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RAZUMIKHIN METHOD TO DELAY DIFFERENTIAL EQUATIONS WITH NON–INSTANTANEOUS IMPULSES

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Abstract. The stability for delay differential equations with non-instantaneous impulses is studied using Lyapunov like functions and the Razumikhin technique. In these differential equations we have impulses, which start abruptly at some points and their action continue on given finite intervals. Sufficient conditions are given and they use comparison results for nonlinear scalar non-instantaneous impulsive equation without any delay. Examples are given to illustrate our stability properties and the influence of non-instantaneous impulses on the behavior of the solution.

1. Introduction

Differential equations with impulses are well known in the literature and widely applied to model various processes and phenomena that are characterized by rapid changes in their state. In the literature there are two popular types of impulses. The first type, the so called *instantaneous impulses*, is used in the case when the duration of the changes is relatively short compared to the overall duration of the whole process (see, for example, [1], [5], [8], [10]–[12], the monographs [9], [13] and the cited references therein). The second type, the so called *non-instantaneous impulses*, is used in the case of impulsive action which start at arbitrary fixed points and remain active on finite time intervals and for recent work we refer the reader to [2], [15], [16], [18], [19]. An overview of the main properties of the presence of non-instantaneous impulses to ordinary differential equations and to fractional differential equations is given in the book [3] (note non-instantaneous impulsive differential equations are natural generalizations of impulsive differential equations). When one studies differential equations with delays and non-instantaneous impulses there are a number of technical and theoretical difficulties related to the phenomena of "beating" of the solutions, bifurcation, loss of the property of autonomy, etc.

In this paper stability of nonlinear non-instantaneous impulsive differential equations with delays is studied. A comparison principle and the Razumikhin method are applied. In particular nonlinear non-instantaneous impulsive differential equations without any delay are used as comparison equations. Several sufficient conditions for stability, uniform stability and asymptotic uniform stability are obtained. Some examples are given to illustrate the theory and the influence of non-instantaneous impulses on the behavior of the solutions is illustrated.

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2. Statement of the problem and auxiliary results

In this paper we will assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $s_0 = 0 < s_i \leqslant t_i < s_{i+1}$, $i = 1,2,\dots$, and $\lim_{k\to\infty} s_k = \infty$.

REMARK 1. The intervals $(s_k, t_k]$, $k = 1,2,\dots$ are called intervals of non - instantaneous impulses.

Let $t_0 \in \mathbb{R}_+$ be a given arbitrary point. Keeping in mind the meaning of t_0 as an initial time of the modeled rate of change of the process, we will assume everywhere in the paper that the initial time t_0 is not in any interval of non-instantaneous impulses. Without loss of generality we will assume that $0 \leqslant t_0 < s_1$.

Consider the system of *non-instantaneous impulsive delay differential equations* (NIDDE)

$$\begin{aligned} & x' = f(t, x_t) \text{ for } t \in (t_k, s_{k+1}],\ k = 0,1,2,\dots \\ & x(t) = \Phi_k(t, x(t), x(s_k - 0)) \quad \text{for } t \in (s_k, t_k],\ k = 1,2,\dots, \end{aligned} \tag{2.1}$$

with initial condition

$$x(t + t_0) = \phi(t) \ \text{ for } t \in [-r, 0], \tag{2.2}$$

where $x, x_0 \in \mathbb{R}^n$, $f : [0, s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1}] \times \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_i : [s_i, t_i] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(i = 1,2,3,\dots)$ $r > 0$ is a given number, $\phi : [-r, 0] \to \mathbb{R}^n$ $x_t = x(t+s), s \in [-r, 0]$.

REMARK 2. The functions $\Phi_k(t, x, y)$, $k = 1,2,\dots$, are called non-instantaneous impulsive functions.

REMARK 3. If $t_k = s_k$, $k = 1,2,\dots$ then the IVP for NIDDE (2.1), (2.2) reduces to an IVP for impulsive delay differential equations. In this case at any point of instantaneous impulse t_k the amount of jump of the solution $x(t)$ is given by $\Delta x|_{t=t_k} = x(t_k + 0) - x(t_k - 0) = \Phi_k(t_k, x(t_k + 0), x(t_k - 0)) - x(t_k - 0)$.

Let $E = \{\phi : [-h, 0] \to \mathbb{R}^n$ is continuous everywhere except at a finite number of points $\tau_j \in [-r, 0]$ in which it is continuous from the left$\}$. For any function $\phi \in E$ we define $||\phi||_0 = \sup_{s\in[-r,0]} ||\phi(s)||$ where $||.||$ is a norm in $\mathbb{R}^n$.

Let $J \subset \mathbb{R}_+$ be a given interval. We introduce the following classes of functions

$$\begin{aligned} PC^1(J) = \{ & u : J \to \mathbb{R}^n : \ u \in C^1[J \cap \big([0, s_1] \bigcup \cup_{k=1}^{\infty} (t_k, s_{k+1}]\big), \mathbb{R}^n] : \\ & u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty,\ \ u(s_k + 0) = \lim_{t \downarrow s_k} u(t) < \infty,\ \ k :\ s_k \in J\}, \\ PC(J) = \{ & u : J \to \mathbb{R}^n : \ u \in C[J\big([0, s_1] \bigcup \cup_{k=1}^{\infty} (t_k, s_{k+1}]\big), \mathbb{R}^n] : \\ & u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty,\ \ u(s_k + 0) = \lim_{t \downarrow s_k} u(t) < \infty,\ \ k :\ s_k \in J\}. \end{aligned}$$

Consider the corresponding IVP for the delay differential equation (DDE)

$$x'(t) = f(t, x_t) \quad \text{for } t \in \ [\tau, s_{p+1}] \quad \text{with } \ x(t - \tau) = \tilde{\phi}(t) \ \text{ for } t \in [-h, 0], \tag{2.3}$$

where $\tau \in [t_p, s_{p+1})$, p is given natural number.

We will say conditions (H1) is satisfied if

(H1) The function $f \in C([0,s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$ is such that for any initial function $\tilde{\phi} \in E$ the IVP for the system of DDE (2.3) has a solution $x(t;\tau,\tilde{\phi}) \in C^1([\tau, s_{p+1}], \mathbb{R}^n)$.

REMARK 4. Some conditions for the function $f \in C([\tau, s_{p+1}] \times \mathbb{R}^n, \mathbb{R}^n)$ satisfying condition (H1) can be found, for example, in the books [6], [7].

In connection with solvability of the impulsive equations in (2.1) we introduce the following condition

(H2) For any natural number k the functions $\Phi_k \in C([s_k,t_k] \times \mathbb{R}^{2n}, \mathbb{R}^n)$, $k = 1,2,\ldots$, are such that for any $v \in \mathbb{R}^n$ and any $t \in [s_k,t_k]$ there exists exactly one function $u_k : [s_k,t_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $u_k = u_k(t,v)$, such that $u_k(t,v) = \Phi_k(t, u_k(t,v), v)$.

EXAMPLE 1. We will give some examples of functions satisfying condition (H2). Let $n = 1$.

Let $\Phi_k(t,u,v) = a_k(t)u + b_k(t)v$, $k = 1,2,\ldots$ where $a_k, b_k : [s_k,t_k] \to \mathbb{R}$. Then if $a_k(t) \neq 1$ then $u_k(t,v) = \frac{b_k(t)v}{1-a_k(t)}$. If there exists at least one point $t^* \in [s_k,t_k]$ such that $a_k(t^*) = 1$ then for $b_k(t) \neq 0$ the function u_k does not exist.

Let $\Phi_k(t,u,v) = u^2 e^v$ for a natural number k. Then since the equation $u = u^2 e^v$ has two solutions $u = 0$ and $u = e^{-v}$, condition (H2) is not satisfied.

Let $\Phi_k(t,u,v) = g_k(t,v)$ where $g_k \in C([s_k,t_k] \times \mathbb{R}, \mathbb{R})$. Then condition (H2) is satisfied with $u_k(t,v) \equiv g_k(t,v)$.

REMARK 5. In the case the impulsive conditions in (2.1) are of the type

$$x(t) = \Psi_k(t, x(s_k - 0)) \quad \text{for } t \in (s_k,t_k],\ k = 1,2,\ldots \tag{2.4}$$

then condition (H2) is satisfied.

In connection with the existence of the zero solution of (2.1) we introduce the condition:

(H3) The functions $\Phi_k(t,x,y)$, $k = 1,2,\ldots$, $f(t,x)$ are such that $\Phi_k(t,0,0) = 0$ for $t \in [s_k,t_k]$ and $f(t,0) = 0$ for $[0,s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1}]$.

REMARK 6. If conditions (H2) and (H3) are satisfied then the functions $u_k(t,v)$ defined in condition (H2) satisfy $u_k(t,0) = 0$ for $t \in [s_k,t_k]$.

