

BOUNDS FOR GLOBAL SOLUTIONS OF A REACTION DIFFUSION SYSTEM WITH THE ROBIN BOUNDARY CONDITIONS

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Abstract. In this paper, we are concerned with the large-time behavior of solutions of a reaction diffusion system arising from a nuclear reactor model with the Robin boundary conditions, which consists of two real-valued unknown functions. It is shown that global solutions of this system are uniformly bounded in a suitable norm with respect to time.

1. Introduction

We consider the asymptotic behavior of global solutions of the initial boundary value problem for a reaction diffusion system:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & t > 0, x \in \Omega, \\ \partial_t u_2 - \Delta u_2 = a u_1, & t > 0, x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta u_2 = 0, & t > 0, x \in \partial\Omega, \\ u_1(0, x) = u_{10}(x) \geq 0, u_2(0, x) = u_{20}(x) \geq 0, & x \in \Omega. \end{cases} \quad (1)$$

Here Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$, and ν denotes the unit outward normal vector on $\partial\Omega$. Furthermore u_1, u_2 are real-valued unknown functions and a, b are given positive constants. We also assume $\alpha \geq 0$ and $\beta > 0$. This problem is introduced in 1968 by Kastenbergh and Chambré [13] for the purpose to give mathematical model of a nuclear reactor, where u_1 represents the neutron flux and u_2 represents the fuel temperature.

This model is studied by many authors under various (linear) boundary conditions, see, e.g., [6], [7], [10], [11], [12], [24] and [25]. They investigated the existence of positive steady-state solutions and the asymptotic behavior of solutions. In our previous

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work [14], we also studied the initial-boundary value problem for this system with nonlinear boundary conditions:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & t > 0, x \in \Omega, \\ \partial_t u_2 - \Delta u_2 = a u_1, & t > 0, x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & t > 0, x \in \partial\Omega, \\ u_1(0, x) = u_{10}(x) \geq 0, u_2(0, x) = u_{20}(x) \geq 0, & x \in \Omega, \end{cases} \tag{2}$$

where $\gamma \geq 2$. We showed the existence and the ordered uniqueness of positive stationary solution for $N \in [1, 5]$. For nonstationary problem, we proved that any positive stationary solution plays a role of threshold to separate global solutions and finite time blowing-up solutions. More precisely, if the initial data is less than or equal to positive stationary solutions, then solutions of (2) exists globally and tends to zero as $t \rightarrow \infty$, and if the initial data is strictly larger than positive stationary solutions, then solutions of (2) blow up in finite time. For general initial data, however, this result does not say anything about the asymptotic behavior of global solutions. When we assume that solutions exist globally, it is natural to ask whether global solutions blow up at ∞ or not. We here restrict ourselves to the case where $\gamma = 2$, for the technical reason. Bounds for global solutions of this system with the homogeneous Dirichlet boundary conditions is already studied by Quittner [22] for the case where $N = 2$. This strong restriction on N arises from applying Hardy type inequality (see [4]). As for the Robin boundary conditions, by making use of the good properties of the first eigenfunction of Laplacian with Robin boundary conditions, we can discuss the case where $N = 2, 3$.

This kind of problem is well known for the scalar problem:

$$\begin{cases} \partial_t u(t, x) - \Delta u(t, x) = f(u(t, x)), & t > 0, x \in \Omega, \\ u(t, x) = 0, & t > 0, x \in \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega. \end{cases} \tag{3}$$

For simplicity, assume that $f(u) = |u|^{p-2}u$ and p is Sobolev subcritical, that is, $p \in (2, p_S)$, where p_S is the Sobolev critical exponent defined by $p_S = \infty$ for $N = 1, 2$; $p_S = \frac{2N}{N-2}$ for $N = 3$. The boundedness of global solutions of (3) was first discussed by [19, 20] in the abstract setting of the form $u_t + \partial\phi^1(u) - \partial\phi^2(u) = 0$ in $L^2(\Omega)$. Here $\partial\phi^i$ are subdifferentials of lower semi-continuous convex and homogeneous functionals ϕ^i ($i = 1, 2$) on $L^2(\Omega)$, where it is shown that every global solution of (3) is uniformly bounded in $H_0^1(\Omega)$ with respect to time. Ni-Sacks-Tavantzis [18] studied (3) for the case where Ω is convex domain and proved every positive global solution of (3) is uniformly bounded in $L^\infty(\Omega)$ with respect to time provided that $p \in (2, 2 + \frac{2}{N})$. Furthermore they also showed that if $p \geq p_S$, then (3) has a global solution whose L^∞ norm goes to ∞ as $t \rightarrow \infty$ in the case where $N \geq 3$. Cazenave-Lions [5] dealt with more general nonlinear term $f(u)$ (including $f(u) = |u|^{p-2}u$) and showed that every global solution allowing sing-changed solution is bounded in $L^\infty(\Omega)$ uniformly in time provided that $p \in (2, p_{CL})$, where $p_{CL} = \infty$ when $N = 1$; $p_{CL} = 2 + \frac{12}{3N-4}$ when $N \geq 2$. (Note that $p_{CL} \leq p_S$ for any $N \in \mathbb{N}$). Giga removed this restriction on p in his paper [9]

for positive global solutions, that is, he showed every positive global solution of (3) is uniformly bounded in $L^\infty(\Omega)$ for any $p \in (2, p_S)$. Quittner [23] removed the restriction of the positivity of solutions, i.e., he proved that every global solution of (3) (allowing sign-changed solution) is uniformly bounded in $L^\infty(\Omega)$ for any $p \in (2, p_S)$.

