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SECOND–ORDER ORDINARY DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH A NEW KIND OF INTEGRAL AND MULTI–STRIP BOUNDARY CONDITIONS

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Abstract. In this paper, we study the existence of solutions for nonlinear second-order ordinary differential equations and inclusions with nonlinearity depending upon the unknown function together with its first derivative, supplemented with a new kind of integral and multi-strip boundary conditions. Krasnoselskii fixed point theorem and Banach contraction mapping principle are employed to prove the existence and uniqueness results for the single-valued boundary value problem. In the multi-valued case the nonlinear alternative of Leray-Schauder type is the key tool for studying convex valued right-hand side, while the case of non-convex valued right-hand side is handled with the aid of a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. Examples are constructed for the illustration of the obtained results.

1. Introduction

In this paper, we introduce a new kind of integral and multi-strip boundary conditions, which relate the distribution of the unknown function (and its derivative) on the given arbitrary interval with a sum of sub-strips conditions defined on finite many segments of the given domain and solve a second order nonlinear ordinary differential equation equipped with these conditions. Precisely, in the first part of the paper, we investigate the following single-valued boundary value problem:

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), \ -\infty < a < t < T < \infty, \\ \displaystyle\int_a^T u(s)ds = \sum_{j=1}^m \gamma_j \int_{\xi_j}^{\eta_j} u(s)ds + \lambda_1, \\ \displaystyle\int_a^T u'(s)ds = \sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j} u'(s)ds + \lambda_2, \end{cases} \tag{1}$$

where $f : [a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $a < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \ldots < \xi_m < \eta_m < T$, $\gamma_j, \rho_j \in \mathbb{R}^+$ $(j = 1, 2, \ldots, m)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. In the second part of

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the paper, we investigate the following multi-valued boundary value problem:

$$\begin{cases} u''(t) \in F(t,u(t),u'(t)), \ -\infty < a < t < T < \infty, \\ \displaystyle\int_a^T u(s)ds = \sum_{j=1}^m \gamma_j \int_{\xi_j}^{\eta_j} u(s)ds + \lambda_1, \\ \displaystyle\int_a^T u'(s)ds = \sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j} u'(s)ds + \lambda_2, \end{cases} \tag{2}$$

where $F : [a,T] \times \mathbb{R} \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

The importance of boundary value problems is well-recognized in view of their extensive applications in applied sciences and engineering [34]. Much of the literature on the topic deals with classical boundary conditions. However, in order to model the physical problem involving the data available at arbitrary interior points or finite many segments of the given domain, one needs to apply the concept of nonlocal conditions. For examples and details of nonlocal nonlinear boundary value problems, see ([23, 19, 2, 31, 11, 24, 18, 17, 4, 6, 33, 5]) and the references cited therein. In case of arbitrarily shaped domain, integral boundary conditions act as more realistic and practical tools. Examples of such conditions can be found in numerous fields such as thermal and hydrodynamic problems, underground water flow, blood flow problems, chemical engineering, population dynamics, thermoelasticity, etc. For a detailed account of these conditions, we refer the reader to a series of papers ([9, 26, 22, 30, 25, 1, 7, 3, 29, 20, 16, 8]) and the references cited therein.

The rest of the paper is organized as follows. In Section 2, we prove an auxiliary lemma related to the linear variant of the problem (1). The existence and uniqueness results for the given single-valued boundary value problem together with illustrative examples are presented in Section 3. Section 4 deals with the existence of solutions for multi-valued boundary value problem (2) involving convex valued as well as non-convex valued maps.

2. Preliminary result

The following lemma plays a key role in defining the solution for the boundary value problem (1).

LEMMA 1. *Let $g \in C([a,T],\mathbb{R})$ and*

$$\Big[T - a - \sum_{j=1}^m \gamma_j(\eta_j - \xi_j)\Big]\Big[T - a - \sum_{j=1}^m \rho_j(\eta_j - \xi_j)\Big] \neq 0.$$

Then the solution of the equation $u''(t) = g(t)$, $t \in [a,T]$ subject to the integral and multi-strip boundary conditions of (1) is given by

$$\begin{aligned} u(t) = \int_a^t (t-s)g(s)ds + \frac{1}{A_1A_2}\Big[-\frac{1}{2}\int_a^T \Big(2\chi(t) + A_1(T-s)\Big)(T-s)g(s)ds \\ + A_1\lambda_1 + \lambda_2\chi(t) + \sum_{j=1}^m \int_{\xi_j}^{\eta_j} \int_a^s \Big(\rho_j\chi(t) + \gamma_j A_1(s-p)\Big)g(p)dpds\Big], \end{aligned} \tag{3}$$

where

$$\chi(t) = A_2(t-a) - A_3, \tag{4}$$

$$\begin{aligned} A_1 &= T-a-\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j),\;\; A_2 = T-a-\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j), \\ A_3 &= \frac{(T-a)^2}{2} - \frac{1}{2}\sum_{j=1}^{m}\gamma_j\Big[(\eta_j-a)^2-(\xi_j-a)^2\Big]. \end{aligned} \tag{5}$$

Proof. Integrating both sides of the equation $u''(t) = g(t)$ from a to t, we obtain

$$u(t) = c_1 + c_2(t-a) + \int_a^t (t-s)g(s)ds, \tag{6}$$

where c_1 and c_2 are unknown arbitrary real constants. Using the boundary conditions of (1) in (6) together with the notations (4) and (5), we find that

$$\begin{aligned} c_1 = \frac{1}{A_1A_2}\Big[&\int_a^T \Big(A_3(T-s) - A_1\frac{(T-s)^2}{2}\Big)g(s)ds + A_1\lambda_1 \\ &+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s \Big(A_1\gamma_j(s-p) - A_3\rho_j\Big)g(p)dpds - A_3\lambda_2\Big], \end{aligned}$$

$$c_2 = \frac{1}{A_1}\Big[-\int_a^T (T-s)g(s)ds + \sum_{j=1}^{m}\rho_j\int_{\xi_j}^{\eta_j}\int_a^s g(p)dpds + \lambda_2\Big].$$

Inserting the values c_1 and c_2 in (6) yields the solution (3). The converse of the Lemma follows by direct computation. This completes the proof. □

