant N_0}\int_{\mathbb{R}^2}|\psi_N|^2\left(\sum_{N'>N_0}N'|\psi_{N'}|^4\right)^{1/2}\left(\sum_{N'\geqslant N_0}\frac{1}{N'}\right)^{1/2}\\
&\lesssim \frac{1}{N_0^{1/2}}\sum_{N\leqslant N_0}\|\psi_N\|_4^2\left(\sum_{N'>N_0}\int_{\mathbb{R}^2}N'|\psi_{N'}|^4\right)^{1/2}\\
&\lesssim \frac{1}{N_0^{1/2}}\sum_{N\leqslant N_0}N\|\psi_N\|_2^2\left(\sum_{N'>N_0}(N')^2\rho_\psi\|\psi_{N'}\|_2\int_{\mathbb{R}^2}N'|\psi_{N'}|^2\right)^{1/2}\\
&\lesssim \rho_\psi^{1/2}N_0^{1/2}\sum_{N\leqslant N_0}\|\psi_N\|_2^2\left(\sum_{N'>N_0}(N')^3\|\psi_{N'}\|_2^3\right)^{1/2}\\
&\lesssim \rho_\psi^{1/2}N_0^{1/2}\|\psi\|_2^2\left(\sum_{N'>N_0}(N')^2\|\psi_{N'}\|_2^2\right)^{1/4}\left(\sum_{N'>N_0}(N')^4\|\psi_{N'}\|_2^4\right)^{1/4}\\
&\lesssim \rho_\psi^{1/2}N_0^{1/2}\|\psi\|_2^2\lambda^{3/2}\\
&\lesssim N_0^{1/2}\|\psi\|_2^2\lambda^{3/2},
\end{aligned}$$

where in the last inequality was used the inequality (2.7).

II) If $N > N_0$.

Using Cauchy-Schwartz inequality two times and inequalities (2.5) and (2.6), we obtain

$$\begin{aligned}
I_2 &= \sum_{N>N_0}\int_{\mathbb{R}^2}|\psi_N|^2\sum_{N'\geqslant N}|\psi_{N'}|^2\\
&\lesssim \sum_{N>N_0}\int_{\mathbb{R}^2}\rho_\psi N^2\|\psi_N\|_2\sum_{N'\geqslant N}|\psi_{N'}|^2\\
&\lesssim \rho_\psi\sum_{N>N_0}\|\psi_N\|_2\sum_{N'\geqslant N}N'^2\int_{\mathbb{R}^2}|\psi_{N'}|^2\\
&\lesssim \rho_\psi\lambda^2\sum_{N>N_0}\|\psi_N\|_2\\
&\lesssim \rho_\psi\lambda^2\left(\sum_{N>N_0}N^2\|\psi_N\|_2^2\right)^{1/2}\left(\sum_{N>N_0}\frac{1}{N^2}\right)^{1/2}\\
&\lesssim \rho_\psi\lambda^2\frac{\lambda}{N_0}.
\end{aligned}$$

Estimate of I_1: We will consider two cases

I) If $N \leqslant N_0$.

The inequality (2.6) gives

$$\begin{aligned} I_1 &= \sum_{N\leqslant N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N'\leqslant N} |\psi_{N'}|^2 \\ &\lesssim \sum_{N\leqslant N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N'\leqslant N} (N')^2 \|\psi_{N'}\|_2^2 \\ &\lesssim N_0^2 \sum_{N\leqslant N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N'\leqslant N_0} \|\psi_{N'}\|_2^2 \\ &\lesssim N_0^2 \|\psi\|_2^4. \end{aligned} \tag{2.8}$$

II) If $N > N_0$.