Proofs for the boundedness of global solutions of (3) deeply rely on the fact that the energy functional $E(u)$, defined by $E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{p} \int_\Omega |u|^p dx$, becomes a Lyapunov function, in other words, (3) possesses the variational structure. In addition to that, in [9] the rescaling argument is introduced and in [23] the bootstrap argument based on the interpolation and the maximal regularity is used.

Unfortunately for our system, we can not apply the arguments similar to those of [9] and [23], since (1) does not possess the variational structure.

To cope with this difficulty, making much use of the special form of our system, we first show the uniform bound for the L^1 -norm with the positive weight φ_1 , the first eigenfunction of the Laplace operator with the Robin boundary condition. To derive the uniform H^1 -bound, we rely on some energy method with a special device (see Lemma 3.2). Furthermore by applying Moser’s iteration scheme such as in Nakao [17], we derive the uniform L^∞ -bound via H^1 -bound.

2. Existence of local solutions

Throughout this paper, we denote by $\|\cdot\|_p$ and $\|\cdot\|$ the norm in $L^p(\Omega)$ ($1 \leq p \leq \infty$) and $H^1(\Omega)$ respectively. We also simply write $u(t)$ instead of $u(t, \cdot)$. In this section, we prepare a couple of results concerning the local well-posedness. The following result is proved in [14] as Theorem 3.1.

THEOREM 2.1. *Let $(u_{10}, u_{20}) \in (L^\infty(\Omega))^2$, then there exists $T = T(\|u_{i0}\|_\infty) > 0$ ($i = 1, 2$) such that (2) possesses a unique solution $(u_1, u_2) \in (L^\infty(0, T; L^\infty(\Omega)) \cap C([0, T]; L^2(\Omega)))^2$ satisfying*

$$\sqrt{t} \partial_t u_1, \sqrt{t} \partial_t u_2, \sqrt{t} \Delta u_1, \sqrt{t} \Delta u_2 \in L^2(0, T; L^2(\Omega)). \tag{4}$$

Furthermore, if the initial data is nonnegative, then the local solution (u_1, u_2) for (2) is nonnegative.

In order to treat the case where the data belong to $H^1(\Omega)$, we need to fix some abstract setting. Let $H := L^2(\Omega) \times L^2(\Omega)$ and for $u = (u_1, u_2) \in H$ we put

$$D(\phi) := \{ u \in H; u_1, u_2 \in H^1(\Omega), u_2 \in L^\gamma(\partial\Omega) \},$$

$$\phi(u) = \begin{cases} \frac{1}{2} \int_\Omega (|\nabla u_1(x)|^2 + b|u_1(x)|^2 + |\nabla u_2(x)|^2) dx \\ \quad + \int_{\partial\Omega} \left(\frac{\alpha}{2} |u_1(x)|^2 + \frac{\beta}{\gamma} |u_2(x)|^\gamma \right) d\sigma & \text{if } u \in D(\phi), \\ +\infty & \text{if } u \notin D(\phi). \end{cases}$$

Then ϕ is a lower semi-continuous convex function from H into $[0, \infty)$ and its subdifferential $\partial\phi$ is given by

$$\partial\phi(u) = \{ w \in H; w = (-\Delta u_1 + bu_1, -\Delta u_2) \} \quad \forall u \in D(\partial\phi),$$

$$D(\partial\phi) = \{ u = (u_1, u_2); u_1, u_2 \in H^2(\Omega), \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0 \}.$$

Then we have

THEOREM 2.2. *Let $N \leq 5$. Assume that $(u_{10}, u_{20}) \in D(\phi)$. Then there exists $T = T(\phi(u_0)) > 0$ such that (2) possesses a unique solution $(u_1, u_2) \in (C([0, T]; L^2(\Omega)))^2$ satisfying*

$$\partial_t u_1, \partial_t u_2, \Delta u_1, \Delta u_2 \in L^2(0, T; L^2(\Omega)). \tag{5}$$

Furthermore, if the initial data is nonnegative, then the local solution (u_1, u_2) for (2) is nonnegative.

Proof. Put $u(t) = (u_1(t), u_2(t))$ and

$$B(u) := \{ b \in H; b = (-u_1 u_2, -\alpha u_1) \},$$

then (2) can be reduced to the following abstract evolution equation in H :

$$\frac{d}{dt} u(t) + \partial\phi(u(t)) + B(u(t)) = 0, \quad u(0) = (u_{10}, u_{20}). \tag{6}$$

We are going to apply Theorem II of [21]. To do this, we have to check three assumptions. The compactness assumption (A.1) requires that the set $\{ u \in H; \phi(u) + |u|_H^2 \leq L \}$ is compact in H for all $L > 0$, which is assured by the Rellich-Kondrachov theorem. The demiclosedness assumption (A.2) on $B(u)$ is assured by the continuity of the mapping $(u_1, u_2) \mapsto (-u_1 u_2, -\alpha u_1)$ in \mathbb{R}^2 .