3. The single-valued case

Let $\Pi = C^1([a,T],\mathbb{R})$ denote the Banach space endowed with the norm defined by $\|u\|_\Pi = \|u\| + \|u'\| = \sup_{t\in[a,T]}|u(t)| + \sup_{t\in[a,T]}|u'(t)|$. In view of Lemma 1, we transform boundary value problem (1) into an equivalent fixed point problem as

$$u = \mathscr{S}u, \tag{7}$$

where $\mathscr{S} : \Pi \to \Pi$ is defined by

$$\begin{aligned} (\mathscr{S}u)(t) = &\int_a^t (t-s)f(s,u(s),u'(s))ds \\ &-\frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t) + A_1(T-s)\Big](T-s)f(s,u(s),u'(s))ds \\ &+\frac{1}{A_1A_2}\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t) + \gamma_jA_1(s-p)\Big]f(p,u(p),u'(p))dpds \\ &+\frac{1}{A_1A_2}\Big[A_1\lambda_1 + \chi(t)\lambda_2\Big]. \end{aligned} \tag{8}$$

Note that

$$(\mathscr{S}u)'(t) = \int_a^t f(s,u(s),u'(s))ds + \frac{1}{A_1}\Big[\sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j}\int_a^s f(p,u(p),u'(p))dpds \\ - \int_a^T (T-s)f(s,u(s),u'(s))ds + \lambda_2\Big]. \tag{9}$$

For computational convenience in the forthcoming analysis, we set

$$Q = \tfrac{(T-a)^2}{2} + \Big|\tfrac{A_2(T-a)-A_3}{A_1A_2}\Big|\Big[\tfrac{(T-a)^2}{2} + \textstyle\sum_{j=1}^m \rho_j\Big(\tfrac{(\eta_j-a)^2}{2} - \tfrac{(\xi_j-a)^2}{2}\Big) + |\lambda_2|\Big] \\ + \frac{1}{|A_2|}\Big[\frac{(T-a)^3}{3!} + \sum_{j=1}^m \gamma_j\Big(\frac{(\eta_j-a)^3}{3!} - \frac{(\xi_j-a)^3}{3!}\Big) + |\lambda_1|\Big], \tag{10}$$

$$Q_1 = (T-a) + \frac{1}{|A_1|}\Big[\frac{(T-a)^2}{2} + \sum_{j=1}^m \rho_j\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big) + |\lambda_2|\Big]. \tag{11}$$

Now we are ready to present our main results. For that, we need the following assumptions:

(H_1) Let $f:[a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that

$$|f(t,u,u') - f(t,v,v')| \leqslant \ell\big(|u-v| + |u'-v'|\big),\ \forall t\in[a,T],\ \ell>0,\ u,v,u',v'\in\mathbb{R};$$

(H_2) $\ell(Q+Q_1)<1$, where Q and Q_1 are defined by (10) and (11) respectively.

In the first result we prove the existence of solutions for the problem (1) by applying Krasnoselskii's fixed point theorem [32].

LEMMA 2. *(Krasnoselskii's fixed point theorem). Let Y be a closed bounded, convex and nonempty subset of a Banach space X. Let ϕ_1,ϕ_2 be the operators mapping Y into X such that*
(i) $\phi_1 y_1 + \phi_2 y_2 \in Y$ whenever $y_1,y_2\in Y$;
(ii) ϕ_1 is compact and continuous; and
(iii) ϕ_2 is a contraction mapping.
Then there exists $y\in Y$ such that $y=\phi_1 y + \phi_2 y$.

THEOREM 1. *Suppose that (H_1), (H_2) and the following condition hold:*

(H_3) *There exist a function $\theta\in C([a,T],\mathbb{R}^+)$ with $\|\theta\| = \sup_{t\in[a,T]}|\theta(t)|$ such that $|f(t,u,u')|\leqslant\theta(t)$, $\forall(t,u,u')\in[a,T]\times\mathbb{R}\times\mathbb{R}$.*

Then the boundary value problem (1) has at least one solution on $[a,T]$.

Proof. Consider $B_{\bar r} = \{u\in\Pi : \|u\|_\Pi \leqslant \bar r\}$ with $\bar r \geqslant \|\theta\|(Q+Q_1)$ and introduce the operators $\mathscr{S}_1$ and $\mathscr{S}_2$ on $B_{\bar r}$ as

$$(\mathscr{S}_1 u)(t) = \int_a^t (t-s)f(s,u(s),u'(s))ds,$$

$$(\mathscr{S}_2u)(t) = -\frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t)+A_1(T-s)\Big](T-s)f(s,u(s),u'(s))ds$$
$$+\frac{1}{A_1A_2}\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\rho_j\chi(t)+\gamma_jA_1(s-p)\Big]f(p,u(p),u'(p))dpds$$
$$+\frac{1}{A_1A_2}\Big[A_1\lambda_1+\chi(t)\lambda_2\Big].$$

Moreover, we have

$$(\mathscr{S}_1u)'(t) = \int_a^t f(s,u(s),u'(s))ds, \quad t\in[a,T],$$

$$(\mathscr{S}_2u)'(t) = \frac{1}{A_1}\Big[\sum_{j=1}^m\rho_j\int_{\xi_j}^{\eta_j}\int_a^s f(p,u(p),u'(p))dpds$$
$$-\int_a^T(T-s)f(s,u(s),u'(s))ds+\lambda_2\Big], \quad t\in[a,T].$$

Observe that $\mathscr{S}=\mathscr{S}_1+\mathscr{S}_2$. For $u,v\in B_{\bar r}$, we find that

$$\|\mathscr{S}_1u+\mathscr{S}_2v\| = \sup_{t\in[a,T]}\Big\{\Big|\int_a^t(t-s)f(s,u(s),u'(s))ds$$
$$-\frac{1}{2A_1A_2}\int_a^T\Big[2\chi(t)+A_1(T-s)\Big](T-s)f(s,v(s),v'(s))ds$$
$$+\frac{1}{A_1A_2}\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\rho_j\chi(t)+\gamma_jA_1(s-p)\Big]f(p,v(p),v'(p))dpds$$
$$+\frac{1}{A_1A_2}\Big[A_1\lambda_1+\chi(t)\lambda_2\Big]\Big|\Big\}$$
$$\leqslant \|\theta\|\sup_{t\in[a,T]}\Big\{\frac{(t-a)^2}{2}$$
$$+\Big|\frac{A_2(t-a)-A_3}{A_1A_2}\Big|\Big[\frac{(T-a)^2}{2}+\frac{1}{2}\sum_{j=1}^m|\rho_j[(\eta_j-a)^2-(\xi_j-a)^2]|+|\lambda_2|\Big]$$
$$+\frac{1}{|A_2|}\Big[\frac{(T-a)^3}{3!}+\frac{1}{3!}\sum_{j=1}^m|\gamma_j[(\eta_j-a)^3-(\xi_j-a)^3]|+|\lambda_1|\Big]\Big\}\leqslant\|\theta\|Q$$

and

$$\|(\mathscr{S}_1u)'+(\mathscr{S}_2v)'\|$$
$$=\sup_{t\in[a,T]}\Big\{\Big|\int_a^t f(s,u(s),u'(s))ds+\frac{1}{A_1}\Big[\sum_{j=1}^m\rho_j\int_{\xi_j}^{\eta_j}\int_a^s f(p,v(p),v'(p))dpds$$
$$-\int_a^T(T-s)f(s,v(s),v'(s))ds+\lambda_2\Big]\Big|\Big\}$$
$$\leqslant\|\theta\|\sup_{t\in[a,T]}\Big\{(t-a)+\frac{1}{2|A_1|}\Big[\sum_{j=1}^m|\rho_j[(\eta_j-a)^2-(\xi_j-a)^2]|+(T-a)^2+2|\lambda_2|\Big]\Big\}$$
$$\leqslant\|\theta\|Q_1.$$