We split I_1 in two terms

$$I_1 = \sum_{N>N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N_0\leqslant N'\leqslant N} |\psi_{N'}|^2 + \sum_{N>N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N'\leqslant N_0} |\psi_{N'}|^2 := L_1 + L_2,$$

the estimate for L_2 is similar with (2.8), thus

$$L_2 \lesssim N_0^2 \|\psi\|_2^4.$$

And in order to estimate L_1 we will use the inequality (2.5), it follows that

$$\begin{aligned} L_1 &= \sum_{N>N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N_0\leqslant N'\leqslant N} |\psi_{N'}|^2 \\ &\lesssim \sum_{N>N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N_0\leqslant N'\leqslant N} (N')^2 \rho_\psi^2 \\ &\lesssim \rho_\psi^2 \sum_{N>N_0} \int_{\mathbb{R}^2} |\psi_N|^2 \sum_{N_0\leqslant N'\leqslant N} (N')^2 \\ &\lesssim \rho_\psi^2 \sum_{N>N_0} \int_{\mathbb{R}^2} N^2 |\psi_N|^2 \\ &\lesssim \rho_\psi^2 \lambda^2. \end{aligned}$$

Now combining the inequality (2.3) with the above estimates of I_1 and I_2 and considering the similar estimates to the other terms obtained of $\|S(\phi)\|_4^4$, we get

$$\lambda^2 \leqslant C\Big((\|\psi\|_2^4 + \|\phi\|_2^4) N_0^2 + N_0^{1/2} \lambda^{3/2} (\|\psi\|_2^2 + \|\phi\|_2^2) + (\rho_\psi + \rho_\phi) \lambda^3 N_0^{-1} + (\rho_\psi^2 + \rho_\phi^2) \lambda^2\Big),$$

where $C > 0$ is a universal constant. Finally considering the L^2 conservation of u and v, let $c_0 = \|\psi\|_2 + \|\phi\|_2 = \|u_0\|_2 + \|v_0\|_2$, using that $N_0 = \lambda k_0$ (see (2.4)) and (2.7), we obtain

$$1 \leqslant C\Big(c_0^4 k_0^2 + k_0^{1/2} c_0^2 + (\rho_\psi + \rho_\phi) k_0^{-1} + \rho_\psi^2 + \rho_\phi^2\Big),$$

and taking k_0 such that

$$c_0^4 k_0^2 + k_0^{1/2} c_0^2 < \frac{1}{C},$$

we arrive to

$$0 < k_1 \leqslant (\rho_\psi + \rho_\phi) k_2 + \rho_\psi^2 + \rho_\phi^2, \tag{2.9}$$

where

$$k_1 = \frac{1 - C c_0^4 k_0^2 - C k_0^{1/2} c_0^2}{C} \quad \text{and} \quad k_2 = k_0^{-1}$$

note that (2.9) implies

$$\rho_\psi \geqslant k_3 = \sqrt{k_1 + \frac{k_2^2}{4}} - \frac{k_2}{2} > 0, \quad \text{and} \quad \rho_\phi \geqslant k_3 > 0.$$

The definition of ρ, implies that there exists $a \in \mathbb{R}^2$ and

$$N > N_0 \gtrsim \frac{1}{(T^* - t)^{1/2}}$$

such that $\dfrac{|\psi_N(a)|}{N} \geqslant \frac{k_3}{4}$ and there exists $b \in \mathbb{R}^2$ and $N > N_0 \gtrsim \dfrac{1}{(T^* - t)^{1/2}}$ such that $\dfrac{|\phi_N(b)|}{N} \geqslant \frac{k_3}{4}$. We consider the first case happens, the second case is similar

$$\frac{|\psi_N(a)|}{N} = \frac{1}{N} \int_{\mathbb{R}^2} \widehat{f}(-\xi) u \left(a - \frac{\xi}{N}\right) d\xi \geqslant \frac{k_3}{4},$$

where

$$f(x) = \chi_{\{1 < |x| \leqslant 2\}}(x),$$

and choosing $M > 0$ such that

$$\left(\int_{|\xi| \geqslant M} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} \leqslant \frac{k_3}{8 c_0}$$

we obtain

$$\frac{1}{N} \int_{|\xi| < M} \widehat{f}(-\xi) u(a - \frac{\xi}{N}) d\xi \geqslant \frac{k_3}{8},$$

using Cauchy-Schwartz inequality we concluded that

$$\frac{1}{N} \left\| u \left(a - \frac{\xi}{N}\right) \chi_{\{|\xi| < M\}} \right\|_2 \gtrsim k_3,$$

or equivalently

$$\int_{|a-x| < \frac{M}{N}} |u(x)|^2 dx \gtrsim k_3^2, \quad N \gtrsim \frac{1}{(T^* - t)^{1/2}}$$

and this inequality proves the Theorem 1. $\square$

3. Proof of Theorem 2

Consider the Cauchy problem associated to the linear parts of (1.1),

$$\begin{cases} iw_t + \Delta w = 0, & x \in \mathbb{R}^2,\ t \in \mathbb{R}, \\ w(0) = w_0. \end{cases} \tag{3.1}$$