The last assumption to check is the boundedness assumption (A.4):

$$|B(u)|_H^2 \leq k |\partial\phi(u)|_H^2 + \ell(\phi(u) + |u|_H) \quad \forall u \in D(\partial\phi), \tag{7}$$

where $k \in [0, 1)$ and $\ell(\cdot) : [0, \infty) \rightarrow [0, \infty)$ is a monotone increasing function. We note that

$$|B(u)|_H^2 \leq \|u_1\|_4^2 \|u_2\|_4^2 + \alpha^2 \|u_1\|_2^2, \quad \exists C > 0 \text{ such that } C(\|u_1\|^2 + \|u_2\|^2) \leq \phi(u) + 1. \tag{8}$$

Hence for $N \leq 4$, (7) holds true with $k = 0$ and $\ell(r) = Cr^2$. As for the case where $N = 5$, Gagliardo-Nirenberg interpolation inequality gives

$$\|v\|_4 \leq C \|v\|_{H^2}^{\frac{1}{4}} \|v\|_2^{\frac{3}{4}}.$$

Then by Young's inequality, (7) is satisfied with $\ell(r) = Cr^3$. Thus the local existence part is verified.

To prove the uniqueness part, let $u^1 = (u_1^1, u_2^1)$, $u^2 = (u_1^2, u_2^2)$ be solutions of (2) and put $\delta u_i = u_i^1 - u_i^2$ ($i = 1, 2$). Then δu_i satisfy

$$\partial_t \delta u_1 - \Delta \delta u_1 + b \delta u_1 = \delta u_1 u_2^1 + \delta u_2 u_1^2, \tag{9}$$

$$\partial_t \delta u_2 - \Delta \delta u_2 = a \delta u_1, \tag{10}$$

$$\partial_\nu \delta u_1 + \alpha \delta u_1 = \partial_\nu \delta u_2 + \beta (|u_2^1|^{\gamma-2} u_2^1 - |u_2^2|^{\gamma-2} u_2^2) = 0. \tag{11}$$

Multiplying (9) by δu_1 and (10) by δu_2 , we have by (11)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\delta u_1(t)\|_2^2 + \|\nabla \delta u_1\|_2^2 + \alpha \|\delta u_1\|_{2, \partial \Omega}^2 + b \|\delta u_1\|_2^2 \\ & \leq \int_{\Omega} (|\delta u_1|^2 |u_2^1| + |\delta u_1| |\delta u_2| |u_1^1|) dx, \end{aligned} \tag{12}$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\delta u_2(t)\|_2^2 + \|\nabla \delta u_2\|_2^2 + \beta \int_{\partial \Omega} (|u_2^1|^{\gamma-2} u_2^1 - |u_2^2|^{\gamma-2} u_2^2) \delta u_2 d\sigma \\ & \leq a \int_{\Omega} |\delta u_1| |\delta u_2| dx, \end{aligned} \tag{13}$$

where $\|v\|_{2, \partial \Omega}^2 = \int_{\partial \Omega} v^2 d\sigma$. Let $N \leq 5$, then since $H^1(\Omega)$ and $H^2(\Omega)$ are embedded in $L^{\frac{10}{3}}(\Omega)$ and $L^{10}(\Omega)$ respectively, by Young’s inequality we find that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} \int_{\Omega} |\delta u_i| |\delta u_j| |w| dx & \leq C \|\delta u_i\| \|\delta u_j\|_2 \|w\|_{H^2(\Omega)} \\ & \leq \varepsilon (\|\nabla \delta u_i\|_2^2 + \|\delta u_i\|_2^2) + C_\varepsilon \|\delta u_j\|_2^2 \|w\|_{H^2(\Omega)}^2. \end{aligned}$$

Hence, by adding (12) and (13), we obtain

$$\frac{d}{dt} (\|\delta u_1(t)\|_2^2 + \|\delta u_2(t)\|_2^2) \leq C (\|u_2^1\|_{H^2(\Omega)}^2 + \|u_1^1\|_{H^2(\Omega)}^2 + 1) (\|\delta u_1(t)\|_2^2 + \|\delta u_2(t)\|_2^2),$$

Thus since $u_2^1, u_1^1 \in L^2(0, T; H^2(\Omega))$, the uniqueness follows from Gronwall’s inequality. The nonnegativity of solutions can be proved by exactly the same argument as in the proof of Theorem 3.1 in [14].

3. Main result and proof

In what follows we always consider the case where $\gamma = 2$ and we are concerned with global solutions of (1). We put $H^1 = \{(w_1, w_2) \in H^1(\Omega) \times H^1(\Omega) ; w_1, w_2 \geq 0, w_1, w_2 \not\equiv 0\}$ and $V = \{(w_1, w_2) \in L^\infty(\Omega) \times L^\infty(\Omega) ; w_1, w_2 \geq 0, w_1, w_2 \not\equiv 0\}$. Our main theorem can be stated as follows.

THEOREM 3.1. *Let $N = 2, 3$ and $\alpha \leq 2\beta$. Assume that $(u_{10}, u_{20}) \in H^1$ and (u_1, u_2) is the corresponding global solution of (1) satisfying the same regularity given*

in Theorem 2.2. Then there exist constants $M_i = M_i(\|u_{10}\|, \|u_{20}\|) > 0$ ($i = 1, 2$) such that

$$\sup_{t \geq 0} \|u_1(t)\| \leq M_1, \quad \sup_{t \geq 0} \|u_2(t)\| \leq M_2. \tag{14}$$

Moreover if $(u_{10}, u_{20}) \in V$ and (u_1, u_2) is the corresponding global solution of (1) satisfying the same regularity given in Theorem 2.1. Then there exist constants $M'_i = M'_i(\|u_{10}\|_\infty, \|u_{20}\|_\infty) > 0$ ($i = 1, 2$) such that

$$\sup_{t \geq 0} \|u_1(t)\|_\infty \leq M'_1, \quad \sup_{t \geq 0} \|u_2(t)\|_\infty \leq M'_2. \tag{15}$$