In consequence, we obtain

$$\|\mathscr{S}_1 u+\mathscr{S}_2 v\|_\Pi \leqslant \|\theta\|(Q+Q_1) \leqslant \bar{r}.$$

Thus $\mathscr{S}_1 u+\mathscr{S}_2 v \in B_{\bar{r}}$, which verifies the condition (i) in Lemma 2. Using the assumption (H_1) and (H_2), we obtain

$$\begin{aligned}
\|\mathscr{S}_2 u-\mathscr{S}_2 v\| \leqslant & \sup_{t\in[a,T]}\Big\{\frac{1}{2|A_1A_2|}\int_a^T \Big[2|\chi(t)|+|A_1|(T-s)\Big](T-s)\\
&\times\Big|f(s,u(s),u'(s))-f(s,v(s),v'(s))\Big|ds\\
&+\frac{1}{|A_1A_2|}\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j|\chi(t)|+\gamma_j|A_1|(s-p)\Big]\\
&\times\Big|f(p,u(p),u'(p))-f(p,v(p),v'(p))\Big|dpds\Big\}\\
\leqslant & \,\ell\Big(\|u-v\|+\|u'-v'\|\Big)\Big\{\Big|\frac{A_2(T-a)-A_3}{2A_1A_2}\Big|\Big[(T-a)^2\\
&+\sum_{j=1}^m|\rho_j[(\eta_j-a)^2-(\xi_j-a)^2]|\Big]\\
&+\frac{1}{6|A_2|}\Big[(T-a)^3+\sum_{j=1}^m|\gamma_j[(\eta_j-a)^3-(\xi_j-a)^3]|\Big]\\
\leqslant & \,\ell\overline{Q}\Big(\|u-v\|+\|u'-v'\|\Big)\leqslant \ell\overline{Q}\|u-v\|_\Pi,
\end{aligned}$$

where

$$\overline{Q}=Q-\frac{(T-a)^2}{2}-\Big|\lambda_2\frac{A_2(T-a)-A_3)}{A_1A_2}\Big|-\frac{|\lambda_1|}{|A_2|}, \tag{12}$$

and

$$\begin{aligned}
&\|(\mathscr{S}_2 u)'-(\mathscr{S}_2 v)'\|\\
\leqslant & \sup_{t\in[a,T]}\Big\{\frac{1}{|A_1|}\Big[\sum_{j=1}^m\rho_j\int_{\xi_j}^{\eta_j}\int_a^s\Big|f(p,u(p),u'(p))-f(p,v(p),v'(p))\Big|dpds\\
&-\int_a^T(T-s)\Big|f(s,u(s),u'(s))-f(s,v(s),v'(s))\Big|ds\Big]\Big\}\\
\leqslant & \,\ell\big(\|u-v\|+\|u'-v'\|\big)\sup_{t\in[a,T]}\Big\{\frac{1}{2|A_1|}\Big[\sum_{j=1}^m|\rho_j[(\eta_j-a)^2-(\xi_j-a)^2]|+(T-a)^2\Big]\Big\}\\
\leqslant & \,\ell\overline{Q}_1\Big(\|u-v\|+\|u'-v'\|\Big)\leqslant \ell\overline{Q}_1\|u-v\|_\Pi,
\end{aligned}$$

where

$$\overline{Q}_1=Q_1-(T-a)-\frac{|\lambda_2|}{|A_1|}. \tag{13}$$

Thus we get

$$\|\mathscr{S}_2 u-\mathscr{S}_2 v\|_\Pi \leqslant \ell\Big(\overline{Q}+\overline{Q}_1\Big)\|u-v\|_\Pi,$$

which implies that $\mathscr{S}_2$ is a contraction as $\ell\left(\overline{Q}+\overline{Q}_1\right)<\ell\left(Q+Q_1\right)<1$ by the condition (H_2). Next, we show that $\mathscr{S}_1$ is compact and continuous. Notice that continuity of f implies that the operator $\mathscr{S}_1$ is continuous. Also, $\mathscr{S}_1$ is uniformly bounded on $B_{\overline{r}}$ as

$$\|\mathscr{S}_1 u\|_\Pi \leqslant \|\theta\|\Big[\frac{(T-a)^2}{2}+(T-a)\Big],$$

where we have used (H_3). Let us fix $\max_{t\in[a,T]\times B_{\overline{r}}\times B_{\overline{r}}}|f(t,u,u')|=\overline{f}$, and take $t_1,t_2\in[a,T]$. Then

$$\begin{aligned}
&|(\mathscr{S}_1u)(t_2)-(\mathscr{S}_1u)(t_1)|\\
&=\left|\int_a^{t_1}[(t_2-s)-(t_1-s)]f(s,u(s),u'(s)ds+\int_{t_1}^{t_2}(t_2-s)f(s,u(s),u'(s))ds\right|\\
&\leqslant \overline{f}\left|(t_2-t_1)(t_1-a)+\frac{(t_2-t_1)^2}{2}\right|\to 0 \text{ as } (t_2-t_1)\to 0, \text{ independently of } u\in B_{\overline{r}},
\end{aligned}$$

and

$$|(\mathscr{S}_1u)'(t_2)-(\mathscr{S}_1u)'(t_1)|\leqslant \overline{f}(t_2-t_1)\to 0 \text{ as } (t_2-t_1)\to 0, \text{ independently of } u\in B_{\overline{r}}.$$

This implies that $\mathscr{S}_1$ is relatively compact on $B_{\overline{r}}$. Hence we deduce by the Arzelá-Ascoli theorem that the operator $\mathscr{S}_1$ is compact on $B_{\overline{r}}$. Thus, all the assumptions of Lemma 2 are satisfied. In consequence, by the conclusion of Lemma 2, the boundary value problem (1) has at least one solution on $[a,T]$. □

REMARK 1. If we interchange the role of the operators $\mathscr{S}_1$ and $\mathscr{S}_2$ in the previous theorem, then the condition (H_2) is replaced with $\ell\dfrac{(T-a)}{2}[T-a+2]<1$.

In the next result, we establish the uniqueness result for the problem (1) by means of the following Banach's contraction mapping principle.