We now give a brief description of the solution of IVP for NIDDE (2.1). The solution $x(t;t_0,x_0)$ of (2.1) is given by

$$x(t;t_0,\phi) = \begin{cases} X_k(t), & \text{for } t \in (t_k, s_{k+1}],\ k = 0,1,2,\ldots, \\ \Phi_k(t, x(t;t_0,\phi), X_k(s_k - 0)), & \text{for } t \in (s_k,t_k] \ \ k = 1,2,\ldots \end{cases} \tag{2.5}$$

where

— on the interval $[t_0-r,t_0]$ the solution satisfies the initial condition (2.2);

— on the interval $[t_0,s_1]$ the solution coincides with $X_0(t)$ which is the solution of IVP for DDE (2.3) for $\tau=t_0$, $p=0$ and $\tilde{\phi}=\phi$;

— on the interval $(s_1,t_1]$ the solution $x(t;t_0,\phi)$ satisfies the equation

$$x(t;t_0,\phi)=\Phi_1(t,x(t;t_0,\phi),X_0(s_1-0));$$

— on the interval $(t_1,s_2]$ the solution coincides with $X_1(t)$ which is the solution of IVP for DDE (2.3) for $\tau=t_1$, $p=1$ and

$$\tilde{\phi}(t)=\begin{cases}\Phi_1(t_1,x(t_1;t_0,\phi),X_0(s_1-0)) & t=0\\ x(t-t_1;t_0,\phi) & t\in[-r,0);\end{cases}$$

— on the interval $(s_2,t_2]$ the solution $x(t;t_0,\phi)$ satisfies the equation

$$x(t;t_0,x_0)=\Phi_2(t,x(t;t_0,\phi),X_1(s_2-0));$$

— on the interval $(t_2,s_3]$ the solution coincides with $X_2(t)$ which is the solution of IVP for DDE (2.3) for $\tau=t_2$, $p=2$ and

$$\tilde{\phi}(t)=\begin{cases}\Phi_2(t_2,x(t_2;t_0,\phi),X_1(s_2-0)) & t=0\\ x(t-t_2;t_0,\phi) & t\in[-r,0);\end{cases}$$

and so on.

REMARK 7. The conditions (H1) and (H2) guarantee the existence of a solution $x(t;t_0,x_0)$ from $PC^1([t_0,\infty),R^n)$ of NIDDE (2.1) for any $t_0\in[0,s_1)\bigcup\cup_{k=1}^{\infty}[t_k,s_{k+1})$ and any initial function $\phi\in E$. If additionally the condition (H3) is satisfied then the IVP for NIDDE (2.1), (2.2) with zero initial function $\phi\equiv 0$ has a zero solution.

REMARK 8. According to the above description any solution $x(t;t_0,\phi)$ of the IVP for NIDDE (2.1) is from the class $PC^1([t_0,b))$, $b\leqslant\infty$.

REMARK 9. Note that the IVP for NIDDE (2.1), (2.2) with nonzero initial function ϕ could have a zero solution for $t\geqslant\tau\geqslant t_0$.

EXAMPLE 2. Consider the IVP for the scalar NIDDE

$$\begin{aligned}&x'(t)=x(t-2\pi)\text{ for } t\in(2k\pi,2(k+1)\pi],\ k=0,1,2,\dots,\\ &x(t)=tx(2(k+1)\pi-0))\text{ for } \ t\in(2(k+1)\pi,2(k+2)\pi],\ k=0,1,2,\dots,\\ &x(s)=\sin(s),\quad s\in[-2\pi,0].\end{aligned}\tag{2.6}$$

The solution of the IVP for the scalar NIDDE (2.6) is given by

$$x(t)=\begin{cases}\sin(t) & t\in[-2\pi,0]\\ 1-\cos(t) & t\in[0,2\pi]\\ 0 & t>2\pi\end{cases}$$

Therefore, in spite of the initial function being nonzero, the solution is zero for $t > 2\pi$.

EXAMPLE 3. Consider the IVP for the scalar NIDDE

$$\begin{aligned} &x'(t) = x(t-1) \text{ for } t \in (t_k, s_{k+1}],\ k = 0,1,2,\dots,\\ &x(t) = \Phi_k(t, x(s_k - 0)) \text{ for } \ t \in (s_k, t_k],\ k = 1,2,\dots,\\ &x(t) = 1, \quad t \in [-1,0] \end{aligned} \tag{2.7}$$

where $x \in \mathbb{R}$, $t_0 = 0$.

Case 1. Let $t_k = k$, $k = 0,1,2,\dots$ and $s_k = k - 0.5$, $k = 1,2,3,\dots$, i.e. the length of the intervals $[t_k, s_{k+1}]$ is $0.5 < 1 = r$.

Case 1.1. Let $\Phi_k(t,y) = 0,5ty$, $k = 1,2,3,\dots$. The graph of the solution of NIDDE (2.7) is given on Figure 1 and it seems to be an increasing function not approaching 0.

Case 1.2. Let $\Phi_k(t,y) = \frac{y}{t}$, $k = 1,2,3,\dots$. The graph of the solution of NIDDE (2.7) is given on Figure 2 and it seems to be a function not approaching 0.

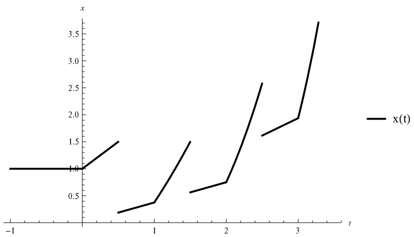

Figure 1. Example 3. Case 1.1. Graph of solution $x(t)$.

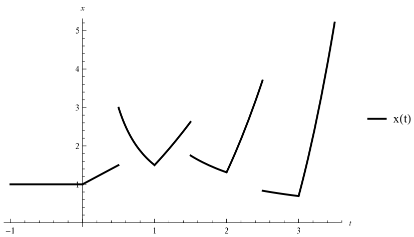

Figure 2. Example 3. Case 1.2. Graph of solution $x(t)$.

Case 2. Let $t_k = 3k - 1$, $k = 1,2,\dots$ and $s_k = 3k$, $k = 1,2,3,\dots$, i.e. the length of the intervals $[t_k, s_{k+1}]$ is $1 = r$.

Case 2.1. Let $\Phi_k(t,y) = 0,5ty$, $k = 1,2,3,\dots$. The graph of solution of NIDDE (2.7) is given on Figure 3 and it seems to be a function not approaching 0.

Case 2.2. Let $\Phi_k(t,y) = \frac{y}{t}$, $k = 1,2,3,\dots$. The graph of solution of NIDDE (2.7) is given on Figure 4 and it seems to be a decreasing function approaching 0.

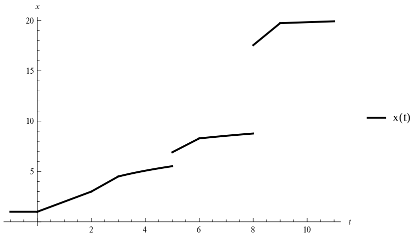

Figure 3. Example 3. Case 2.1. Graph of solution $x(t)$.

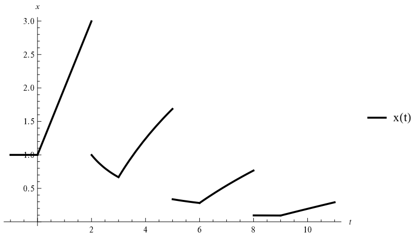

Figure 4. Example 3. Case 2.2. Graph of solution $x(t)$.

Therefore, the behavior of the solution depends significantly on both the impulsive functions and the length of the intervals of the acting impulses.

We give a definition for different types of stability of the zero solution of NIDDE (2.1).

DEFINITION 1. The zero solution of the system NIDDE (2.1) (with $\phi \equiv 0$) is said to be

- *stable* if for every $\varepsilon > 0$ and every initial time $t_0 \in [0, s_0) \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1})$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ such that for any initial function $\psi \in E:\ ||\psi||_0 < \delta$ the inequality $||x(t; t_0, \psi)|| < \varepsilon$ holds for $t \geqslant t_0$;
- *uniformly stable* if for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ such that for any initial point $t_0 \in [0, s_0) \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1})$ and any initial function $\psi \in E$ with $||\psi||_0 < \delta$ the inequality $||x(t; t_0, \psi)|| < \varepsilon$ holds for $t \geqslant t_0$;
- *uniformly attractive* if there exists $\beta > 0$ such that for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that for any initial point $t_0 \in [0, s_0) \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1})$ and any initial function $\psi \in E$ with $||\psi||_0 < \beta$ the inequality $||x(t; t_0, \psi)|| < \varepsilon$ holds for $t \geqslant t_0 + T$;
- *uniformly asymptotically stable* if it is is uniformly stable and uniformly attractive.

Define sets:

$$\mathscr{K} = \{\sigma \in C(\mathbb{R}_+, \mathbb{R}_+), \text{strictly increasing and } \sigma(0) = 0\},$$
$$S(A) = \{x \in \mathbb{R}^n :\ ||x|| \leqslant A\}, \quad A > 0..$$

We now introduce the class Λ of Lyapunov-like functions which will be used to investigate the stability of the zero solution of the system NIDDE (2.1).

DEFINITION 2. Let $\alpha < \beta \leqslant \infty$ be given numbers and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ be a given set. We will say that the function $V(t,x) : [\alpha - r, \beta) \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda([\alpha - r, \beta), \Delta)$ if

1. The function $V(t,x)$ is continuous on $[\alpha, \beta)/\{s_k\} \times \Delta$ and it is locally Lipschitz with respect to its second argument;
2. For each $s_k \in (\alpha, \beta)$ and $x \in \Delta$ there exist finite limits

$$V(s_k, x) = V(s_k - 0, x) = \lim_{t \uparrow s_k} V(t,x), \quad \text{and } V(s_k + 0, x) = \lim_{t \downarrow s_k} V(t,x).$$

We will define the *generalized Dini derivative* of the function $V(t,x) \in \Lambda([t_0 - r, T), E)$ along trajectories of solutions of IVP for the system NIDDE (2.1) by:

$$D^+_{(2.1)} V(t, \psi(0), \psi) = \limsup_{h \to 0^+} \frac{1}{h} \Big\{ V(t, \psi(0)) - V(t-h, \psi(0) - hf(t, \psi)) \Big\} \quad \text{for } t \in [t_0, T) \bigcap \cup_{k=0}^{\infty} (t_k, s_{k+1}) \tag{2.8}$$

where $\psi \in E$.