We divide the proof into several steps. We first derive the L^1 -estimate of the solutions. In this step, we rely on the properties of the first eigenvalue and the corresponding eigenfunction of $-\Delta$ with the Robin boundary conditions :

LEMMA 3.2. ([8]) Let λ_1 and φ_1 be the first eigenvalue and the corresponding eigenfunction for the problem:

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega, \\ \partial_\nu\varphi + \gamma\varphi = 0, & x \in \partial\Omega, \end{cases} \tag{16}$$

where Ω is smooth bounded domain in \mathbb{R}^N and $\gamma > 0$. Then $\lambda_1 > 0$ and there exists a constant $C_\gamma > 0$ such that

$$\varphi_1(x) \geq C_\gamma \quad x \in \overline{\Omega}.$$

Actually, it is easy to see that $\varphi_1 > 0$ in Ω by the strong maximum principle as the same method for the eigenvalue problem with the Dirichlet Laplacian. Furthermore suppose that there exists $x_0 \in \partial\Omega$ such that $\varphi_1(x_0) = 0$. Then the boundary condition assures $\partial_\nu\varphi_1(x_0) = -\gamma\varphi_1(x_0) = 0$. On the other hand, we know $\partial_\nu\varphi_1(x_0) < 0$ by Hopf’s strong maximum principle. This is contradiction, i.e., $\varphi_1(x) > 0$ on $\overline{\Omega}$.

The second step is to derive uniform L^2 -estimates and third one is to derive uniform H^1 -estimates. In the last step, we get uniform L^∞ bounds for global solutions of (1) applying Moser’s iteration scheme (see [1] and [17]).

(1) Uniform estimates in L^1

Let λ_1 and φ_1 be the first eigenvalue and the corresponding eigenfunction of (16) respectively. We here normalize φ_1 so that $\|\varphi_1\|_1 = 1$. Multiplying φ_1 by the first and second equations of (1), we get

$$\left(\int_\Omega u_1\varphi_1 dx\right)_t + (b + \lambda_1) \int_\Omega u_1\varphi_1 dx + (\alpha - \gamma) \int_{\partial\Omega} u_1\varphi_1 d\sigma = \int_\Omega u_1 u_2 \varphi_1 dx, \tag{17}$$

$$\left(\int_\Omega u_2\varphi_1 dx\right)_t + \lambda_1 \int_\Omega u_2\varphi_1 dx + (\beta - \gamma) \int_{\partial\Omega} u_2\varphi_1 d\sigma = a \int_\Omega u_1\varphi_1 dx. \tag{18}$$

Multiplying (17) by a and substituting (18) and equation (1) to the second term of the left-hand side and the right-hand side respectively, we have

$$\begin{aligned}
 & a\left(\int_{\Omega} u_1 \varphi_1 dx\right)_t + (b + \lambda_1)\left(\left(\int_{\Omega} u_2 \varphi_1 dx\right)_t + \lambda_1 \int_{\Omega} u_2 \varphi_1 dx + (\beta - \gamma) \int_{\partial\Omega} u_2 \varphi_1 d\sigma\right) \\
 & + a(\alpha - \gamma) \int_{\partial\Omega} u_1 \varphi_1 d\sigma = \int_{\Omega} (\partial_t u_2 - \Delta u_2) u_2 \varphi_1 dx
 \end{aligned} \tag{19}$$

Then differentiating (18) with respect to t once and substituting (19) to the right-hand side, we obtain

$$\begin{aligned}
 & \left(\int_{\Omega} u_2 \varphi_1 dx\right)_{tt} + (b + 2\lambda_1)\left(\int_{\Omega} u_2 \varphi_1 dx\right)_t + \lambda_1(b + \lambda_1) \int_{\Omega} u_2 \varphi_1 dx \\
 & + a(\alpha - \gamma) \int_{\partial\Omega} u_1 \varphi_1 d\sigma + (\beta - \gamma)\left(\int_{\partial\Omega} u_2 \varphi_1 d\sigma\right)_t + (\beta - \gamma)(b + \lambda_1) \int_{\partial\Omega} u_2 \varphi_1 d\sigma \\
 & = \int_{\Omega} (\partial_t u_2 - \Delta u_2) u_2 \varphi_1 dx \\
 & = \frac{1}{2}\left(\int_{\Omega} u_2^2 \varphi_1 dx\right)_t + \int_{\Omega} |\nabla u_2|^2 \varphi_1 dx + \frac{\lambda_1}{2} \int_{\Omega} u_2^2 \varphi_1 dx + \left(\beta - \frac{\gamma}{2}\right) \int_{\partial\Omega} u_2^2 \varphi_1 d\sigma.
 \end{aligned} \tag{20}$$

Finally choosing $\gamma = \frac{\alpha + 2\beta}{2} > 0$, we deduce

$$\begin{aligned}
 & \left(\int_{\Omega} u_2 \varphi_1 dx\right)_{tt} + (b + 2\lambda_1)\left(\int_{\Omega} u_2 \varphi_1 dx\right)_t + \lambda_1(b + \lambda_1) \int_{\Omega} u_2 \varphi_1 dx \\
 & - \frac{\alpha}{2}\left(\int_{\partial\Omega} u_2 \varphi_1 d\sigma\right)_t - \frac{\alpha}{2}\lambda_1 \int_{\partial\Omega} u_2 \varphi_1 d\sigma \\
 & \geq \frac{1}{2}\left(\int_{\Omega} u_2^2 \varphi_1 dx\right)_t + \frac{\lambda_1}{2} \int_{\Omega} u_2^2 \varphi_1 dx.
 \end{aligned} \tag{21}$$