LEMMA 3. (Banach fixed point theorem) [13] *Let X be a Banach space, $D\subset X$ closed and $F:D\to D$ a strict contraction, i.e. $|Fx-Fy|\leqslant k|x-y|$ for some $k\in(0,1)$ and all $x,y\in D$. Then F has a fixed point in D.*

THEOREM 2. *Assume that (H_1) and (H_2) are satisfied. Then there exists a unique solution for the problem (1) on $[a,T]$.*

Proof. Define a set $B_w=\{u\in\Pi:\|u\|_\Pi\leqslant w\}$ with $w\geqslant\dfrac{(Q+Q_1)K}{1-\ell(Q+Q_1)}$ and $\sup_{t\in[a,T]}|f(t,0,0)|=K$. In the first step, we show that $\mathscr{S}B_w\subset B_w$, where the operator $\mathscr{S}$ is defined by (8). For any $u\in B_w, t\in[a,T]$, we find that

$$\begin{aligned}
|f(s,u(s),u'(s))| &= |f(s,u(s),u'(s))-f(s,0,0)+f(s,0,0)|\\
&\leqslant |f(s,u,u')-f(s,0,0)|+|f(s,0,0))|\\
&\leqslant \ell(\|u\|+\|u'\|)+K\leqslant \ell\|u\|_\Pi+K\leqslant \ell w+K.
\end{aligned}$$

Therefore, for $u \in B_w,$ we obtain

$$\begin{aligned}
\|(\mathscr{S}u)\| = & \sup_{t\in[a,T]}\Big\{\Big|\int_a^t (t-s)f(s,u(s),u'(s))ds \\
& -\frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t)+A_1(T-s)\Big](T-s)f(s,u(s),u'(s))ds \\
& +\frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t)+\gamma_jA_1(s-p)\Big]f(p,u(p),u'(p))dpds \\
& +\frac{1}{A_1A_2}\Big[A_1\lambda_1+\chi(t)\lambda_2\Big]\Big|\Big\} \\
\leqslant & [\ell w+K]\sup_{t\in[a,T]}\Big\{\frac{(t-a)^2}{2} \\
& +\Big|\frac{A_2(t-a)-A_3}{2A_1A_2}\Big|\Big[(T-a)^2+\sum_{j=1}^m|\rho_j[(\eta_j-a)^2-(\xi_j-a)^2]|+2|\lambda_2|\Big] \\
& +\frac{1}{6|A_2|}\Big[(T-a)^3+\sum_{j=1}^m|\gamma_j[(\eta_j-a)^3-(\xi_j-a)^3]|+6|\lambda_1|\Big]\Big\} \\
\leqslant & [\ell w+K]Q,
\end{aligned}$$

where Q is given by (10) and

$$\begin{aligned}
\|(\mathscr{S}u)'\| = & \sup_{t\in[a,T]}\Big\{\Big|\int_a^t f(s,u(s),u'(s))ds+\frac{1}{A_1}\Big[\sum_{j=1}^m\rho_j\int_{\xi_j}^{\eta_j}\int_a^s f(p,u(p),u'(p))dpds \\
& -\int_a^T (T-s)f(s,u(s),u'(s))ds+\lambda_2\Big]\Big|\Big\} \\
\leqslant & [\ell w+K]\sup_{t\in[a,T]}\Big\{(t-a) \\
& +\frac{1}{2|A_1|}\Big[\sum_{j=1}^m|\rho_j[(\eta_j-a)^2-(\xi_j-a)^2]|+(T-a)^2+2|\lambda_2|\Big]\Big\} \\
\leqslant & [\ell w+K]Q_1,
\end{aligned}$$

where Q_1 is given by (11). Consequently we have

$$\|(\mathscr{S}u)\|_\Pi \leqslant [\ell w+K](Q+Q_1)\leqslant w.$$

This shows that $\mathscr{S}B_w\subset B_w$. Next we show that the operator $\mathscr{S}$ is a contraction. Let

$u, v \in \Pi$. Then

$$\begin{aligned}
\|\mathscr{S}u-\mathscr{S}v\| &= \sup_{t\in[0,T]}\Big|\mathscr{S}u(t)-\mathscr{S}v(t)\Big| \\
&\leqslant \sup_{t\in[a,T]}\Big\{\int_a^t (t-s)\Big|f(s,u(s),u'(s))-f(s,v(s),v'(s))\Big|ds \\
&\quad +\frac{1}{2|A_1A_2|}\int_a^T\Big[2|\chi(t)|+|A_1|(T-s)\Big](T-s) \\
&\quad \times\Big|f(s,u(s),u'(s))-f(s,v(s),v'(s))\Big|ds \\
&\quad +\frac{1}{|A_1A_2|}\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\rho_j|\chi(t)|+\gamma_j|A_1|(s-p)\Big] \\
&\quad \times\Big|f(p,u(p),u'(p))-f(p,v(p),v'(p))\Big|dpds\Big\} \\
&\leqslant \ell\Big(\|u-v\|+\|u'-v'\|\Big)\Big\{\frac{(T-a)^2}{2}+\Big|\frac{A_2(T-a)-A_3}{2A_1A_2}\Big|\Big[(T-a)^2 \\
&\quad +\sum_{j=1}^m|\rho_j[(\eta_j-a)^2-(\xi_j-a)^2]|\Big] \\
&\quad +\frac{1}{|6A_2|}\Big[(T-a)^3+\sum_{j=1}^m|\gamma_j[(\eta_j-a)^3-(\xi_j-a)^3]|\Big]\Big\} \\
&\leqslant \ell Q(\|u-v\|+\|u'-v'\|)\leqslant \ell Q\|u-v\|_\Pi,
\end{aligned}$$

and

$$\begin{aligned}
\|(\mathscr{S}u)'-(\mathscr{S}v)'\| &= \sup_{t\in[0,T]}\Big|(\mathscr{S}u)'(t)-(\mathscr{S}v)'(t)\Big| \\
&\leqslant \sup_{t\in[a,T]}\Big\{\int_a^t\Big|f(s,u(s),u'(s))-f(s,v(s),v'(s))\Big|ds \\
&\quad +\frac{1}{|A_1|}\Big[\sum_{j=1}^m\rho_j\int_{\xi_j}^{\eta_j}\int_a^s\Big|f(p,u(p),u'(p))-f(p,v(p),v'(p))\Big|dpds \\
&\quad -\int_a^T(T-s)\Big|f(s,u(s),u'(s))-f(s,v(s),v'(s))\Big|ds\Big]\Big\} \\
&\leqslant \ell\Big(\|u-v\|+\|u'-v'\|\Big)\sup_{t\in[a,T]}\Big\{(t-a) \\
&\quad +\frac{1}{2|A_1|}\Big[\sum_{j=1}^m\rho_j[(\eta_j-a)^2-(\xi_j-a)^2]|+(T-a)^2\Big]\Big\} \\
&\leqslant \ell Q_1(\|u-v\|+\|u'-v'\|)\leqslant \ell Q_1\|u-v\|_\Pi,
\end{aligned}$$

where Q and Q_1 are defined by (10) and (11) respectively. In view of the foregoing inequalities, it follows that

$$\|\mathscr{S}u-\mathscr{S}v\|_\Pi\leqslant \ell(Q+Q_1)\|u-v\|_\Pi,$$

which, by assumption (H_2), implies that the operator $\mathscr{S}$ is a contraction. Thus, by Banach's contraction mapping principle, we deduce that the operator $\mathscr{S}$ has a fixed point, which corresponds to a unique solution of the problem (1) on $[a,T]$. This completes the proof. □