The solution to (3.1) is given by $w(x,t) = U(t)w_0(x)$, where $\widehat{w(\cdot,t)}(\xi) = e^{-it|\xi|^2}\widehat{w_0}(\xi)$. In order to prove the Theorem 2 we need the following lemmas of Harmonic Analysis that have been proven by Bourgain in [2]

LEMMA 1. *Let $f \in L^2$, $\|f\|_2 = 1$. For given $\varepsilon > 0$, there are functions $(f_r)_{1 \leqslant r \leqslant R}$ such that*

$$R < R(\varepsilon)$$

each $\widehat{f_r}$ is supported by a square box

$$\tau_r \subset \mathbb{R}^2 \quad \text{of size} \quad A_r$$

and

$$|\widehat{f_r}| < \frac{1}{A_r}, \quad \|f_r\|_2 > \varepsilon'(\varepsilon),$$

$$\|U(t)f - \sum_r U(t)f_r\|_{L^4(dxdt)} < \varepsilon.$$

LEMMA 2. *Let $\operatorname{supp}\widehat{g} \subseteq \tau \subseteq \mathbb{R}^2$ where τ is a square of size A with center ξ_0 and $|\widehat{g}| < \frac{1}{A}$. Then, for given $\varepsilon > 0$, there is a collection $(Q_r)_{1 \leqslant r \leqslant R(\varepsilon)}$ of regions of the form*

$$Q_r = \left\{(x,t) \in \mathbb{R}^3; x + 2t\xi_0 \in I_r,\ t \in J_r\right\}, \tag{3.2}$$

where I_r is an interval in $\mathbb{R}^2$ of size $1/A$ and J_r an interval in $\mathbb{R}$ of size $1/A^2$ such that

$$\left(\int_{\mathbb{R}^3 \setminus \cup Q_r} |U(t)g|^4 dxdt\right)^{1/4} < \varepsilon. \tag{3.3}$$

Following the ideias of Borgain in [2], we consider $0 < T_0 < T^*$, $\psi = u(T_0)$ and $\phi = v(T_0)$, the integral equations give

$$u(t) = U(t - T_0)\psi + \mathbb{L}(u,v),$$

$$v(t) = U(t - T_0)\phi + \mathbb{L}(v,u),$$

where

$$\mathbb{L}(u,v) = -i\alpha \int_{T_0}^{t} U(t-s)(|u|^2u + |v|^2u)ds.$$

Let $0<\gamma\ll 1$ and we take $T_0<T_1<T*$ such that

$$\|u\|_{L^4_x L^4_{[T_0,T_1]}} := \|u\|_{L^4} = \gamma, \quad \text{and} \quad \|v\|_{L^4_x L^4_{[T_0,T_1]}} := \|v\|_{L^4[T_0,T_1]} = \gamma, \tag{3.4}$$

the Hölder inequality gives

$$\||v|^2 u\|_{L^{4/3}} \leqslant \||v|^2\|_{L^2}\|u\|_{L^4} = \|v\|_{L^4}^2\|u\|_{L^4},$$

and using that $(4,4)$ is a pair of exponents admissible, applying the Strichartz' inequality

$$\|u(t)-U(t-T_0)\psi\|_{L^4[T_0,T_1]} \lesssim \gamma^3, \tag{3.5}$$

triangle inequality, (3.4) and (3.5) implies

$$\|U(t-T_0)\psi\|_{L^4[T_0,T_1]} \sim \gamma.$$

Similarly we have

$$\|U(t-T_0)\phi\|_{L^4[T_0,T_1]} \sim \gamma, \quad \text{and} \quad \|v(t)-U(t-T_0)\phi\|_{L^4[T_0,T_1]} \lesssim \gamma^3.$$