We will use some comparison results for NIDDE (2.1) by applying Lyapunov functions. As a comparison equation we will consider the following scalar non- instantaneous impulsive differential equation (NIDE)

$$\begin{aligned} & u' = g(t,u), \quad \text{for } t \in \cup_{k=0}^{\infty}(t_k, s_{k+1}], \\ & u(t) = \Xi_k(t, u(s_k - 0)) \text{ for } t \in (s_k, t_k], \quad k = 1, 2, \dots, \\ & u(t_0) = u_0 \end{aligned} \tag{2.9}$$

and the IVP for its corresponding scalar ordinary differential equation (ODE)

$$u' = g(t,u), \quad t \in [\tau, s_{k+1}], \quad u(\tau) = \tilde{u}_0 \tag{2.10}$$

where $u, \tilde{u}_0 \in \mathbb{R}$, $\tau \in [t_k, s_{k+1})$.

We will use maximal solutions of the IVP for ODE (2.10).

We will use the following conditions:

(H4) The function $g \in C([0,s_1] \bigcup \cup_{k=1}^{\infty}[t_k, s_{k+1}] \times \mathbb{R}, \mathbb{R}_+)$, $g(t,0) = 0$ and for any initial point $(\tau, \tilde{u}_0)$: $\tau \in [t_k, s_{k+1})$, $k = 0, 1, 2 \dots$, and $\tilde{u}_0 \in \mathbb{R}$ the IVP for ODE (2.10) has a maximal solution $\tilde{u}(t; \tau, \tilde{u}_0) \in C^1([\tau, s_{k+1}], \mathbb{R})$.

(H5) For all natural numbers k the functions $\Xi_k \in C([s_k, t_k] \times \mathbb{R}, \mathbb{R})$ are such that $\Xi_k(t, 0) = 0$ for $t \in [s_k, t_k]$ and $\Xi_k(t, u) \leqslant \Xi_k(t, v)$ for $u \leqslant v$, $t \in [s_k, t_k]$.

DEFINITION 3. Let p be a natural number and $T \in (t_p, s_p]$ be a given number. The function $u^*(t)$ will be called a maximal solution of the IVP for NIDE (2.9) on the interval $[t_0, T]$ if

— it is a solution of the IVP for NIDE (2.9) on $[t_0, T]$;

— for any $k = 0, 1, 2, \dots, p-1$ and any solution $u(t) \in C^1([t_k, s_{k+1}], \mathbb{R})$ of IVP for ODE (2.10) with $\tau = t_k$, $\tilde{u}_0 = u^*(t_k)$ the inequalities

$$u^*(t) \geqslant u(t) \ \text{ for } t \in [t_k, s_{k+1}] \cap [t_0, T]$$

and for any $k = 1, 2, \dots, p-1$

$$\Xi_k(t, u^*(s_k - 0)) \geqslant \Xi_k(t, u(s_k)) \ \text{ for } t \in (s_k, t_k]$$

hold.

The existence of a maximal solution of (2.9) is established in [4], Lemma 1.

LEMMA 1. (Lemma 1 [4]) *Let:*
1. Condition (H4) be satisfied on $\cup_{k=0}^{p}(t_k, s_{k+1}]$ *where* $p \leqslant \infty$ *is a positive integer.*
2. Condition (H5) be satisfied for all $k = 1, 2, \dots, p-1$.
Then there exists a maximal solution of (2.9) on the interval $[t_0, s_{p+1}]$.

Applying the scalar NIDE (2.9) as a comparison equation we will obtain a comparison result for NIDDE (2.1).

LEMMA 2. *Suppose:*

1. *The function* $x^*(t) = x(t;t_0,\phi) \in PC^1([t_0,\Theta],\Delta)$ *is a solution of the NIDDE (2.1) where* $\Delta \subset \mathbb{R}^n$, $\Theta \in (t_p, s_{p+1}]$ *is a given number,* p *is a natural number.*

2. *For all* $k = 1,\dots,p-1$ *condition (H5) is satisfied.*

3. *Condition (H4) is satisfied on the interval* $[t_p,\Theta]\bigcup\cup_{k=0}^{p-1}[t_k,s_{k+1}]$.

4. *The function* $V \in \Lambda([t_0-r,\Theta],\Delta)$ *and*

   (i) for any $t \in (t_p,\Theta]\bigcup\cup_{k=0}^{p-1}(t_k,s_{k+1}]$ *such that* $V(t,\psi(0)) \geqslant \sup_{s\in[-r,0]} V(t+s,\psi(s))$ *the inequality*

$$D^+_{(2.1)}V(t,\psi(0),\psi) \leqslant g(t,V(t,\psi(0)))$$

   holds where $\psi(s) = x^*(t+s), s \in [-r,0]$;

   (ii) for any number $k = 1,2,\dots,p-1$ *the inequality*

$$V(t,\Phi_k(t,x^*(t),x^*(s_k-0))) \leqslant \Xi_k(t,V(s_k-0,x^*(s_k-0))) \text{ for } t \in (s_k,t_k]$$

   holds.

Then the inequality $\max_{s\in[-r,0]} V(t_0+s,\phi(s)) \leqslant u_0$ *implies* $V(t,x^*(t)) \leqslant u^*(t)$ *for* $t \in [t_0,\Theta]$ *where* $u^*(t)$ *is the maximal solution of IVP for NIDE (2.9) on* $[t_0,\Theta]$.

Proof. Note that according to Lemma 1 from conditions 2 and 3 of Lemma 2 there exists a maximal solution $u^*(t) = u^*(t;t_0,u_0)$ of IVP for NIDE (2.9) on $[t_0,\Theta]$.

Let $\max_{s\in[-r,0]} V(t_0+s,\phi(s)) \leqslant u_0$. We use induction to prove Lemma 2. Denote $m(t) = V(t,x^*(t))$ for $t \in [t_0-r,\Theta]$.

Let $t \in (t_p,\Theta]\bigcup\cup_{k=0}^{p-1}(t_k,s_k]$. Denote $\psi(s) = x^*(t+s), s \in [-r,0]$, and use condition 4(i) and we obtain

$$\begin{aligned} D_+m(t) &= \lim_{h\to 0+} \frac{V(t,\psi(0)) - V(t-h,\psi(-h))}{h} \\ &= \lim_{h\to 0+} \frac{V(t,\psi(0)) - V(t-h,\psi(0)-hf(t,\psi))}{h} \\ &\quad + \lim_{h\to 0+} \frac{V(t-h,\psi(0)-hf(t,\psi)) - V(t-h,\psi(-h))}{h} \\ &\leqslant D^+_{(2.1)}V(t,\psi(0),\psi) + L\limsup_{h\to 0+}\frac{1}{h}||x^*(t)-x^*(t-h)-hf(t,x^*_t)|| \\ &= D^+_{(2.1)}V(t,\psi(0),\psi) \end{aligned} \tag{2.11}$$

where $L > 0$ is the Lipschitz constant of the Lyapunov function $V(t,x)$.

Case 1. Let $t \in [t_0,s_1]$. Then the function $m(t) \in C([t_0,s_1],\mathbb{R}_+)$ and the inequality $m(t_0) = V(t_0,\phi(0)) \leqslant \sup_{s\in[-r,0]} V(t_0+s,\phi(s)) \leqslant u_0$ holds. We will prove that

$$m(t) \leqslant u^*(t), \quad t \in [t_0,s_1] \tag{2.12}$$

Assume (2.12) is not true. Therefore, there exists $t^* \in (t_0, s_1)$ such that

$$m(t) \leqslant u^*(t),\ \ t \in [t_0, t^*),\ \ m(t^*) = u^*(t^*)\ \ \text{and}\ \ m(t) > u^*(t),\ t \in (t^*, t^* + \delta] \quad (2.13)$$

where $\delta > 0$ is a small enough number. Therefore applying condition (H4) we get that for every $t \in [t^*, t^* + \delta]$ the inequality $m(t) = V(t, x^*(t)) > u^*(t) \geqslant u^*(\xi_t) = \max_{s\in[-r,0]} V(t+s, x^*(t+s))$ holds where $\xi_t \in [t-r, t]$. According to condition 4(i) and inequality (2.11) the inequality $D_+ m(t) \leqslant g(t, m(t))$ holds. The function $u^*(t)$ is a maximal solution of IVP for ODE (2.10) with $\tau = t^*$, $\tilde{u}_0 = m(t^*)$ on $[t^*, t^* + \delta]$. According to the comparison result for ordinary differential equations (see, for example [14]) we obtain $m(t) \leqslant u^*(t)$ on $[t^*, t^* + \delta]$. The contradiction proves inequality (2.12) and the claim of Lemma 2 on $[t_0, s_1]$.

Case 2. Let $t \in (s_1, t_1]$. Then $x^*(t) = \Phi_1(t, x^*(t), x^*(s_1 - 0))$. From conditions 4(ii) for $k = 1$, condition (H4) and Case 1 we get $V(t, x^*(t)) = V(t, \Phi_1(t, x^*(t), x^*(s_1 - 0))) \leqslant \Xi_1(t, V(s_1 - 0, x^*(s_1 - 0)) \leqslant \Xi_1(t, u^*(s_1 - 0)) = u^*(t)$, $t \in (s_1, t_1]$. The claim of Lemma 2 is true on $[s_1, t_1]$.

Case 3. Let $t \in (t_1, s_2]$ (if $\Theta < s_2$ then we consider the interval $(t_1, \Theta]$). Define the function $m(t) = V(t, x^*(t))$ for $t \in (t_1, s_2]$ and $m(t_1) = V(t_1, \Phi_1(t_1, x^*(t_1), x^*(s_1 - 0)))$. The function $m(t) \in C([t_1, s_2], \mathbb{R}^n)$, satisfies the inequality (2.11). Similarly as in Case 1 we prove the validity of inequality (2.12) on $[t_1, s_2]$.