EXAMPLE 1. Consider the following boundary value problem:

$$\begin{cases} u''(t) = f(t,u(t),u'(t)),\ t\in[1,3], \\ \displaystyle\int_1^3 u(s)ds = \frac{1}{6}\int_{(3/2)}^{(7/4)} u(s)ds + \frac{1}{3}\int_{(2)}^{(9/4)} u(s)ds + \frac{1}{2}\int_{(5/2)}^{(11/4)} u(s)ds + 2, \\ \displaystyle\int_1^3 u'(s)ds = \frac{2}{3}\int_{(3/2)}^{(7/4)} u(s)ds + \frac{19}{24}\int_{(2)}^{(9/4)} u(s)ds + \frac{11}{12}\int_{(5/2)}^{(11/4)} u(s)ds + 1. \end{cases} \tag{14}$$

Here $a=1$, $T=3, m=3$, $\lambda_1=2$, $\lambda_2=1$, $\gamma_1=1/6$, $\gamma_2=1/3$, $\gamma_3=1/2$, $\rho_1=2/3$, $\rho_2=19/24$, $\rho_3=11/12$, $\xi_1=3/2$, $\eta_1=7/4$, $\xi_2=2$, $\eta_2=9/4$, $\xi_3=5/2$, $\eta_3=11/4$, and

$$f(t,u(t),u'(t)) = \frac{1}{25+2t^3}\left[\frac{|u|}{1+|u|}+\frac{|u'|}{1+|u'|}+e^{-t}\right].$$

Observe that $|f(t,u,u')| \leqslant \frac{1}{27}[2+e^{-t}]$ and $|f(t,u,u')-f(t,v,v')| \leqslant \ell\|u-v\|_\Pi$ with $\ell=\frac{2}{27}$. Using the given values, we find that $A_1=1.40625\neq 0, A_2=1.75\neq 0$, and $A_3=1.677083$ (A_1, A_2, and A_3, are respectively given by (5)), $Q=6.774391, Q_1=4.630556$ (Q and Q_1 are defined by (10) and (11) respectively). Further, it is easy to find that $\ell(Q+Q_1)\approx 0.844811<1$. Hence the conclusion of Theorem 1 applies to the boundary value problem (14).

We also see that all the conditions of Theorem 2 are satisfied with $\ell(Q+Q_1)\approx 0.844811<1$. Hence it follows by the conclusion of Theorem 2 that there exists a unique solution for boundary value problem (14) on $[1,3]$.

4. The multi-valued case

In this section we prove the existence of solutions for the multi-valued boundary value problem (2). We consider two cases (a) Carathéodory case (convex multi-valued maps) and (b) Lipschitz case (nonconvex multi-valued maps).

For the convenience of the reader, let us briefly describe some basic concepts of multivalued analysis [14, 21].

A multi-valued map $G: X\to\mathscr{P}(X)$ (i) *is convex (closed) valued* if $G(x)$ is convex (closed) for all $x\in X$; (ii) is *bounded* on bounded sets if $G(Y)=\cup_{x\in Y}G(x)$ is bounded in X for all $Y\in\mathscr{P}_b(X)$ (i.e. $\sup_{x\in Y}\{\sup\{|y|:y\in G(x)\}\}<\infty$); (iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $G(\mathscr{N}_0)\subseteq N$; (iv) *is completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B}\in\mathscr{P}_b(X)$; (v) *is measurable* if for every $y\in X$, the function $t\longmapsto d(y,G(t))=\inf\{|y-z|:z\in G(t)\}$ is measurable.

If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

A multi-valued map $G : X \to \mathscr{P}(X)$ has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

For a normed space $(X, \|\cdot\|)$, we define $\mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is closed}\}$, $\mathscr{P}_b(X) = \{Y \in \mathscr{P}(X) : Y \text{ is bounded}\}$, $\mathscr{P}_{cl,b}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is closed and bounded}\}$, $\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact}\}$ and $\mathscr{P}_{cp,c}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact and convex}\}$.

4.1. The Carathéodory case

In this subsection we consider the case when F has convex values and prove an existence result based on nonlinear alternative of Leray-Schauder type for multi-valued maps, assuming that F is Carathéodory.

DEFINITION 1. A multi-valued map $F : [a,T] \times \mathbb{R} \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,x,y)$ is measurable for each $x, y \in \mathbb{R}$;

(ii) $(x,y) \longmapsto F(t,x,y)$ is upper semicontinuous for almost all $t \in [a,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([a,T],\mathbb{R})$ such that

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y)\} \leqslant \varphi_\rho(t)$$

for all $x, y \in \mathbb{R}$ with $\|x\|, \|y\| \leqslant \rho$ and for a.e. $t \in [a,T]$.

For each $x \in \Pi$, define the set of selections of F by

$$S_{F,u} := \{v \in L^1([a,T],\mathbb{R}) : v(t) \in F(t,u(t),u'(t)) \text{ a.e } t \in [a,T]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$.

We need the following results to establish the main result in this subsection.

LEMMA 4. *([14, Proposition 1.2]) If $G : X \to \mathscr{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

LEMMA 5. *([28]) Let X be a separable Banach space. Let $F : J \times X^2 \to \mathscr{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathscr{P}_{cp,c}(C(J,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

LEMMA 6. *(Nonlinear alternative for Kakutani maps* [15]*). Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathscr{P}_{cp,c}(C)$ is an upper semicontinuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.