We now set

$$y(t) := w'(t) + (b + \lambda_1) w(t) - \frac{1}{2} \int_{\Omega} u_2^2 \varphi_1 dx - \frac{\alpha}{2} \int_{\partial\Omega} u_2 \varphi_1 d\sigma, \quad w(t) := \int_{\Omega} u_2 \varphi_1 dx.$$

Since $\partial_t u_2 \in L^2(0, T; L^2(\Omega))$ implies that there exists $s_0 \in (0, 1)$ such that $|y(s_0)| < \infty$. Then (21) yields

$$y'(t) \geq -\lambda_1 y(t), \quad \text{hence} \quad y(t) \geq y(s_0) e^{-\lambda_1(t-s_0)} \geq -|y(s_0)| =: -C_0 \quad \forall t \geq s_0.$$

Hence by virtue of Schwarz’s inequality and Young’s inequality, we get

$$\begin{aligned}
 -C_0 \leq y(t) &= w'(t) + (b + \lambda_1) w(t) - \frac{1}{2} \int_{\Omega} u_2^2 \varphi_1 dx - \frac{\alpha}{2} \int_{\partial\Omega} u_2 \varphi_1 d\sigma \\
 &\leq w'(t) + (b + \lambda_1) w(t) - \frac{1}{2} w^2(t) \\
 &\leq w'(t) - \frac{1}{4} w^2(t) + (b + \lambda_1)^2 \quad \forall t \geq s_0,
 \end{aligned}$$

i.e.,

$$w'(t) \geq \frac{1}{4}w^2(t) - C_1, \quad C_1 := C_0 + (b + \lambda_1)^2 > 0 \quad \forall t \geq s_0, \tag{22}$$

whence follows

$$w(t) \leq 2C_1^{\frac{1}{2}} =: C_2 \quad \forall t \geq s_0, \tag{23}$$

Indeed, if there exists $t_1 \geq s_0$ such that

$$\frac{1}{4}w^2(t_1) - C_1 > 0, \tag{24}$$

then from (22), (24) we can deduce that there exists $t_2 > t_1$ such that

$$\lim_{t \rightarrow t_2} w(t) = +\infty,$$

which contradicts the assumption that $w(t)$ exists globally. Thus (23) holds and the following global bound for $w(t)$ is established.

$$\sup_{t \geq 0} \int_{\Omega} u_2 \varphi_1 dx \leq \bar{C}_2 := \max\left(C_2, \max_{0 \leq s \leq s_0} w(s)\right). \tag{25}$$

Next we derive a uniform estimate for $\int_{\Omega} u_1 \varphi_1 dx$. Using the facts that $u_1 = \frac{1}{a}(\partial_t u_2 - \Delta u_2)$ and (u_1, u_2) are nonnegative in (17), we can get

$$\begin{aligned} \frac{d}{dt} \left(\int_{\Omega} u_1 \varphi_1 dx \right) &\geq -(b + \lambda_1) \int_{\Omega} u_1 \varphi_1 dx = -(b + \lambda_1) \frac{1}{a} \int_{\Omega} (\partial_t u_2 - \Delta u_2) \varphi_1 dx \\ &= -\frac{b + \lambda_1}{a} w'(t) - \frac{(b + \lambda_1)\lambda_1}{a} w(t) + \frac{(b + \lambda_1)\alpha}{2a} \int_{\partial\Omega} u_2 \varphi_1 d\sigma \\ &\geq -\frac{b + \lambda_1}{a} w'(t) - \frac{(b + \lambda_1)\lambda_1}{a} w(t). \end{aligned}$$

For $\eta \in (0, 1)$, integrating this inequality over $(t, t + \eta)$ and using (25), we obtain

$$\begin{aligned} \left[\int_{\Omega} u_1 \varphi_1 dx \right]_t^{t+\eta} &\geq -\frac{b + \lambda_1}{a} (w(t + \eta) - w(t)) - \frac{(b + \lambda_1)\lambda_1}{a} \int_t^{t+\eta} w(\tau) d\tau \\ &\geq -\frac{b + \lambda_1}{a} \bar{C}_2 - \frac{(b + \lambda_1)\lambda_1}{a} \bar{C}_2 =: -C_3, \end{aligned}$$

where $C_3 > 0$ is independent of t and η . This implies that

$$\int_{\Omega} u_1(t) \varphi_1 dx \leq C_3 + \int_{\Omega} u_1(t + \eta) \varphi_1 dx. \tag{26}$$

Integrating (26) over $\eta \in (0, 1)$ and using integration by parts, we get

$$\begin{aligned} \int_{\Omega} u_1(t)\varphi_1 dx &\leq C_3 + \int_0^1 \int_{\Omega} u_1(t+\eta)\varphi_1 dx d\eta \\ &= C_3 + \int_t^{t+1} \int_{\Omega} u_1(\tau)\varphi_1 dx d\tau \\ &= C_3 + \frac{1}{a} \int_t^{t+1} \int_{\Omega} (\partial_t u_2 - \Delta u_2)\varphi_1 dx d\tau \\ &= C_3 + \frac{1}{a} (w(t+1) - w(t)) + \frac{\lambda_1}{a} \int_t^{t+1} w(\tau) d\tau - \frac{\alpha}{2a} \int_t^{t+1} \int_{\partial\Omega} u_2 \varphi_1 d\sigma d\tau \\ &\leq C_3 + \frac{1 + \lambda_1}{a} \bar{C}_2 =: C_4, \end{aligned}$$

which concludes that

$$\sup_{t \geq 0} \int_{\Omega} u_1 \varphi_1 dx \leq C_4. \tag{27}$$

Thus, from (25), (27) and Lemma 3.2, we can derive the following estimates:

$$\sup_{t \geq 0} \|u_1(t)\|_1 \leq C_5, \quad \sup_{t \geq 0} \|u_2(t)\|_1 \leq C_6. \tag{28}$$