Also an agument similar to that in Case 2 yields $m(t) = V(t, x^*(t)) \leqslant u^*(t)$, $t \in (s_2, t_2]$.

Continue this process and an induction argument proves the claim in Lemma 2 is true for $t \in [t_0, \Theta]$. □

REMARK 10. The result of Lemma 2 is also true on the half line, i.e. $\Theta = \infty$.

REMARK 11. The conditions 4(i) and 4(ii) of Lemma 2 are satisfied only for the particular given solution $x^*(t)$ and the condition 4(i) is satisfied only at some particular points t from the studied interval.

COROLLARY 1. *Let the condition 1 of Lemma 2 be satisfied and the function* $V \in \Lambda([t_0 - r, \Theta], \Delta)$ *be such that*

(i) for any $t \in (t_p, \Theta] \bigcup \cup_{k=0}^{p-1} (t_k, s_{k+1}]$ *such that*

$$V(t, \psi(0)) \geqslant \sup_{s\in[-r,0]} V(t+s, \psi(s))$$

the inequality

$$D^+_{(2.1)} V(t, \psi(0), \psi) \leqslant 0$$

holds where $\psi(s) = x^*(t+s), s \in [-r, 0]$*;*

(ii) for any number $k = 1, 2, \dots, p-1$ *the inequality* $V(t, \Phi_k(t, x^*(t), x^*(s_k - 0))) \leqslant V(s_k - 0, x^*(s_k - 0)))$ *for* $t \in (s_k, t_k]$ *holds.*

Then the inequality $V(t, x^*(t)) \leqslant \max_{s\in[-r,0]} V(t_0 + s, \phi(s))$ *holds on* $[t_0, T]$.

Note in the case $T=\infty$ the interval is $[t_0,\infty)$.

LEMMA 3. (Comparison result for NIDDE, negative Dini derivative) *Assume the following conditions are satisfied:*

1. *Condition 1 of Lemma 2 is satisfied.*
2. *The function $V\in\Lambda([t_0-r,\Theta],\Delta)$ and*

   (i) for any $t\in(t_p,\Theta]\bigcup\cup_{k=0}^{p-1}(t_k,s_{k+1}]$ such that $V(t,\psi(0))\geqslant\sup_{s\in[-r,0]}V(t+s,\psi(s))-\int_{-r}^{0}c(||\psi(\sigma)||)d\sigma$ the inequality

   $$D^+_{(2.1)}V(t,\psi(0),\psi)\leqslant -c(||\psi(0)||)$$

   holds where $\psi(s)=x^*(t+s), s\in[-r,0]$, $c\in\mathscr{K}$;

   (ii) for any number $k=1,2,\dots,p-1$ the inequality

   $$V(t,\Phi_k(t,x^*(t),x^*(s_k-0)))\leqslant V(s_k-0,x^*(s_k-0))\ \textit{for}\ t\in(s_k,t_k]$$

   holds.

Then for $t\in[t_0,\Theta]$ the inequality

$$V(t,x^*(t))\leqslant\begin{cases}V(t_0,x_0)-\int_{t_0}^{t}c(||x^*(s)||)ds & \textit{for}\ t\in[t_0,s_1],\\ V(t_0,x_0)-\sum_{i=0}^{k-1}\int_{t_i}^{s_{i+1}}c(||x^*(s)||)ds & \textit{for}\ t\in(s_k,t_k]\cap[t_0,T],\ k\geqslant 1\\ V(t_0,x_0)-\Big(\sum_{i=0}^{k-1}\int_{t_i}^{s_{i+1}}c(||x^*(s)||)ds & \\ \qquad\qquad +\int_{t_k}^{t}c(||x^*(s)||)ds\Big) & \textit{for}\ t\in(t_k,s_{k+1}]\cap[t_0,\Theta],\ k\geqslant 1\end{cases}$$

holds.

Proof. We use induction to prove Lemma 3.

Case 1. Let $t\in[t_0,s_1]$. We will assume that $\Theta>s_1$. Let $\max_{s\in[-r,0]}V(t_0+s,\phi(s))=B$.

Define

$$m(t)\leqslant\begin{cases}V(t_0,\phi(0)) & \text{for } t\in[t_0-r,t_0],\\ V(t,x^*(t))+\int_{t_0}^{t}c(||x^*(s)||)ds & \text{for } t\in[t_0,s_1].\end{cases}$$

Then $m\in C([t_0-r,s_1],\mathbb{R}_+)$ and $m(t)\leqslant B$ for $t\in[t_0-r,t_0]$.

Let $t \in [t_0, s_1]$ be a fixed point and denote $\psi(s) = x^*(t+s)$, $s \in [-r, 0]$, and use condition 2(i) of Lemma 3 and we obtain

$$\begin{aligned} D_+m(t) = &\lim_{h\to 0+} \frac{V(t,\psi(0)) - V(t-h,\psi(-h))}{h} \\ &+ \lim_{h\to 0+} \frac{1}{h}\int_{t-h}^{t} \Big(c(||x^*(s)||) - c(||x^*(t)||)\Big)ds + c(||\psi(0)||) \\ = &\lim_{h\to 0+} \frac{V(t,\psi(0)) - V(t-h,\psi(0) - hf(t,\psi))}{h} \\ &+ \lim_{h\to 0+} \frac{V(t-h,\psi(0)-hf(t,\psi)) - V(t-h,\psi(-h))}{h} \\ &+ \lim_{h\to 0+} \frac{1}{h}\int_{-h}^{0} \Big(c(||\psi(\sigma)||) - c(||\psi(0)||)\Big)d\sigma + c(||\psi(0)||) \\ \leqslant &D^+_{(2.1)}V(t,\psi(0),\psi) + L\limsup_{h\to 0+}\frac{1}{h}||x^*(t) - x^*(t-h) - hf(t,x^*_t)|| \\ &+ c(||\psi(0)||) \\ = &D^+_{(2.1)}V(t,\psi(0),\psi) + c(||\psi(0)||), \quad t \in [t_0, s_1] \end{aligned} \tag{2.14}$$

where $L > 0$ is the Lipschitz constant of the Lyapunov function $V(t,x)$.

We now prove that

$$m(t) \leqslant B, \quad t \in [t_0, s_1] \tag{2.15}$$

Assume (2.15) is not true. Therefore, there exists $t^* \in (t_0, s_1)$ such that

$$m(t) \leqslant B, \ t \in [t_0 - r, t^*), \ m(t^*) = B \ \text{ and } \ m(t) > B, \ t \in (t^*, t^* + \delta] \tag{2.16}$$

where $\delta > 0$ is a small enough number.

From $m \in C([t_0 - r, s_1], \mathbb{R}_+)$ and (2.16) it follows that the function $m(t)$ is non-decreasing on $t \in [t^*, t^* + \delta]$ and inequality $m(t) \geqslant \max_{s\in[-r,0]} m(t+s)$ holds. Then $m(t) = V(t, x^*(t)) + \int_{t_0}^{t} c(||x^*(\xi)||)d\xi \geqslant V(t+s, x^*(t+s)) + \int_{t_0}^{t+s} c(||x^*(\xi)||)d\xi$, $s \in [-r, 0]$ holds, i.e. $V(t,x^*(t)) \geqslant V(t+s, x^*(t+s)) - \int_{t+s}^{t} c(||x^*(\xi)||)d\xi \geqslant V(t+s, x^*(t+s)) - \int_{t-r}^{t} c(||x^*(\xi)||)d\xi = V(t+s, x^*(t+s)) - \int_{-r}^{0} c(||x^*(t+\sigma)||)d\sigma$, $s \in [-r, 0]$, holds. Thus from condition 2(i) and (2.14) it follows that $D_+m(t) \leqslant 0$ on $[t^*, t^* + \delta]$. Therefore, $B < m(t^* + \delta) \leqslant m(t^*) = B$. The contradiction proves inequality (2.15) and the claim of Lemma 3 on $[t_0, s_1]$.

Case 2. Let $t \in (s_1, t_1]$. Then $x^*(t) = \Phi_1(t, x^*(t), x^*(s_1 - 0))$ and from condition 2(ii) of Lemma 3 we get $V(t, x^*(t)) = V(t, \Phi_k(t, x^*(t), x^*(s_1 - 0))) \leqslant V(s_1 - 0, x^*(s_1 - 0)) \leqslant V(t_0, x_0) - \int_{t_0}^{s_1} c(||x^*(s)||)ds$.

Case 3. Let $t \in [t_1, s_2]$. We will assume $\Theta > s_2$.

Define

$$m_1(t) \leqslant \begin{cases} V(t_1, x^*(t_1)) & \text{for } t \in [t_1 - r, t_1], \\ V(t, x^*(t)) + \int_{t_1}^{t} c(||x^*(s)||)ds & \text{for } t \in [t_1, s_2]. \end{cases}$$

The function $m_1 \in C([t_1 - r, s_2], \mathbb{R}_+)$.

From condition 2(i) of Lemma 3 the inequality (2.14) is satisfied on $[t_1,s_2]$ where the function $m(t)$ is replaced by $m_1(t)$ and $\psi(s)=x^*(t+s), s\in[-r,0]$ for any fixed point $t\in[t_1,s_2]$.