DEFINITION 2. A function $u \in C^1([a,T],\mathbb{R})$ is a solution of the problem (1) if $\int_a^T u(s)ds = \sum_{j=1}^m \gamma_j \int_{\xi_j}^{\eta_j} u(s)ds + \lambda_1$, $\int_a^T u'(s)ds = \sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j} u'(s)ds + \lambda_2$, and there exists function $v \in L^1([a,T],\mathbb{R})$ such that $v(t) \in F(t,u(t),u'(t))$ a.e. on $[a,T]$ and

$$\begin{aligned} u(t) = & \int_a^t (t-s)v(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t) + A_1(T-s)\Big](T-s)v(s)ds \\ & + \frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t) + \gamma_j A_1(s-p)\Big]v(p)dpds \\ & + \frac{1}{A_1A_2}\Big[A_1\lambda_1 + \chi(t)\lambda_2\Big]. \end{aligned} \tag{15}$$

THEOREM 3. *Assume that $F : [a,T]\times\mathbb{R}\times\mathbb{R} \to \mathscr{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory satisfying assumptions:*

(H_4) *there exist functions $p \in C([a,T],\mathbb{R}^+)$, and nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|F(t,u,u')\| = \sup\{|v| : v \in F(t,u,u')\} \leqslant \|p\|\Psi(\|u\|_\Pi),\ (t,u,u') \in [a,T]\times\mathbb{R}\times\mathbb{R};$$

(H_5) *there exists a constant $N > 0$ such that*

$$\frac{N}{\|p\|\Psi(N)(Q+Q_1)} > 1, \tag{16}$$

where Q and Q_1 are defined by (10) and (11) respectively.

Then the boundary value problem (1) has at least one solution on $[a,T]$.

Proof. To transform the problem (2) into a fixed point problem, we define an operator $\mathscr{F} : \Pi \longrightarrow \mathscr{P}(\Pi)$ by

$$\mathscr{F}(u) = \left\{ \begin{array}{l} h \in \Pi : \\ h(t) = \left\{ \begin{array}{l} \int_a^t (t-s)v(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t) + A_1(T-s)\Big](T-s)v(s)ds \\ + \frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t) + \gamma_j A_1(s-p)\Big]v(p)dpds \\ + \frac{1}{A_1A_2}\Big[A_1\lambda_1 + \chi(t)\lambda_2\Big], \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,u}$. It is obvious that the fixed points of $\mathscr{F}$ are solutions of the boundary value problem (2).

We will show that $\mathscr{F}$ satisfies the assumptions of Leray-Schauder nonlinear alternative (Lemma 6). The proof consists of several steps.

Step 1. $\mathscr{F}(u)$ *is convex for each* $u \in \Pi$.

This step is obvious since $S_{F,u}$ is convex (F has convex values), and therefore we omit the proof.

Step 2. $\mathscr{F}$ *maps bounded sets (balls) into bounded sets in* Π.

For the positive number r, let $B_r = \{u : u \in \Pi \text{ and } \|u\|_\Pi \leqslant r\}$ be a bounded set in Π. Then, for each $h \in \mathscr{F}(u), u \in B_r$, there exists $v \in S_{F,u}$ such that

$$\begin{aligned} h(t) = &\int_a^t (t-s)v(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t) + A_1(T-s)\Big](T-s)v(s)ds \\ &+ \frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t) + \gamma_j A_1(s-p)\Big]v(p)dpds + \frac{1}{A_1A_2}\Big[A_1\lambda_1 + \chi(t)\lambda_2\Big]. \end{aligned}$$

Then, for $t \in [a,T]$, we have

$$\begin{aligned} |h(t)| \leqslant &\sup_{t\in[a,T]}\Big\{\Big|\int_a^t (t-s)v(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t) + A_1(T-s)\Big](T-s)v(s)ds \\ &+ \frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t) + \gamma_j A_1(s-p)\Big]v(p)dpds \\ &+ \frac{1}{A_1A_2}\Big[A_1\lambda_1 + \chi(t)\lambda_2\Big]\Big|\Big\} \\ \leqslant &\|p\|\Psi(\|u\|_\Pi)Q \leqslant \|p\|\Psi(r)Q, \end{aligned}$$

and

$$\begin{aligned} |h'(t)| \leqslant &\sup_{t\in[a,T]}\Big\{\Big|\int_a^t v(s)ds + \frac{1}{A_1}\Big[\sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j}\int_a^s v(p)dpds \\ &- \int_a^T (T-s)v(s)ds + \lambda_2\Big]\Big|\Big\} \\ \leqslant &\|p\|\Psi(\|u\|_\Pi)Q_1 \leqslant \|p\|\Psi(r)Q_1, \end{aligned}$$

which yield

$$\|h\|_\Pi \leqslant \|p\|\Psi(r)(Q+Q_1).$$

Step 3. $\mathscr{F}$ *maps bounded sets into equicontinuous sets of* Π.

Let $t_1,t_2 \in [a,T]$ with $t_1 < t_2$ and $u \in B_r$. Then, for each $h \in \mathscr{F}(u)$, we obtain

$$\begin{aligned}
|h(t_2)-h(t_1)| &\leqslant \Big| \int_a^{t_1} \Big[(t_2-s)-(t_1-s)\Big]v(s)ds + \int_{t_1}^{t_2}(t_2-s)v(s)ds\Big| \\
&\quad + \Big|\frac{(t_2-t_1)}{A_1}\Big|\Big[\int_a^T (T-s)v(s)ds + \sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j}\int_a^s v(p)dpds + \lambda_2\Big] \\
&\leqslant \|p\|\Psi(r)\Big\{\Big[(t_2-t_1)(t_1-a)+\frac{(t_2-t_1)^2}{2}\Big] \\
&\quad + \frac{1}{|A_1|}(t_2-t_1)\Big[\frac{(T-a)^2}{2} + \sum_{j=1}^m \rho_j\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big) + |\lambda_2|\Big]\Big\} \\
&\quad \to 0 \text{ as } (t_2-t_1)\to 0, \text{ independently of } u \in B_r,
\end{aligned}$$

and

$$\begin{aligned}
|h'(t_2)-h'(t_1)| &\leqslant \Big|\int_{t_1}^{t_2} f(s,u(s),u'(s))ds\Big| \\
&\leqslant \|p\|\Psi(r)(t_2-t_1) \to 0 \text{ as } (t_2-t_1)\to 0, \text{ independently of } u \in B_r.
\end{aligned}$$

Therefore it follows by the Ascoli-Arzelá theorem that $\mathscr{F}:\Pi \to \mathscr{P}(\Pi)$ is completely continuous.

Since $\mathscr{F}$ is completely continuous, in order to prove that it is u.s.c., it is enough to prove that $\mathscr{F}$ has a closed graph. We establish it in the following step.

Step 4. $\mathscr{F}$ has a closed graph.