By (3.4), (3.5), Hölder inequality and by the definition of γ it follows that

$$\begin{aligned}
\gamma^4 &= \int_{T_0}^{T_1}\int_{\mathbb{R}^2} u(t)\left[u(t)\overline{u(t)}^2\right] dxdt \\
&= \int_{T_0}^{T_1}\int_{\mathbb{R}^2} u(t)\left(U(t-T_0)\psi+\mathbb{L}(u,v)\right)\overline{\left(U(t-T_0)\psi+\mathbb{L}(u,v)\right)}^2 dxdt \\
&= \int_{T_0}^{T_1}\int_{\mathbb{R}^2} u(t)\left(U(t-T_0)\psi\right)\overline{\left(U(t-T_0)\psi\right)}^2 dxdt + O(\gamma^6).
\end{aligned} \tag{3.6}$$

From now on the rest of the proof is a consequence of the above lemmas. We will give some details: In fact using the Lemma 1 with $\varepsilon=\gamma^2$ and $f=U(-T_0)\psi$, then

$$U(t-T_0)\psi = U(t)f = \sum_r U(t)f_r + \mathscr{L},$$

where $\mathscr{L}=U(t)f-\sum_r U(t)f_r$ is such that $\|\mathscr{L}\|_{L^4(dxdt)}<\varepsilon=\gamma^2$. Similarly as in (3.6) we get

$$\gamma^4 = \int_{T_0}^{T_1}\int_{\mathbb{R}^2} u(t)\left(\sum_{r_1<R(\gamma^2)} U(t)f_{r_1}\right)\overline{\left(\sum_{r_2<R(\gamma^2)} U(t)f_{r_2}\right)\left(\sum_{r_3<R(\gamma^2)} U(t)f_{r_3}\right)} dxdt + O(\gamma^5). \tag{3.7}$$

The number of terms into above integral is smaller than $R(\gamma^2)^3$. Thus, there is $r_1,r_2,r_3<R(\gamma^2)$ such that

$$\int_{T_0}^{T_1}\int_{\mathbb{R}^2} u(t)\left(U(t)f_{r_1}\right)\overline{\left(U(t)f_{r_2}\right)}\overline{\left(U(t)f_{r_3}\right)} dxdt > \frac{\gamma^4}{R(\gamma^2)^3)} := \eta. \tag{3.8}$$

In the proof of Lemma 1, we have $\operatorname{supp}\widehat{f_r} \subset \tau_r \subset \operatorname{supp}\widehat{f} = \operatorname{supp}\widehat{\psi}$, where τ_r is a square of size A_r. Supossing that $A_{r_1} = \max\{A_{r_1}, A_{r_2}, A_{r_3}\}$, let be τ a square of size $A \sim A_{r_1}$, such that $\tau_{r_j} \subset \tau$, $j = 1,2,3$. Let P_τ the Fourier restriction wrt x-variable $\widehat{P_\tau \psi} = \chi_\tau \widehat{\psi}$, using Plancherel's formula and properties of the Fourier transform we have

$$\int_{T_0}^{T_1}\int_{\mathbb{R}^2} u(t)\,(\cdots)\,\overline{(\cdots)(\cdots)}\,dxdt = \int_{T_0}^{T_1}\int_{\mathbb{R}^2} P_\tau u(t)\,(\cdots)\,\overline{(\cdots)(\cdots)}\,dxdt. \tag{3.9}$$

Since $(4,4)$ is a pair of exponents admissible, applying the Strichartz' estimate we obtain

$$\|U(t) f_{r_j}\|_4 \leqslant \|f_{r_j}\|_2 \leqslant \|\psi\|_{L^2}, \quad j = 1,2,3.$$

Using (3.8), (3.9) and Hölder inequality we can show that

$$\int_{T_0}^{T_1}\int_{\mathbb{R}^2} |P_\tau u(t)\,(U(t) f_{r_1})\,|^2 dxdt \gtrsim \eta^2. \tag{3.10}$$