We will prove that

$$m_1(t)\leqslant V(t_1,\psi(0)),\quad t\in[t_1,s_2] \tag{2.17}$$

Assume (2.17) is not true. Therefore, there exists $t^*\in(t_1,s_2)$ such that

$$\begin{aligned} &m_1(t)\leqslant V(t_1,\psi(0)),\ \ t\in[t_1-r,t^*),\ \ m_1(t^*)=V(t_1,\psi(0))\\ &\text{and}\ \ m_1(t)>V(t_1,\psi(0)),\ t\in(t^*,t^*+\delta] \end{aligned} \tag{2.18}$$

where $\delta>0$ is a small enough number.

From $m_1\in C([t_1-r,s_2],\mathbb{R}_+)$ and (2.16) it follows that the function $m_1(t)$ is nondecreasing on $t\in[t^*,t^*+\delta]$ and inequality $m_1(t)\geqslant\max_{s\in[-r,0]}m_1(t+s)$ holds. Then as in Case 1 the inequality $V(tx^*(t))\geqslant\sup_{s\in[-r,0]}V(t+s,x^*(t+s))-\int_{-r}^{0}c(||x^*(t+\xi)||)d\xi$ holds for all $t\in[t^*,t^*+\delta]$. Thus from condition 2(i) and (2.14) with $m=m_1$ it follows that $D_+m_1(t^)\leqslant 0$ on $[t^*,t^*+\delta]$ and we obtain a contradiction which proves inequality (2.17). From inequality (2.17) it follows that $V(t,x^*(t))\leqslant V(t_1,x^*(t_1))-\int_{t_1}^{t}c(||x^*(s)||)ds\leqslant V(t_0,x_0)-\int_{t_0}^{s_1}c(||x^*(s)||)ds-\int_{t_1}^{t}c(||x^*(s)||)ds$, i.e. the claim of Lemma 3 is true on $[t_1,s_2]$.

Continue this process and an induction argument proves the claim in Lemma 3 is true for $t\in[t_0,\Theta]$. □

3. Main results

We study the stability properties of the zero solution of nonlinear differential equations with non-instantaneous impulses.

THEOREM 1. (Stability) *Suppose:*

1. *Conditions (H1)-(H5) are satisfied.*
2. *There exists a function $V\in\Lambda([-r,\infty),\mathbb{R}^n)$ such that*

   (i) *for any point $t\in[0,s_1]\bigcup\cup_{k=1}^{\infty}(t_k,s_{k+1}]$ and any function $\psi\in E$ such that $V(t,\psi(0))\geqslant\sup_{s\in[-r,0]}V(t+s,\psi(s))$ the inequality*

   $$D^+_{(2.1)}V(t,\psi(0),\psi)\leqslant g(t,V(t,\psi(0)))$$

   holds;

   (ii) *for any point $v\in\mathbb{R}^n$ and any $t\in[s_k,t_k]$, $k=1,2,3,\dots$ the inequality*

   $$V(t,\Psi_k(t,u_k(t,v),v))\leqslant\Xi_k(t,V(s_k-0,v))$$

   holds where the functions $u_k(t,v)$ are defined in condition (H2);

(iii) the inequalities $a(||x||) \leqslant V(t,x) \leqslant b(||x||),\ t \geqslant -r,\ \ x \in \mathbb{R}^n$, *hold where* $a, b \in \mathcal{K}$.

3. *The zero solution of the scalar NIDE (2.9) is stable.*

Then the zero solution of NIDDE (2.1) with zero initial function is stable.

Proof. Let $\varepsilon > 0$ and $t_0 \in [0,s_1) \bigcup \cup_{k=1}^{\infty} [s_k,t_{k+1})$ be arbitrary given numbers. Without loss of generality assume $t_0 \in [0,s_1)$.

From condition 3 there exists a number $\delta_1 = \delta_1(t_0,\varepsilon) > 0$ such that the inequality $|u_0| < \delta_1$ provided $|u(t;t_0,u_0)| < a(\varepsilon),\ t \geqslant t_0$.

Since $b \in \mathcal{K}$ there exists $\delta_2 > 0$ such that $b(\delta_2) < \delta_1$.

Let $\phi \in E$ with $||\phi||_0 < \delta$ with $\delta = \min\{\delta_1, \delta_2, \varepsilon\}$.

Let $u_0^* = \sup_{s \in [-r,0]} V(t_0+s,\phi(s))$. Then for all $s \in [-r,0]$ the inequality $V(t_0 + s,\phi(s)) \leqslant b(||\phi(s)||) < b(\delta_2) < \delta_1$ holds. Therefore, the maximal solution $u^*(t) = u(t;t_0,u_0^*)$ of the IVP for NIDE (2.9) satisfies $|u^*(t)| < a(\varepsilon),\ t \geqslant t_0$. Consider the solution $x^*(t) = x(t;t_0,\phi) \in PC^1([t_0,\infty),\mathbb{R}^n)$ of NIDDE (2.1). From conditions 2(i) and 2(ii) it follows that the conditions 4(i) and 4(ii) of Lemma 2 are satisfied. Therefore, according to Lemma 2 for $\Theta = \infty$ and $\Delta = \mathbb{R}^n$ we get $V(t,x^*(t)) \leqslant u^*(t),\ t \geqslant t_0$.

From condition 2(*iii*) we obtain

$$a(||x^*(t)||) \leqslant V(t,x^*(t)) \leqslant u^*(t) < a(\varepsilon),\ t \geqslant t_0$$

so the result follows. □

THEOREM 2. (Uniform stability) *Suppose:*

1. *Conditions (H1)-(H5) are satisfied.*

2. *There exists a function* $V \in \Lambda([-r,\infty),S(\lambda))$ *such that*

   (i) for any $t \in [0,s_1] \bigcup \cup_{k=1}^{\infty} (t_k,s_{k+1}]$ *and any function* $\psi \in E:\ ||\psi||_0 \leqslant \lambda$ *such that* $V(t,\psi(0)) \geqslant \sup_{s \in [-r,0]} V(t+s,\psi(s))$ *the inequality*

   $$D^{+}_{(2.1)} V(t,\psi(0),\psi) \leqslant g(t,V(t,\psi(0)))$$

   holds;

   (ii) for any natural number k, any point $t \in [s_k,t_k]$ *and any* $v \in S(\lambda)$ *such that* $\Psi_k(t,u_k(t,v),v) \in S(\lambda)$ *the inequality*

   $$V(t,\Psi_k(t,u_k(t,v),v)) \leqslant \Xi_k(t,V(s_k-0,v))$$

   holds where the functions $u_k(t,v)$ *are defined in condition (H2);*

   (iii) the inequalities $a(||x||) \leqslant V(t,x) \leqslant b(||x||),\ t \geqslant -r,\ \ x \in S(\lambda)$, *hold where* $a, b \in \mathcal{K}$.

3. *The zero solution of the scalar NIDE (2.9) is uniformly stable.*

Then the zero solution of NIDDE (2.1) with zero initial function is uniformly stable.

Proof. Let $\varepsilon \in (0,\lambda]$ and $t_0 \in [0,s_1)\bigcup\cup_{k=1}^{\infty}[t_k,s_{k+1})$ be arbitrary given numbers. Without loss of generality assume $t_0 \in [0,s_1)$.

From condition 3 there exists a number $\delta_1 = \delta_1(\varepsilon) > 0$ such that the inequality $|u_0| < \delta_1$ provided $|u(t;t_0,u_0)| < a(\varepsilon)$, $t \geqslant t_0$.

From $b \in \mathscr{K}$ there exists a number $\delta_2 = \delta_2(\varepsilon) > 0$ such that $b(\delta_2) < \delta_1$. Let $\delta = \min\{\varepsilon,\delta_2\}$. Choose the initial function $\phi \in E$ such that $||\phi||_0 < \delta$. Then from condition 2(iii) for all $s \in [-r,0]$ the inequalities $||\phi(s)|| < \delta \leqslant \varepsilon \leqslant \lambda$ and $V(t_0 + s,\phi(s)) \leqslant b(||\phi(s)||) \leqslant b(\delta_2) < \delta_1$ hold. Let $u_0^* = \sup_{s\in[-r,0]} V(t_0+s,\phi(s)) < \delta_1$. Therefore, the maximal solution $u^*(t) = u(t;t_0,u_0^*)$ of the IVP for NIDE (2.9) satisfies $|u^*(t)| < a(\varepsilon)$, $t \geqslant t_0$. Consider any solution $x^*(t) = x(t;t_0,\phi) \in PC^1([t_0,\infty),\mathbb{R}^n)$ of NIDDE (2.1). We now prove that

$$||x^*(t)|| < \varepsilon, \quad t \geqslant t_0. \tag{3.1}$$

For $t = t_0$ we get $||x^*(t_0)|| = ||\phi(t_0)|| \leqslant ||\phi||_0 < \delta \leqslant \varepsilon$.

Assume inequality (3.1) is not true and let $t^* > t_0$ be such that

$$||x^*(t)|| < \varepsilon \text{ for } t \in [t_0,t^*) \quad \text{and } ||x^*(t^*)|| = \varepsilon.$$

Case 1. Let there exists an integer $p:\ t^* \in (t_p,s_{p+1}]$. Therefore, $x^*(t) \in S(\lambda)$ on $[t_0,t^*]$ and conditions 4(i) and 4(ii) of Lemma 2 are satisfied on $[t_0,t^*]$. According to Lemma 2 applied to the solution $x^*(t)$ for $\Theta = t^*$ and $\Delta = S(\lambda)$ we get $V(t,x^*(t)) \leqslant u^*(t)$ on $[t_0,t^*]$. Then from condition 2 (iii) and the choice of the initial function ϕ we obtain $a(\varepsilon) = a(||x^*(t^*)||) \leqslant V(t^*,x^*(t^*)) \leqslant u^*(t^*) < a(\varepsilon)$. The contradiction proves (3.1) and therefore, the zero solution of NIDDE (2.1) with zero initial function is uniformly stable.