Let $u_n \to u_*, h_n \in \mathscr{F}(u_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathscr{F}(u_*)$. Associated with $h_n \in \mathscr{F}(u_n)$, there exists $v_n \in S_{F,u_n}$ such that for each $t \in [a,T]$,

$$\begin{aligned}
h_n(t) &= \int_a^t (t-s)v_n(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t)+A_1(T-s)\Big](T-s)v_n(s)ds \\
&\quad + \frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t)+\gamma_j A_1(s-p)\Big]v_n(p)dpds + \frac{1}{A_1A_2}\Big[A_1\lambda_1+\chi(t)\lambda_2\Big].
\end{aligned}$$

Thus it suffices to show that there exists $v_* \in S_{F,u_*}$ such that for each $t \in [a,T]$,

$$\begin{aligned}
h_*(t) &= \int_a^t (t-s)v_*(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t)+A_1(T-s)\Big](T-s)v_*(s)ds \\
&\quad + \frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t)+\gamma_j A_1(s-p)\Big]v_*(p)dpds + \frac{1}{A_1A_2}\Big[A_1\lambda_1+\chi(t)\lambda_2\Big].
\end{aligned}$$

Let us consider the linear operator $\Theta : L^1([a,T],\mathbb{R}) \to \Pi$ given by

$$\begin{aligned}
v \mapsto \Theta(v)(t) &= \int_a^t (t-s)v(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t)+A_1(T-s)\Big](T-s)v(s)ds \\
&\quad + \frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t)+\gamma_j A_1(s-p)\Big]v(p)dpds \\
&\quad + \frac{1}{A_1A_2}\Big[A_1\lambda_1+\chi(t)\lambda_2\Big].
\end{aligned}$$

Oberve that $\|h_n(t)-h_*(t)\| \to 0$ as $n\to\infty$, and thus, it follows by Lemma 5 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,u_n})$. Since $u_n\to u_*$, we have

$$h_*(t) = \int_a^t (t-s)v_*(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t)+A_1(T-s)\Big](T-s)v_*(s)ds + \frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t)+\gamma_j A_1(s-p)\Big]v_*(p)dpds + \frac{1}{A_1A_2}\Big[A_1\lambda_1+\chi(t)\lambda_2\Big],$$

for some $v_*\in S_{F,u_*}$.

Step 5. We show there exists an open set $U\subseteq\Pi$ with $u\notin\lambda\mathscr{F}(u)$ for any $\lambda\in(0,1)$ and all $u\in\partial U$.

Let $\lambda\in(0,1)$ and $u\in\lambda\mathscr{F}(u)$. Then there exists $v\in L^1([0,1],\mathbb{R})$ with $v\in S_{F,u}$ such that, for $t\in[a,T]$, we have

$$u(t) = \lambda\int_a^t (t-s)v(s)ds - \frac{\lambda}{2A_1A_2}\int_a^T \Big[2\chi(t)+A_1(T-s)\Big](T-s)v(s)ds + \frac{\lambda}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t)+\gamma_j A_1(s-p)\Big]v(p)dpds + \frac{\lambda}{A_1A_2}\Big[A_1\lambda_1+\chi(t)\lambda_2\Big].$$

Then, for $t\in[a,T]$, using the computations in the first step leads to

$$\|u\|_\Pi \leqslant \|p\|\Psi(\|u\|_\Pi)(Q+Q_1),$$

which can alternatively be expressed as

$$\frac{\|u\|_\Pi}{\|p\|\Psi(\|u\|_\Pi)(Q+Q_1)} \leqslant 1.$$

By the condition (H_5), we can find a positive number N such that $\|u\|_\Pi\neq N$. Let us set

$$U=\{u\in\Pi : \|u\|_\Pi < N\}.$$

Note that the operator $\mathscr{F}:\overline{U}\to\mathscr{P}(\Pi)$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $u\in\partial U$ such that $u\in\lambda\mathscr{F}(u)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 6), we deduce that $\mathscr{F}$ has a fixed point $u\in\overline{U}$ which is a solution of the problem (2). This completes the proof. □

4.2. The Lipschitz case

In this subsection we prove the existence of solutions for the problem (2) for non-convex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [12].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathscr{P}(X)\times\mathscr{P}(X)\to\mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B)=\max\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(A,b)\},$$

where $d(A,b)=\inf_{a\in A} d(a;b)$ and $d(a,B)=\inf_{b\in B} d(a;b)$. Then $(\mathscr{P}_{cl,b}(X),H_d)$ is a metric space (see [27]).

DEFINITION 3. A multivalued operator $N : X\to\mathscr{P}_{cl}(X)$ is called

(a) $\gamma-$Lipschitz if and only if there exists $\gamma>0$ such that

$$H_d(N(x),N(y))\leqslant \gamma d(x,y) \text{ for each } x,y\in X;$$

(b) a contraction if and only if it is $\gamma-$Lipschitz with $\gamma<1$.

LEMMA 7. *([12]) Let (X,d) be a complete metric space. If $N : X\to\mathscr{P}_{cl}(X)$ is a contraction, then $FixN\neq\emptyset$.*

THEOREM 4. *Assume that:*

(A_1) *$F:[a,T]\times\mathbb{R}^2\to\mathscr{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,u,v):[a,T]\to\mathscr{P}_{cp}(\mathbb{R})$ is measurable for each $u,v\in\mathbb{R}$;*

(A_2) *For almost all $t\in[a,T]$ and $u_1,u_2,w_1,w_2\in\mathbb{R}$ we have*

$$H_d(F(t,u_1,u_2),F(t,w_1,w_2))\leqslant m(t)(|u_1-w_1|+|u_2-w_2|)$$

with $m\in C(J,\mathbb{R}^+)$ and $d(0,F(t,0,0))\leqslant m(t)$, for almost all $t\in[a,T]$.

Then the boundary value problem (1) has at least one solution on $[a,T]$ if

$$\|m\|(Q+Q_1)<1,$$

where Q and Q_1 are defined by (10) and (11) respectively.

Proof. Consider the operator $\mathscr{F}$ defined at the begin of the proof of Theorem 3. Observe that the set $S_{F,u}$ is nonempty for each $u\in\Pi$ by the assumption (A_1), so F has a measurable selection (see Theorem III.6 [10]). Now we show that the operator $\mathscr{F}$ satisfies the assumptions of Lemma 7. We show that $\mathscr{F}(u)\in\mathscr{P}_{cl}(\Pi)$ for each $u\in\Pi$. Let $\{u_n\}_{n\geqslant 0}\in\mathscr{F}(u)$ be such that $u_n\to u$ $(n\to\infty)$ in Π. Then $u\in\Pi$ and there exists $v_n\in S_{F,u_n}$ such that, for each $t\in[a,T]$,

$$\begin{aligned} u_n(t) = &\int_a^t (t-s)v_n(s)ds-\frac{1}{2A_1A_2}\int_a^T\Big[2\chi(t)+A_1(T-s)\Big](T-s)v_n(s)ds\\ &+\frac{1}{A_1A_2}\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\rho_j\chi(t)+\gamma_jA_1(s-p)\Big]v_n(p)dpds+\frac{1}{A_1A_2}\Big[A_1\lambda_1+\chi(t)\lambda_2\Big]. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([a,T],\mathbb{R})$. Thus, $v \in S_{F,u}$ and for each $t \in [a,T]$, we have

$$\begin{aligned}u_n(t) \to u(t) = &\int_a^t (t-s)v(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t) + A_1(T-s)\Big](T-s)v(s)ds \\ &+\frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t) + \gamma_j A_1(s-p)\Big]v(p)dpds \\ &+\frac{1}{A_1A_2}\Big[A_1\lambda_1 + \chi(t)\lambda_2\Big].\end{aligned}$$

Hence, $u \in \mathscr{F}(u)$.