Now, applying Lemma 2 with $g = f_{r_1}$, $A = A_{r_1}$ and $\varepsilon = \eta$ there are $\{Q_s\}$, $1 \leqslant s \leqslant R(\varepsilon)$ be the regions (3.2). From (3.4), (3.3) and (3.10) it follows that

$$\iint_{Q\cap(\mathbb{R}^2\times[T_0,T_1])} |P_\tau u(t)\,(U(t) f_{r_1})\,|^2 dxdt \gtrsim \frac{\eta^2}{R(\eta)} = \eta_1. \tag{3.11}$$

In the same way as in [2] we can show that there exist $t \in [T_0, T_1]$ and an interval $I_1 = I - 2t\xi_0$ of size $1/A \lesssim \eta_1^{-1}(T-t)^{1/2}$ such that

$$\int_{I_1} |P_\tau u(t)|^2 dx \gtrsim \eta_1^2.$$

As $\widehat{P_\tau u} = \chi_\tau \widehat{u}$, then $P_\tau u = \mathscr{F}^{-1}(\chi_\tau) * u$, also since $\chi_\tau(\xi) = \chi_{\tau_0}(A^{-1/2}\xi)$, where τ_0 is a square of size 1, then

$$P_\tau u = \theta_A * u,$$

where $\theta_A(\xi) = \mathscr{F}^{-1}(\chi_\tau)(\xi) = A\mathscr{F}^{-1}(\chi_{\tau_0})(A^{1/2}\xi)$. It's not difficult to see that

$$\|\theta_A\|_{L^\infty} = \|\mathscr{F}^{-1}(\chi_\tau)\|_{L^\infty} \leqslant \|\chi_\tau\|_{L^1} = A, \tag{3.12}$$

and

$$\|\theta_A\|_{L^2} = \|\mathscr{F}^{-1}(\chi_\tau)\|_{L^2} = \|\chi_\tau\|_{L^2} = A^{1/2}. \tag{3.13}$$

Thus let $M = M(\eta_1, \|u_0\|_{L^2}) \gg 1$ very large, such that

$$\int_{|y|>\frac{M}{A^{1/2}}} |\theta_A|^2(y)dy = A\int_{|y|>M} |\mathscr{F}^{-1}(\chi_{\tau_0})|^2(y)dy \leqslant \frac{CA\eta_1^2}{16\|u_0\|_{L^2}^2}. \tag{3.14}$$

We have

$$\begin{aligned} P_\tau u(x) &= \int_{\mathbb{R}^2} \theta_A(y)u(x-y)dy = \int_{|y|\leqslant\frac{M}{A^{1/2}}} \theta_A(y)u(x-y)dy \\ &\quad + \int_{|y|>\frac{M}{A^{1/2}}} \theta_A(y)u(x-y)dy := L_1 + L_2, \end{aligned} \tag{3.15}$$

and from (3.14)

$$\begin{aligned} L_2 &= \int_{|y|>\frac{M}{A^{1/2}}} \theta_A(y)u(x-y)dy \\ &\leqslant \left(\int_{|y|>\frac{M}{A^{1/2}}} \theta_A^2(y)dy \right)^{1/2} \|u_0\|_{L^2} \leqslant \frac{A^{1/2}\eta_1 C^{1/2}}{4}, \end{aligned} \tag{3.16}$$

and in L_1 using Cauchy-Schwartz inequality and (3.12) we obtain

$$\begin{aligned} L_1 = \int_{|y|\leqslant\frac{M}{A^{1/2}}} \theta_A(y)u(x-y)dy &\lesssim \frac{M}{A^{1/2}} \left(\int_{|y|\leqslant\frac{M}{A^{1/2}}} \theta_A^2(y)|u(x-y)|^2 dy \right)^{1/2} \\ &\lesssim M \left(\int_{|y|\leqslant\frac{M}{A^{1/2}}} \theta_A(y)|u(x-y)|^2 dy \right)^{1/2}. \end{aligned} \tag{3.17}$$