Case 2. Let there exists a natural number $p:\ t^* \in (s_p,t_p]$. Then

$$||x^*(t^*)|| = ||\Phi_p(t^*,x^*(t^*),x^*(s_p-0))|| = \varepsilon \leqslant \lambda.$$

As in Case 1 applying Lemma 2 with $\Theta = s_p$ we get

$$V(t,x^*(t)) \leqslant u^*(t),\ \ t \in [t_0,s_p]. \tag{3.2}$$

Then applying inequality (3.2), conditions 2(ii) and (H5) we get

$$\begin{aligned} a(\varepsilon) &= a(||\Phi_p(t^*,x^*(t^*),x^*(s_p-0))||) \leqslant V(t^*,\Phi_p(t^*,x^*(t^*),x^*(s_p-0))) \\ &\leqslant \Xi_p(t^*,V(s_p-0,x^*(s_p-0)) \leqslant \Xi_p(t^*,u(s_p-0)) = u(t^*) < a(\varepsilon). \end{aligned} \tag{3.3}$$

The contradiction proves (3.1) in Case 2 and therefore, the zero solution of NIDDE (2.1) with zero initial function is uniformly stable. □

COROLLARY 2. (Uniform stability) *Let conditions (H1), (H2), (H3) be satisfied and there exists a function $V \in \Lambda([-r,\infty),S(\lambda))$ such that* *(i)* *for any* $t \in [0,s_1]\bigcup\cup_{k=1}^{\infty}(t_k,s_{k+1}]$ *and any function* $\psi \in E: \ ||\psi||_0 \leqslant \lambda$ *such that* $V(t,\psi(0)) \geqslant \sup_{s\in[-r,0]} V(t+s,\psi(s))$ *the inequality*

$$D^+_{(2.1)}V(t,\psi(0),\psi) \leqslant 0$$

holds;

(ii) *for any natural number* k*, any* $t \in [s_k,t_k]$ *and point* $v \in S(\lambda): \ \Psi_k(t,u_k(t,v),v) \in S(\lambda)$ *the inequality*

$$V(t,\Psi_k(t,u_k(t,v),v)) \leqslant V(s_k - 0,v)$$

holds where the functions $u_k(t,v)$ *are defined in condition (H2);*

(iii) *the inequalities* $a(||x||) \leqslant V(t,x) \leqslant b(||x||)$, $t \geqslant -r$, $x \in S(\lambda)$, *hold where* $a,b \in \mathscr{K}$.

Then the zero solution of NIDDE (2.1) with zero initial function is uniformly stable.

EXAMPLE 4. Consider the scalar IVP for NIDDE

$$\begin{aligned} &x' = -\sin(t)(2x(t) - x(t-\pi)), \quad \text{for } t \in (t_k,s_{k+1}],\ k = 0,1,2,\dots,\\ &x(t) = \Phi_k(t,x(t),x(s+k-0)) \text{ for } t \in (s_k,t_k],\ k = 1,2,\dots,\\ &x(t_0+s) = \phi(s),\ s \in [-\pi,0]. \end{aligned} \tag{3.4}$$

where $t_k = 2k\pi$, $s_k = (2k-1)\pi$, $\Phi_k : \ [(2k-1)\pi,2k\pi]\times\mathbb{R}^2 \to \mathbb{R}$, $i = 1,2,\dots$, $\Phi_k(t,0,0) = 0$.

The scalar IVP for NIDDE with $\phi(s) \equiv 0$ has a zero solution.

Let $V(t,x) = x^2$.

Consider the delay differential equation without impulses $x' = -\sin(t)(2x(t) - x(t-\pi))$ for $t \geqslant t_0$.

Let $\psi \in C([-\pi,0],\mathbb{R})$ and $t \geqslant t_0$ be such that $V(t,\psi(0)) \geqslant \sup_{s\in[-\pi,0]} V(t+s,\psi(s))$, i.e. $\psi^2(0) \geqslant \sup_{s\in[pi,0]}\psi^2(s)$. Then the inequality

$$D^+_{(3.4)}V(t,\psi(0),\psi) = -2\psi(0)\sin(t)\big(2\psi(0) - \psi(-\pi)\big) \leqslant (2-4\sin t)\psi^2(0) \leqslant 0$$

does not hold for all $t \geqslant t_0$. Its zero solution seems to be stable, but not uniformly stable (see Figures 5, 6).

Let $t \in [2k\pi,(2k+1)\pi]$ then $\sin(t) > 0$. Let the function $\psi \in C([-\pi,0],\mathbb{R})$ be such that $|\psi(0)| \geqslant |\psi(s)|$, $s \in [-\pi,0]$. Using $2xy \leqslant x^2 + y^2$ we get

$$\begin{aligned} D^+_{(3.4)}V(t,\psi(0),\psi) &= -4\sin(t)(\psi(0))^2 + 2\sin(t)\psi(0)\psi(-\pi)\\ &\leqslant -3\sin(t)(\psi(0))^2 + \sin(t)(\psi(-h))^2 \leqslant -2\sin(t)(\psi(0))^2\\ &\leqslant 0 \quad \text{for } t \in [2k\pi,(2k+1)\pi],\ k = 0,1,2,\dots, \end{aligned}$$

i.e. condition (i) of Corollary 2 is satisfied.

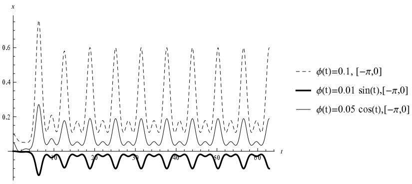

Figure 5. Example 4. Graph of the solutions of $x' = -\sin(t)(2x(t) - x(t-\pi))$ with different initial functions ϕ and $t_0 = 0$.

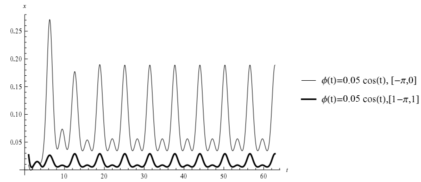

Figure 6. Example 4. Graph of the solutions of $x' = -\sin(t)(2x(t) - x(t-\pi))$ with different initial functions ϕ and $t_0 = 1$.

Let $t \in [(2k-1)\pi, 2k\pi]$ where k is a natural number.

Case 1. Let $\Phi_k(t,x,y) = x+y$. Then condition (H2) is not satisfied since the equation $x = x+y$ has no unique solution and the function $u_k(t,y)$ does not exist.

Case 2. Let $\Phi_k(t,x,y) = \frac{x^2}{ty}$. Then $u_k(t,y) = ty$ and conditions (H2) and (H3) are satisfied. However for any point $v \in \mathbb{R}$ the inequality $V(t, \Psi_k(t, u_k(t,v), v)) = \left(\frac{u_k(t,v)^2}{tv}\right)^2 = \left(tv\right)^2 \leqslant v^2 = V((2k-1)\pi - 0, v)$ does not hold, i.e. condition (ii) of Corollary 2 is not satisfied.

Case 3. Let $\Phi_k(t,x,y) = y\frac{1}{t}$. Then $u_k(t,y) = y\frac{1}{t}$, the conditions (H2),(H3) are satisfied and for any point $v \in \mathbb{R}$ the inequality $V(t, \Psi_k(t, u_k(t,v), v)) = \left(v\frac{1}{t}\right)^2 \leqslant v^2 = V((2k-1)\pi - 0, v)$ holds.

According to Corollary 2 the zero solution of the scalar NIDDE (3.4) with impulsive functions $\Phi_k(t,x,y) = y\frac{1}{t}$ is uniformly stable.

The above example illustrates that the behavior of the solutions of non-instantaneous impulsive delay fractional equations depends significantly on the type of the impulsive functions and there may be significant changes in the stability properties of the solutions of the corresponding delay equation without any impulses.

Now we present some sufficient conditions for the uniform asymptotic stability of the zero solution of the NIDDE.

THEOREM 3. (Uniform asymptotic stability) *Let the following conditions be satisfied:*

1. *Conditions (H1), (H2), (H3) are satisfied.*
2. *There exists a positive constant $M < \infty$ such that $\sum_{i=1}^{\infty}(t_i - s_i) \leqslant M$.*
3. *There exists a function $V \in \Lambda([-r,\infty), S(\lambda))$ such that*
   (i) *for any $t \in [0, s_1] \bigcup \cup_{k=1}^{\infty}(t_k, s_{k+1}]$ and any function $\psi \in E : ||\psi||_0 \leqslant \lambda$ such that $V(t, \psi(0)) \geqslant \sup_{s\in[-r,0]} V(t+s, \psi(s))$ the inequality*

$$D^{+}_{(2.1)} V(t, \psi(0), \psi) \leqslant -c(||\psi(0)||) \tag{3.5}$$

holds where $\lambda > 0$ is a given number, $c \in \mathscr{K}$;

(ii) for any natural number k, any $t \in [s_k, t_k]$ and point $v \in S(\lambda)$: such that $\Psi_k(t, u_k(t, v), v) \in S(\lambda)$ the inequality

$$V(t, \Psi_k(t, u_k(t, v), v)) \leqslant V(s_k - 0, v)$$

holds where the functions $u_k(t, v)$ are defined in condition (H2);

(iii) the inequalities $a(||x||) \leqslant V(t, x) \leqslant b(||x||)$, $t \geqslant -r$, $x \in S(\lambda)$, hold where $a, b \in \mathscr{K}$.

Then the zero solution of NIDDE (2.1) with zero initial function is uniformly asymptotically stable.