Next we show that there exists $\delta < 1$ $\Big(\delta := \|m\|(Q+Q_1)\Big)$ such that

$$H_d(\mathscr{F}(u), \mathscr{F}(\bar{u})) \leqslant \delta\|u-\bar{u}\|_\Pi \text{ for each } u, \bar{u} \in C^2([a,T],\mathbb{R}).$$

Let $u, \bar{u} \in C^2([a,T],\mathbb{R})$ and $h_1 \in \mathscr{F}(u)$. Then there exists $v_1(t) \in F(t,u(t),u'(t))$ such that, for each $t \in [a,T]$,

$$\begin{aligned}h_1(t) = &\int_a^t (t-s)v_1(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t) + A_1(T-s)\Big](T-s)v_1(s)ds \\ &+\frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t) + \gamma_j A_1(s-p)\Big]v_1(p)dpds + \frac{1}{A_1A_2}\Big[A_1\lambda_1 + \chi(t)\lambda_2\Big].\end{aligned}$$

By (A_2), we have

$$H_d(F(t,u(t),u'(t)), F(t,\bar{u}(t),\bar{u}'(t))) \leqslant m(t)(|u(t)-\bar{u}(t)| + |u'(t)-\bar{u}'(t)|)$$

so, there exists $z \in F(t,u(t),u'(t))$ such that

$$|v_1(t) - z| \leqslant m(t)(|u(t)-\bar{u}(t)| + |u'(t)-\bar{u}'(t)|)$$

for almost all $t \in [a,T]$. Define the multifunction $U : [a,T] \to \mathscr{P}(\mathbb{R})$ by

$$U(t) = \{z \in \mathbb{R} : |v_1(t) - z| \leqslant m(t)(|u(t)-\bar{u}(t)| + |u'(t)-\bar{u}'(t)|) \text{ for almost all } t \in [a,T]\}.$$

It is easy to check that the multifunction $U(\cdot) \cap F(\cdot,u(\cdot),u'(\cdot))$ is measurable. Hence, we can choose $v_2 \in S_{F,u}$ such that

$$|v_1(t) - v_2(t)| \leqslant m(t)(|u(t)-\bar{u}(t)| + |u'(t)-\bar{u}'(t)|)$$

for almost all $t \in [a,T]$.

For each $t \in [a,T]$, let us define

$$\begin{aligned}h_2(t) = &\int_a^t (t-s)v_2(s)ds - \frac{1}{2A_1A_2}\int_a^T \Big[2\chi(t) + A_1(T-s)\Big](T-s)v_2(s)ds \\ &+\frac{1}{A_1A_2}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j\chi(t) + \gamma_j A_1(s-p)\Big]v_2(p)dpds + \frac{1}{A_1A_2}\Big[A_1\lambda_1 + \chi(t)\lambda_2\Big].\end{aligned}$$

Thus,

$$\begin{aligned}
|h_1(t)-h_2(t)| \leqslant & \sup_{t\in[a,T]}\Big\{\int_a^t (t-s)|v_1(s)-v_2(s)|ds \\
& +\frac{1}{2|A_1A_2|}\int_a^T \Big[2|\chi(t)|+|A_1|(T-s)\Big](T-s)|v_1(s)-v_2(s)|ds \\
& +\frac{1}{|A_1A_2|}\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\rho_j|\chi(t)|+\gamma_j|A_1|(s-p)\Big]|v_1(p)-v_2(p)|dpds\Big]\Big\} \\
\leqslant & \|m\|\Big(\|u-\bar{u}\|+\|u'-\bar{u}'\|\Big)\Big\{\frac{(T-a)^2}{2}+\Big|\frac{A_2(T-a)-A_3}{2A_1A_2}\Big|\Big[(T-a)^2 \\
& +\sum_{j=1}^m |\rho_j[(\eta_j-a)^2-(\xi_j-a)^2]|\Big] \\
& +\frac{1}{|6A_2|}\Big[(T-a)^3+\sum_{j=1}^m |\gamma_j[(\eta_j-a)^3-(\xi_j-a)^3]|\Big]\Big\} \\
\leqslant & \|m\|Q(\|u-\bar{u}\|+\|u'-\bar{u}'\|) \leqslant \|m\|Q\|u-\bar{u}\|_\Pi,
\end{aligned}$$

$$\begin{aligned}
|h_1'(t)-h_2'(t)| \leqslant & \sup_{t\in[a,T]}\Big\{\int_a^t |v_1(s)-v_2(s)|ds+\frac{1}{|A_1|}\Big[\sum_{j=1}^m \rho_j\int_{\xi_j}^{\eta_j}\int_a^s |v_1(p)-v_2(p)|dpds \\
& -\int_a^T (T-s)|v_1(s)-v_2(s)|ds\Big]\Big\} \\
\leqslant & \|m\|\Big(\|u-\bar{u}\|+\|u'-\bar{u}'\|\Big)\sup_{t\in[a,T]}\Big\{(t-a) \\
& +\frac{1}{2|A_1|}\Big[\sum_{j=1}^m \rho_j[(\eta_j-a)^2-(\xi_j-a)^2]+(T-a)^2\Big]\Big\} \\
\leqslant & \|m\|Q_1(\|u-\bar{u}\|+\|u'-\bar{u}'\|) \leqslant \|m\|Q_1\|u-\bar{u}\|_\Pi.
\end{aligned}$$

Hence,

$$\|h_1-h_2\|_\Pi \leqslant \|m\|(Q+Q_1)\|u-\bar{u}\|_\Pi.$$

Analogously, interchanging the roles of u and $\bar{u}$, we obtain

$$H_d(\mathscr{F}(u),\mathscr{F}(\bar{u})) \leqslant \|m\|(Q+Q_1)\|u-\bar{u}\|_\Pi.$$

So $\mathscr{F}$ is a contraction. Therefore, it follows by Lemma 7 that $\mathscr{F}$ has a fixed point x which is a solution of (2). This completes the proof. □

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