(2) Uniform estimates in L^2

We here try to get L^2 uniform bounds of solutions of (1). Since (17) gives

$$\int_{\Omega} u_1 u_2 \varphi_1 dx \leq \frac{d}{dt} \left(\int_{\Omega} u_1 \varphi_1 dx \right) + (b + \lambda_1) \int_{\Omega} u_1 \varphi_1 dx,$$

it follows from (27) that

$$\sup_{t \geq 0} \int_t^{t+1} \int_{\Omega} u_1 u_2 dx d\tau \leq C_7. \tag{29}$$

Multiplying the second equation of (1) by u_2 and using integration by parts, we get

$$\frac{1}{2} \frac{d}{dt} \|u_2(t)\|_2^2 + \|\nabla u_2(t)\|_2^2 + \beta \|u_2(t)\|_{2,\partial\Omega}^2 = a \int_{\Omega} u_1 u_2 dx.$$

Hence by virtue of Poincaré - Friedrichs' inequality $\|v\|_2^2 \leq C_F^{-1} (\|\nabla v\|_2^2 + \beta \|v\|_{2,\partial\Omega}^2)$, we have

$$\frac{1}{2} \frac{d}{dt} \|u_2(t)\|_2^2 + C_F \|u_2(t)\|_2^2 \leq a \int_{\Omega} u_1 u_2 dx. \tag{30}$$

Applying Gronwall's inequality to (30), we get

$$\|u_2(t)\|_2^2 \leq e^{-2C_F t} \|u_{20}\|_2^2 + \int_0^t 2a \left(\int_{\Omega} u_1 u_2 dx \right) e^{-2C_F(t-\tau)} d\tau. \tag{31}$$

In order to obtain uniform bounds of L^2 -norm for u_2 with respect to t , we need to confirm that the second term of right hand side of (31) is bounded. For any $t \geq 0$, we can express $t = n + \varepsilon$ with some $n \in \mathbb{N} \cup \{0\}$ and $\varepsilon \in [0, 1)$. Then, by virtue of (29), it follows that

$$\begin{aligned} & \int_0^t \left(\int_{\Omega} u_1 u_2 dx \right) e^{-2C_F(t-\tau)} d\tau \\ &= \int_{t-1}^t \left(\int_{\Omega} u_1 u_2 dx \right) e^{-2C_F(t-\tau)} d\tau + \int_{t-2}^{t-1} \left(\int_{\Omega} u_1 u_2 dx \right) e^{-2C_F(t-\tau)} d\tau \\ & \quad + \dots + \int_{t-n}^{t-(n-1)} \left(\int_{\Omega} u_1 u_2 dx \right) e^{-2C_F(t-\tau)} d\tau + \int_0^{t-n} \left(\int_{\Omega} u_1 u_2 dx \right) e^{-2C_F(t-\tau)} d\tau \\ &\leq e^{-0} \int_{t-1}^t \left(\int_{\Omega} u_1 u_2 dx \right) d\tau + e^{-2C_F} \int_{t-2}^{t-1} \left(\int_{\Omega} u_1 u_2 dx \right) d\tau \\ & \quad + \dots + e^{-2(n-1)C_F} \int_{t-n}^{t-(n-1)} \left(\int_{\Omega} u_1 u_2 dx \right) d\tau + e^{-2nC_F} \int_0^{t-n} \left(\int_{\Omega} u_1 u_2 dx \right) d\tau \\ &\leq C_7 \left(1 + e^{-2C_F} + e^{-4C_F} + \dots + e^{-2nC_F} \right) \\ &= C_7 \frac{1 - e^{-2(n+1)C_F}}{1 - e^{-2C_F}} \leq \frac{C_7}{1 - e^{-2C_F}}. \end{aligned}$$

Therefore we obtain from (31)

$$\|u_2(t)\|_2^2 \leq e^{-2C_F t} \|u_{20}\|_2^2 + \frac{2aC_7}{1 - e^{-2C_F}} \quad \forall t \geq 0.$$

This implies that there exists $C_8 > 0$ such that

$$\sup_{t \geq 0} \|u_2(t)\|_2 \leq C_8. \tag{32}$$

Note that the above argument can be done without any restriction on dimension N .