Proof. From Corollary 2 the zero solution of the NIDDE (2.1) is uniformly stable. Therefore, for the number λ there exists $\alpha = \alpha(\lambda) \in (0, \lambda)$ such that for any $\tilde{t}_0 \in [0, s_1) \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1})$ and $\phi_0 \in E$ the inequality $||\phi_0||_0 < \alpha$ implies

$$||x(t; \tilde{t}_0, \phi_0)|| < \lambda \quad \text{for } t \geqslant \tilde{t}_0 \tag{3.6}$$

where $x(t; \tilde{t}_0, \phi_0)$ is any solution of the NIDDE (2.1).

Now we prove that the zero solution of NIDDE (2.1) is uniformly attractive. Consider the constant $\beta \in (0, \alpha]$ such that $a(\beta) \leqslant b(\alpha)$. Let $\varepsilon \in (0, \lambda]$ and $t_0 \in [0, s_1) \bigcup \cup_{k=1}^{\infty} [s_k, t_{k+1})$ be arbitrary given numbers. Without loss of generality assume $t_0 \in [0, s_1)$.

Let the function $\phi \in E$, $||\phi||_0 < \beta$ and $x^*(t) = x(t; t_0, \phi)$ be any solution of (2.1). Then $||\phi||_0 < \alpha$ and according to (3.6) the inequality

$$||x^*(t)|| < \lambda \quad \text{for } t \geqslant t_0 \tag{3.7}$$

holds, i.e. the inclusion $x^*(t) \in S(\lambda)$ is satisfied on $[t_0, \infty)$.

Choose a constant $\gamma = \gamma(\varepsilon) \in (0, \varepsilon]$ such that $a(\gamma) \leqslant b(\varepsilon)$. Let $T > \frac{a(\alpha)}{c(\gamma)} + M$ and m be a natural number such that $t_m < t_0 + T \leqslant s_{m+1}$. Note T depends only on ε but not on t_0. We now prove that

$$||x^*(t)|| < \varepsilon \ \text{ for } t \geqslant t_0 + T. \tag{3.8}$$

Assume

$$||x^*(t)|| \geqslant \gamma \ \text{ for every } t \in [t_0, t_0 + T]. \tag{3.9}$$

For any $t \in [0, s_1] \bigcup \cup_{k=1}^{\infty} (t_k, s_{k+1}]$ such that $V(t, x^*(t)) \geqslant \sup_{s \in [-r,0]} V(t+s, x^*(t+s))$ we have the inequality $V(t, x^*(t)) \geqslant \sup_{s \in [-r,0]} V(t+s, x^*(t+s)) - \int_{-r}^{0} c(||x^*(t+\sigma)d\sigma\Big)$. Thus from condition 3(i) of Theorem 3 it follows that condition 2 (i) of Lemma 3 is satisfied. According to Lemma 3 (applied to the solution $x^*(t)$ for the interval

$[t_0, t_0+T]$ and $\Delta = S(\lambda)$), conditions 2, 3(ii) of Theorem 3 and the choice of T we get

$$\begin{aligned} & V(t_0+T, x^*(t_0+T)) \\ & \leqslant V(t_0,x_0) - \Big(\sum_{i=0}^{m-1} \int_{t_i}^{s_{i+1}} c(||x^*(s)||)ds + \int_{t_m}^{t_0+T} c(||x^*(s)||)ds \Big) \\ & \leqslant a(||x_0||) - c(\gamma)\Big(\sum_{i=0}^{m-1} (s_{i+1}-t_i) + (T+t_0-t_m) \Big) \\ & \leqslant a(\alpha) - c(\gamma)\Big(-\sum_{i=1}^{m} (t_i - s_i) + T \Big) \leqslant a(\alpha) - c(\gamma)\Big(-M+T \Big) < 0. \end{aligned}$$

The above contradiction proves there exists $t^* \in [t_0, t_0+T]$ such that $||x^*(t^*)|| < \gamma$.

Consider the interval $[t^*, \infty)$. From inequality (3.5) it follows that we have the inequality $D^+_{(2.1)}V(t,\psi(0),\psi) \leqslant 0$ for $t \in \cup_{k=0}^{\infty}(t_k, s_{k+1}) \cap [t^*,\infty)$ with $\psi(s) = x^*(t+s)$, $s \in [-r, 0]$, i.e. condition 2(i) of Lemma 2 with $\Delta = S(\lambda)$ is satisfied. Therefore, according to Lemma 2 applied to the solution $x^*(t)$ for $\Delta = S(\lambda)$ and $t \geqslant t^*$ the following inequality is satisfied:

$$V(t, x^*(t)) \leqslant V(t^*, x^*(t^*)). \tag{3.10}$$

Then for any $t \geqslant t^*$ applying (3.10), condition 3(iii) of Theorem 3 and inequality (3.7) we get the inequalities

$$b(||x^*(t)||) \leqslant V(t,x^*(t)) \leqslant V(t^*,x^*(t^*)) \leqslant a(||x^*(t^*)||) < a(\gamma) \leqslant b(\varepsilon).$$

Therefore, inequality (3.8) holds for all $t \geqslant t^*$ (hence for $t \geqslant t_0+T$). □

EXAMPLE 5. Consider the following IVP for the system of NIDDE

$$\begin{aligned} & x'(t) = (-2.5 + \frac{cos(y(t))}{1+t^2})x(t) + x(t-1) + y(t) \text{ for } t \in (t_k, s_{k+1}],\ k = 0,1,2,\dots, \\ & y'(t) = -x(t) + (\sin x(t) - 2.5)y(t) + y(t-1) \text{ for } t \in (t_k, s_{k+1}],\ k = 0,1,2,\dots, \\ & x(t) = x(s_k - 0)\frac{t}{t+1}, \qquad y(t) = \frac{y(s_k-0)}{t+1} \quad \text{for} \quad t \in (s_k, t_k],\ k = 1,2,\dots, \\ & x(t_0+s) = \phi_1(s), \quad y(t_0+s) = \phi_2(s), \quad s \in [-1,0] \end{aligned} \tag{3.11}$$

where $n = 2$, $x, y \in \mathbb{R}$, $s_k = k - \frac{1}{2^k}$ and $t_k = k + \frac{1}{2^k}$, $k = 1,2,\dots$.

Then $\sum_{i=1}^{\infty}(t_i - s_i) = \sum_{i=1}^{\infty} \frac{1}{2^{k-1}} = 2$, i.e. condition 2 of Theorem 3 is satisfied for $M = 2$.

Let $V(t,x,y) = x^2+y^2$, $t \in [0, s_1] \bigcup \cup_{k=1}^{\infty}(t_k, s_{k+1}]$ and $\psi \in E: \ ||\psi||_0 \leqslant \lambda$, $\psi = (\psi_1, \psi_2)$ be a function such that $V(t,\psi(0)) \geqslant \sup_{s\in[-r,0]} V(t+s,\psi(s))$, i.e. $\psi_1(0)^2 + \psi_2(0)^2 \geqslant \sup_{s\in[-r,0]} \Big(\psi_1(s)^2 + \psi + 2(s)^2\Big)$. Then applying

$$\psi_1(0)^2 + \psi_2(0)^2 \geqslant \sup_{s\in[-r,0]} \Big(\psi_1(s)^2 + \psi_2(s)^2\Big) \geqslant \psi_1(-1)^2 + \psi_2(-1)^2$$

we obtain the inequality

$$\begin{aligned}
&D^{+}_{(3.11)}V(t,\psi(0),\psi)\\
&=\phi_1(0)\Big((-2.5+\frac{cos(\phi_2(0))}{1+t^2})\phi_1(0)+\phi_1(1)+\phi_2(0)\Big)\\
&\quad+\phi_2(0)\Big(-\phi_1(0)+\sin(\phi_1(0))\phi_2(0)+\phi_2(-1)\Big)\\
&=-2.5\phi_1^2(0)+\frac{cos(\phi_2(0))}{1+t^2})\phi_1^2(0)+\phi_1(0)\phi_1(-1)+(\sin(\phi_1(0))-2.5)\phi_2^2(0)\\
&\qquad+\phi_2(0)\phi_2(-1)\\
&\leqslant -2.5\phi_1^2(0)+\phi_1^2(0)+\phi_2^2(0)+0.5\phi_1^2(0)+0.5\phi_1^2(-1)+0.5\phi_2^2(0)+0.5\phi_2^2(-1)\\
&\leqslant -2.5(\phi_1^2(0)+\phi_2^2(0))+\phi_1^2(0)+\phi_2^2(0)+\phi_1^2(0)+\phi_2^2(0)\\
&\leqslant -0.5(\phi_1^2(0)+\phi_2^2(0))=-c(||\psi(0)||).
\end{aligned}$$

where $c(s)=0.5\sqrt{s}\in\mathscr{K}$, i.e. condition 3(i) of Theorem 3 is satisfied.

Let k be a natural number and $t\in[s_k,t_k]$. Then for any point $v\in S(\lambda):\ v=(v_1,v_2)$ the inequality $(v_1\frac{t}{t+1})^2+(\frac{v_2}{t+1})^2\leqslant(v_1)^2+(v_2)^2\leqslant\lambda^2$ holds, i.e. condition 3(ii) of Theorem 3 is satisfied and therefore, the zero solution of the system of NIDDE (3.11) with zero initial function is uniformly asymptotically stable.

4. Special case – instantaneous impulses

As noted above the instantaneous impulses are a special case of non-instantaneous impulses. As a result from the above results we obtain some stability results for impulsive delay differential equations.

Let the increasing sequence of points $\{t_i\}_{i=1}^{\infty}$ be given suchs that $0<t_i<t_{i+1}$, $i=1,2,\dots$, and $\lim_{k\to\infty}t_k=\infty$.