Let $\theta_A \chi_{|y|\leqslant\frac{M}{A^{1/2}}} = \mathscr{J}$. Now combining (3.15), (3.16) and (3.17) we hold

$$\begin{aligned} C\eta_1^2 &\leqslant 2\int_{I_1} L_1^2 dx + 2\int_{I_1} L_2^2 dx \\ &\lesssim 2M^2 \int_{I_1} \mathscr{J} * |u|^2 dx + \frac{C\eta_1^2}{2}, \end{aligned} \tag{3.18}$$

and from this inequality we obtain

$$\frac{C\eta_1^2}{2} \leqslant 2M^2 \int_{\mathbb{R}^2} \chi_{I_1} \left(\mathscr{J} * |u|^2 \right) dx, \tag{3.19}$$

using Fubinni equality observe that

$$\int_{\mathbb{R}^2} f(g*h) = \int_{\mathbb{R}^2} h(f*\tilde{g}), \quad \tilde{g}(x) = g(-x),$$

therefore from (3.19) it follows that

$$\frac{C\eta_1^2}{4M^2} \leqslant \int_{\mathbb{R}^2} |u|^2 \left(\chi_{I_1} * \tilde{\mathscr{J}} \right) dx, \tag{3.20}$$

and

$$\mathrm{supp}\left(\chi_{I_1} * \tilde{\mathscr{J}} \right) \subseteq \mathrm{supp}\chi_{I_1} + \mathrm{supp}\tilde{\mathscr{J}},$$

as $\mathrm{supp}\chi_{I_1} \lesssim \frac{1}{A}$ and $\mathrm{supp}\tilde{\mathscr{J}} \lesssim \frac{1}{A}$ we complete the proof of theorem. □

4. Proof of the inequality (2.1)

This proof follows the same ideas in [5], for the sake of completeness we make all details here.

We will make the details to $\alpha=\beta$, the general case follows in similar way. We started by noting that

$$|(\nabla(|v|^2u,|u|^2v))|\lesssim|(u,v)|^2|\nabla(u,v)|, \tag{4.1}$$

where $|(u,v)|^2=|u|^2+|v|^2$, the Hölder's inequality gives

$$\|\nabla(|v|^2u,|u|^2v)\|_{4/3}\lesssim\||(u,v)|^2\|_2\|\nabla(u,v)\|_4\leqslant\|(u,v)\|_4^2\|\nabla(u,v)\|_4. \tag{4.2}$$

The Hamiltonian conservation:

$$E(u,v)=\frac{1}{2}\|\nabla(u,v)\|_2^2-\frac{\alpha}{4}\|(u,v)\|_4^4-\frac{\alpha}{2}\|uv\|_2^2=E(u_0,v_0),$$

implies that

$$\|(u,v)\|_4^2\lesssim(1+\|\nabla(u,v)\|_2). \tag{4.3}$$

Considering $0<t<\tau<T^*$, from (4.2), (4.3) and Hölder inequality, we obtain

$$\begin{aligned}\|\nabla(|v|^2u,|u|^2v)\|_{L^{4/3}((t,\tau);L^{4/3})}&\lesssim(1+\|\nabla(u,v)\|_{L^\infty((t,\tau);L^2)})\|\nabla(u,v)\|_{L^{4/3}((t,\tau);L^4)}\\&\lesssim(1+\|\nabla(u,v)\|_{L^\infty((t,\tau);L^2)})(\tau-t)^{1/2}\|\nabla(u,v)\|_{L^4((t,\tau);L^4)}.\end{aligned} \tag{4.4}$$

Deriving in the integral solution of the system (1.3) gives

$$\nabla u(t')=U(t'-t)\nabla u(t)-i\alpha\int_t^{t'}U(t'-s)\nabla(|u|^2u+|v|^2u)ds,$$

$$\nabla v(t')=U(t'-t)\nabla v(t)-i\alpha\int_t^{t'}U(t'-s)\nabla(|v|^2v+|u|^2v)ds.$$

As $u(t),v(t)\in H^1(\mathbb{R}^2)$, $t\in[0,T^*)$ and since $(4,4)$ is a pair of exponents admissible, applying the Strichartz' estimate (see [4], [5] or Theorem 2.3 in [16]) we have

$$\|\nabla u\|_{L^4((t,\tau);L^4)}\lesssim\|\nabla u(t)\|_2+\|\nabla(|u|^2u)\|_{L^{4/3}((t,\tau);L^{4/3})}+\|\nabla(|v|^2u)\|_{L^{4/3}((t,\tau);L^{4/3})}, \tag{4.5}$$