We next derive a uniform L^2 -estimate of u_1 for $N \leq 3$. Multiplying the first equation of (1) by u_1 and using integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \|u_1(t)\|_2^2 + \|\nabla u_1(t)\|_2^2 + \alpha \|u_1(t)\|_{2,\partial\Omega}^2 + b \|u_1(t)\|_2^2 = \int_{\Omega} u_1^2 u_2 dx.$$

We here adopt $(\|\nabla v\|_2^2 + b \|v\|_2^2)^{1/2}$ as the H^1 norm for u_1 . By using Hölder's inequality, the interpolation inequality and the embedding theorem ($\|v\|_6 \leq C_9 \|v\|$), it holds

that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_1(t)\|_2^2 + \|u_1(t)\|^2 &\leq \int_{\Omega} u_1^2 u_2 \, dx \\ &\leq \|u_1(t)\|_4^2 \|u_2(t)\|_2 \\ &\leq \|u_1(t)\|_1^{\frac{1}{3}} \|u_1(t)\|_6^{\frac{9}{5}} \|u_2(t)\|_2 \\ &\leq C_5^{\frac{1}{3}} C_8 C_9^{\frac{9}{5}} \|u_1(t)\|^{\frac{9}{5}} \leq \frac{1}{2} \|u_1(t)\|^2 + C_{10}, \end{aligned}$$

which implies

$$\frac{1}{2} \frac{d}{dt} \|u_1(t)\|_2^2 + \frac{1}{2} \|u_1(t)\|^2 \leq C_{10}.$$

Hence we obtain

$$\|u_1(t)\|_2^2 \leq e^{-t} \|u_{10}\|_2^2 + 2C_{10} (1 - e^{-t}),$$

i.e.,

$$\sup_{t \geq 0} \|u_1(t)\|_2 \leq C_{11}. \tag{33}$$

(3) Uniform estimates in H^1

Now we are in the position to derive a uniform H^1 bounds of solutions of (1). Multiplying the second equation of (1) by $-\Delta u_2$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla u_2(t)\|_2^2 + \beta \|u_2(t)\|_{2,\partial\Omega}^2) + \|\Delta u_2(t)\|_2^2 &= -a \int_{\Omega} u_1 \Delta u_2 \, dx \\ &\leq \frac{1}{2} \|\Delta u_2(t)\|_2^2 + \frac{a^2}{2} \|u_1(t)\|_2^2. \end{aligned}$$

Here we define the H^1 -norm of u_2 by

$$\|u_2\|^2 := \|\nabla u_2\|_2^2 + \beta \|u_2\|_{2,\partial\Omega}^2.$$

Then it holds that $C_F \|u_2\|^2 \leq \|\Delta u_2\|_2^2$, since

$$(C_F)^{\frac{1}{2}} \|u_2\|_2 \|u_2\| \leq \|\nabla u_2\|_2^2 + \beta \|u_2\|_{2,\partial\Omega}^2 = (-\Delta u_2, u_2) \leq \|\Delta u_2\|_2 \|u_2\|_2,$$

where (\cdot, \cdot) denotes the inner product of L^2 . Hence we obtain

$$\frac{d}{dt} \|u_2(t)\|^2 + C_F \|u_2(t)\|^2 \leq a^2 C_{11}^2,$$

whence follows

$$\sup_{t \geq 0} \|u_2(t)\| \leq C_{12}. \tag{34}$$

In order to derive the uniform H^1 -estimate for u_1 , we prepare the following functional $\phi_1(u_1)$:

$$\phi_1(u_1) := \frac{1}{2}(\|\nabla u_1\|_2^2 + \alpha \|u_1\|_{2,\partial\Omega}^2 + b \|u_1\|_2^2) \quad u_1 \in H^1(\Omega).$$

Then it is easy to see

$$\phi_1(u_1) = \frac{1}{2}(\|\nabla u_1\|_2^2 + \alpha \|u_1\|_{2,\partial\Omega}^2 + b \|u_1\|_2^2) \geq \frac{b}{2} \|u_1\|_2^2, \tag{35}$$

$$\|-\Delta u_1 + b u_1\|_2 \|u_1\|_2 \geq |(-\Delta u_1 + b u_1, u_1)| = 2 \phi_1(u_1) \geq 2 \sqrt{\phi_1(u_1)} \sqrt{\frac{b}{2}} \|u_1\|_2,$$

whence follows

$$2b \phi_1(u_1) \leq \|-\Delta u_1 + b u_1\|_2^2. \tag{36}$$

Multiplication of the first equation of (1) by $-\Delta u_1 + b u_1$ and integration over Ω yield

$$\begin{aligned} (\partial_t u_1, -\Delta u_1 + b u_1) + \|-\Delta u_1 + b u_1\|_2^2 &= (u_1 u_2, -\Delta u_1 + b u_1) \\ &\leq \frac{1}{2}(\|u_1 u_2\|_2^2 + \|-\Delta u_1 + b u_1\|_2^2). \end{aligned} \tag{37}$$

Here we note

$$(\partial_t u_1, -\Delta u_1 + b u_1) = \frac{d}{dt} \phi_1(u_1(t)).$$

Hence, in view of (37) and (36), we obtain

$$\frac{d}{dt} \phi_1(u_1(t)) + b \phi_1(u_1(t)) \leq \frac{1}{2} \|u_1 u_2\|_2^2.$$

Here by Hölder’s inequality, (32), (33), (34),(35) and Young’s inequality, we get

$$\begin{aligned} \|u_1 u_2\|_2^2 &= \int_{\Omega} u_1^2 u_2^2 dx = \int_{\Omega} u_1^{\frac{1}{2}} u_2^{\frac{1}{2}} u_1^{\frac{3}{2}} u_2^{\frac{3}{2}} dx \\ &\leq \left(\int_{\Omega} u_1 u_2 dx\right)^{\frac{1}{2}} \left(\int_{\Omega} u_1^3 u_2^3 dx\right)^{\frac{1}{2}} \\ &\leq C_{11}^{\frac{1}{2}} C_8^{\frac{1}{2}} \|u_1(t)\|_6^{\frac{3}{2}} \|u_2(t)\|_6^{\frac{3}{2}} \\ &\leq b \phi_1(u_1(t)) + C_{13}. \end{aligned}$$