Consider the system of *impulsive delay differential equations* (IDDE)

$$\begin{aligned}
&x'=f(t,x_t)\text{ for }t\neq t_k,\ k=1,2,\dots\\
&x(t_k+0)=I_k(x(t_k-0))\quad\text{for }k=1,2,\dots,
\end{aligned}\tag{4.1}$$

and the scalar impulsive differential equation (IDE)

$$\begin{aligned}
&u'=g(t,u)\text{ for }t\neq t_k,\ k=1,2,\dots\\
&u(t_k+0)=J_k(u(t_k-0))\quad\text{for }k=1,2,\dots,
\end{aligned}\tag{4.2}$$

We introduce the following conditions:

(H6) The function $f\in C([0,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$ is such that for any initial function $\tilde{\phi}\in E$ the IVP for the system of DDE (2.3) has a solution $x(t;\tau,\tilde{\phi})$ and $f(t,0)=0$.

(H7) For any natural number k the function $I_k\in C(\mathbb{R}^n,\mathbb{R}^n)$, $k=1,2,\dots$ and $I_k(0)=0$.

(H8) The function $g\in C([0,\infty)\times\mathbb{R},\mathbb{R}_+)$ is such that $g(t,0)=0$ and for any initial point $(\tau,\tilde{u}_0)$ the IVP for ODE $u'=g(t,u)$, $u(\tau)=\tilde{u}_0$ has a maximal solution $\tilde{u}(t;\tau,\tilde{u}_0)$.

(H9) For all natural numbers k the functions $J_k \in C(\mathbb{R},\mathbb{R})$ are such that $J_k(0)=0$ and $J_k(u) \leqslant J_k(v)$ for $u \leqslant v$.

As a special case of Lemma 2 we obtain the following comparison result for IDDE (4.1).

LEMMA 4. *Suppose:*

1. *The function* $x^*(t)=x(t;t_0,\phi) \in PC^1([t_0,\Theta],\Delta)$ *is a solution of the IDDE (4.1) where* $\Delta \subset \mathbb{R}^n$, $\Theta > t_0$, $t_p < \Theta \leqslant t_{p+1}$ *is a given number.*
2. *For all* $k=1,\dots,p$ *condition (H9) is satisfied.*
3. *Condition (H8) is satisfied on the interval* $[t_0,\Theta]$.
4. *The function* $V \in \Lambda([t_0-r,\Theta],\Delta)$ *and*

   (i) for any $t \in [t_0,\Theta]/\{t_k\}_{k=1,}^{\infty}$ *such that* $V(t,\psi(0)) \geqslant \sup_{s\in[-r,0]} V(t+s,\psi(s))$ *the inequality*

$$D^+_{(4.1)} V(t,\psi(0),\psi) \leqslant g(t,V(t,\psi(0))) \tag{4.3}$$

   holds where $\psi(s)=x^*(t+s), s\in[-r,0]$*;*

   (ii) for any number $k=1,2,\dots,p$ *the inequality*

$$V(t_k,I_k(x^*(t_k-0))) \leqslant J_k(V(t_k-0,x^*(t_k-0))) \tag{4.4}$$

   holds.

Then $\max_{s\in[-r,0]} V(t_0+s,\phi(s)) \leqslant u_0$ *implies the inequality* $V(t,x^*(t)) \leqslant u^*(t)$ *on the interval* $[t_0,\Theta]$ *where* $u^*(t)$ *is the maximal solution of IVP for IDE (4.2) on* $[t_0,\Theta]$.

REMARK 12. Note comparison results for IDDE (4.1) are given in Lemma 2 [17] but the inequalities (4.3) and (4.4) are satisfied for all $x \in \Omega_1$ where

$$\Omega_1 = \{x \in PC([t_0,\infty),\Delta):\ V(s,x(s)) \leqslant V(t,x(t)) \text{for } s \in [t-r,t], \text{ and } t \geqslant t_0\}$$

Conditions 4(i) and 4(ii) of Lemma 4 are less restrictive.

Also, in Lemma 2 [17] the inequality $\max_{s\in[-r,0]} V(t_0+s,\phi(s)) \leqslant u_0$ is replaced by the inequality $V(t_0,\phi(0)) \leqslant u_0$ which is not enough for the validity of the claim $V(t,x^*(t)) \leqslant u^*(t)$ on $[t_0,\Theta]$. The same remark concerns Corollary 1 [17] where the claim $V(t,x(t;t_0,\phi_0)) \leqslant V(t_0+0,\phi_0(0),\ t \geqslant t_0$ is not true. Consider, for example, the initial function $\phi_0(s)=\sin(s)$. Then the inequality $V(t,x(t;t_0,\phi_0)) \leqslant V(t_0+0,\phi_0(0)) = V(t_0+0,0)=0$ reduces to $V(t,x(t;t_0,\phi_0))=0$ which is not true for any solution and any function f.

As a special case of Theorem 1 we obtain:

THEOREM 4. (Stability of IDDE) *Suppose:*

1. *Conditions (H6)–(H9) are satisfied.*
2. *There exists a function* $V \in \Lambda([-r,\infty),\mathbb{R}^n)$ *such that*
   - *(i) for any* $t \in \mathbb{R}_+/\{t_k\}_{k=0}^{\infty}$ *and any function* $\psi \in E$ *such that* $V(t,\psi(0)) \geqslant \sup_{s\in[-r,0]} V(t+s,\psi(s))$ *the inequality*
   $$D^+_{(4.1)} V(t,\psi(0),\psi) \leqslant g(t,V(t,\psi(0))) \tag{4.5}$$
   holds;
   - *(ii) for any* $v \in \mathbb{R}^n$ *and* $k = 1,2,3,\ldots$ *the inequality*
   $$V(t_k-0,I_k(v)) \leqslant J_k(V(t_k-0,v)) \tag{4.6}$$
   holds;
   - *(iii) the inequalities* $a(||x||) \leqslant V(t,x) \leqslant b(||x||)$, $t \geqslant -r$, $x \in \mathbb{R}^n$, *hold where* $a,b \in \mathscr{K}$.
3. *The zero solution of the scalar IDE (4.2) is stable.*

Then the zero solution of IDDE (4.1) with zero initial function is stable.

THEOREM 5. (Uniform stability of IDDE) *Suppose:*

1. *Conditions (H6)–(H9) are satisfied.*
2. *There exists a function* $V \in \Lambda([-r,\infty),S(\lambda))$ *such that*
   - *(i) for any* $t \in \mathbb{R}_+/\{t_k\}_{k=1,}^{\infty}$ *and any function* $\psi \in E:\ ||\psi||_0 \leqslant \lambda$ *such that* $V(t,\psi(0)) \geqslant \sup_{s\in[-r,0]} V(t+s,\psi(s))$ *the inequality*
   $$D^+_{(4.1)} V(t,\psi(0),\psi) \leqslant g(t,V(t,\psi(0))) \tag{4.7}$$
   holds;
   - *(ii) for any natural number* k, *and any point* $v \in S(\lambda):\ I_k(v) \in S(\lambda)$ *the inequality*
   $$V(t_k-0,I_k(v)) \leqslant J_k(V(t_k-0,v))$$
   holds;
   - *(iii) the inequalities* $a(||x||) \leqslant V(t,x) \leqslant b(||x||)$, $t \geqslant -r$, $x \in S(\lambda)$, *hold where* $a,b \in \mathscr{K}$.
3. *The zero solution of the scalar IDE (4.2) is uniformly stable.*

Then the zero solution of IDDE (4.1) with zero initial function is uniformly stable.

THEOREM 6. (Uniform asymptotic stability of IDDE) *Let the following conditions be satisfied:*

1. *Conditions (H6) and (H7) are satisfied.*
2. *There exists a positive constant $M < \infty$ such that $\sum_{i=1}^{\infty}(t_i - s_i) \leqslant M$.*
3. *There exists a function $V \in \Lambda([-r,\infty), S(\lambda))$ such that*
   - *(i) for any $t \in \mathbb{R}_+/\{t_k\}_{k=1,}^{\infty}$ and any function $\psi \in E: \ ||\psi||_0 \leqslant \lambda$ such that $V(t,\psi(0)) \geqslant \sup_{s\in[-r,0]} V(t+s,\psi(s))$ the inequality*

$$D^+_{(4.1)} V(t,\psi(0),\psi) \leqslant -c(||\psi(0)||) \tag{4.8}$$

   holds where $\lambda > 0$ is a given number, $c \in \mathscr{K}$;
   - *(ii) for any natural number k and point $v \in S(\lambda): \ I_k(v) \in S(\lambda)$ the inequality*

$$V(t_k - 0, I_k(v)) \leqslant V(t_k - 0, v) \tag{4.9}$$

   holds;
   - *(iii) the inequalities $a(||x||) \leqslant V(t,x) \leqslant b(||x||)$, $t \geqslant -r$, $x \in S(\lambda)$, hold where $a,b \in \mathscr{K}$.*

Then the zero solution of IDDE (4.1) with zero initial function is uniformly asymptotically stable.

REMARK 13. Sufficient conditions for stability, uniform stability and asymptotic uniform stability of IDDE (4.1) are given in Theorem 1, Theorem 2, Theorem 3 [17]. In these Theorems the main conditions (4.5), (4.6) with $g \equiv 0$, $J_k(u) \equiv u$, and (4.8), (4.9) are satisfied for all $x \in \Omega_1$ and these are stronger than the corresponding conditions in Theorem 4, Theorem 5 and Theorem 6. Also condition 2 of Theorem 1 [17] which is weaker than condition 2(iii) of Theorem 4, is not enough for stability of the zero solution.

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