and

$$\|\nabla v\|_{L^4((t,\tau);L^4)}\lesssim\|\nabla v(t)\|_2+\|\nabla(|u|^2v)\|_{L^{4/3}((t,\tau);L^{4/3})}+\|\nabla(|v|^2v)\|_{L^{4/3}((t,\tau);L^{4/3})}, \tag{4.6}$$

from (4.5) and (4.6) we get

$$\begin{aligned}\|\nabla(u,v)\|_{L^4((t,\tau);L^4)}\lesssim&\|\nabla u(t)\|_2+\|\nabla v(t)\|_2+\|\nabla(|u|^2u)\|_{L^{4/3}((t,\tau);L^{4/3})}\\&+\|\nabla(|v|^2u)\|_{L^{4/3}((t,\tau);L^{4/3})}+\|\nabla(|u|^2v)\|_{L^{4/3}((t,\tau);L^{4/3})}\\&+\|\nabla(|v|^2v)\|_{L^{4/3}((t,\tau);L^{4/3})}.\end{aligned} \tag{4.7}$$

Applying the Strichartz' estimate with exponents admissible $(\infty,2)$ and adding with (4.7), we get

$$\begin{aligned}&\|\nabla(u,v)\|_{L^\infty((t,\tau);L^2)}+\|\nabla(u,v)\|_{L^4((t,\tau);L^4)}\\ \lesssim&\|\nabla(u,v)\|_2+\|\nabla(|u|^2u)\|_{L^{4/3}((t,\tau);L^{4/3})}+\|\nabla(|v|^2u)\|_{L^{4/3}((t,\tau);L^{4/3})}\\ &+\|\nabla(|u|^2v)\|_{L^{4/3}((t,\tau);L^{4/3})}+\|\nabla(|v|^2v)\|_{L^{4/3}((t,\tau);L^{4/3})},\end{aligned} \tag{4.8}$$

for all $0<t<\tau<T^*$.

Let us define the function

$$f_t(\tau)=1+\|\nabla(u,v)\|_{L^\infty((t,\tau);L^2)}+\|\nabla(u,v)\|_{L^4((t,\tau);L^4)}. \tag{4.9}$$

The inequality (4.4) implies

$$\|\nabla(|v|^2u,|u|^2v)\|_{L^{4/3}((t,\tau);L^{4/3})}\lesssim(\tau-t)^{1/2}f_t(\tau)^2. \tag{4.10}$$

Analogously as in (4.4) we can show that

$$\begin{aligned}\|\nabla(|u|^2u,|v|^2v)\|_{L^{4/3}((t,\tau);L^{4/3})}&\lesssim(\tau-t)^{1/2}(1+\|\nabla(u,v)\|_{L^\infty((t,\tau);L^2)})\|\nabla(u,v)\|_{L^4((t,\tau);L^4)}\\ &\lesssim(\tau-t)^{1/2}f_t(\tau)^2,\end{aligned} \tag{4.11}$$

combining (4.8)–(4.11), follow that

$$f_t(\tau)\leqslant C(1+\|\nabla(u,v)\|_2)+C(\tau-t)^{1/2}f_t(\tau)^2, \tag{4.12}$$

as f_t is a continuous and increasing function on $]0,T^*[$, if $T^*<\infty$ the blowup alternative said that $f_t(\tau)\to\infty$ when $\tau\nearrow T^*$ and from definition of f_t also we have that $f_t(\tau)\to 1+\|\nabla(u,v)(t)\|_{L^2}$ when $\tau\searrow t$, thus there exists $\tau_0\in]t,T^*[$ such that

$$f_t(\tau_0)=(C+1)(1+\|\nabla(u,v)(t)\|_2). \tag{4.13}$$

In consequence

$$1+\|\nabla(u,v)(t)\|_2\leqslant C(1+C)^2(T^*-t)^{1/2}(1+\|\nabla(u,v)(t)\|_2)^2,$$

and

$$1+\|\nabla(u,v)(t)\|_2\gtrsim\frac{1}{(T^*-t)^{1/2}},\quad 0\leqslant t<T^*.\quad\square \tag{4.14}$$

Acknowledgement. The first author thanks supports from CNPq under grant 304036/2014-5.

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(Received August 3, 2018)

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