Hence it follows that

$$\frac{d}{dt} \phi_1(u_1(t)) + \frac{b}{2} \phi_1(u_1(t)) \leq \frac{C_{13}}{2}.$$

Therefore, applying Gronwall’s inequality, we deduce

$$\phi_1(u_1(t)) \leq \phi_1(u_1(0)) e^{-\frac{b}{2}t} + \frac{C_{13}}{b}.$$

which implies that

$$\sup_{t \geq 0} \|u_1(t)\| \leq C_{14}. \tag{38}$$

(4) Uniform estimates in L^∞

Since Theorem 2.1 assures that there exists $s_1 \in (0, 1)$ such that $u(s_1) \in H^1(\Omega)$ and $\|u(t)\|_\infty$ is bounded on $[0, s_1]$, we can assume without loss of generality that $(u_{10}, u_{20}) \in H^1 \cap V$. To derive L^∞ bounds via H^1 bounds, we rely on the following Alikakos - Moser’s iteration scheme, which plays an essential role in our argument.

LEMMA 3.3. ([17]) Assume that $v \in W_{loc}^{1,2}([0, \infty); L^2(\Omega)) \cap L_{loc}^\infty([0, \infty); L^\infty(\Omega) \cap H^1(\Omega))$ satisfies

$$\frac{d}{dt} \|v(t)\|_r^r + c_1 r^{-\theta_1} \| |v(t)|^{\frac{r}{2}} \|^2 \leq c_2 r^{\theta_2} (\|v(t)\|_r^r + 1) \quad a.e. t \in [0, \infty), \tag{39}$$

for all $r \in [2, \infty)$, where $c_1 > 0$ and $c_2, \theta_1, \theta_2 \geq 0$. Then there exist some constants d_1, d_2, d_3 and $d_4 \geq 0$ such that

$$\sup_{t \geq 0} \|v(t)\|_\infty \leq d_1 2^{\theta_2 + (\theta_1 + \theta_2)d_2} M_0,$$

where $M_0 = \max(1, d_3 \|v_0\|_\infty, \sup_{t \geq 0} \|v(t)\|_2^{d_4})$.

In order to apply Lemma 3.3, we deform (1) in the following way:

$$\partial_t u_1 - \Delta u_1 + u_1 = u_1 u_2 - b u_1 + u_1, \tag{40}$$

$$\partial_t u_2 - \Delta u_2 + u_2 = a u_1 + u_2. \tag{41}$$

Hereafter we employ the usual H^1 norm $(\|\nabla v\|_2^2 + \|v\|_2^2)^{1/2}$ for u_1 and u_2 . Multiplying (40) by $|u_1|^{r-2} u_1$ ($r \geq 2$) and using integration by parts, we obtain

$$\begin{aligned} & \frac{1}{r} \frac{d}{dt} \|u_1(t)\|_r^r + (r-1) \int_\Omega |\nabla u_1|^2 |u_1|^{r-2} dx + \int_{\partial\Omega} |u_1|^r d\sigma + \|u_1(t)\|_r^r \\ & = \int_\Omega u_1^r u_2 dx - b \|u_1(t)\|_r^r + \|u_1(t)\|_r^r. \end{aligned}$$

Hence we have

$$\frac{1}{r} \frac{d}{dt} \|u_1(t)\|_r^r + (r-1) \int_\Omega |\nabla u_1|^2 |u_1|^{r-2} dx + \|u_1(t)\|_r^r \leq \int_\Omega |u_1|^r |u_2| dx + \|u_1(t)\|_r^r.$$

Moreover we note

$$\begin{aligned} (r-1) \int_\Omega |\nabla u_1|^2 |u_1|^{r-2} dx + \|u_1(t)\|_r^r & = \frac{4(r-1)}{r^2} \int_\Omega |\nabla |u_1|^{\frac{r}{2}}|^2 dx + \| |u_1(t)|^{\frac{r}{2}} \|_2^2 \\ & \geq \frac{4(r-1)}{r^2} \| |u_1(t)|^{\frac{r}{2}} \|^2, \end{aligned}$$

where we used the fact that $r \geq 2$ implies $\frac{4(r-1)}{r^2} \in (0, 1]$ to the last inequality. Hence we obtain

$$\frac{1}{r} \frac{d}{dt} \|u_1(t)\|_r^r + \frac{4(r-1)}{r^2} \| |u_1(t)|^{\frac{r}{2}} \|^2 \leq \int_{\Omega} |u_1|^r |u_2| dx + \|u_1(t)\|_r^r. \tag{42}$$

By using Hölder’s inequality, interpolation inequality, Sobolev’s embedding theorem and Young’s inequality, we can get

$$\begin{aligned} \int_{\Omega} |u_1|^r |u_2| dx &\leq \|u_1(t)\|_{\frac{3r}{2}}^r \|u_2(t)\|_3 \\ &\leq \|u_1(t)\|_r^{\frac{r}{2}} \|u_1(t)\|_{\frac{3r}{2}}^{\frac{r}{2}} \|u_2(t)\|_3 \\ &\leq \|u_2(t)\|_3 \| |u_1(t)|^{\frac{r}{2}} \| \| |u_1(t)|^{\frac{r}{2}} \|_6 \\ &\leq C_{15} \|u_1(t)\|_r^{\frac{r}{2}} \| |u_1(t)|^{\frac{r}{2}} \| \\ &\leq \frac{2(r-1)}{r^2} \| |u_1(t)|^{\frac{r}{2}} \|^2 + \frac{C_{15}^2 r^2}{8(r-1)} \|u_1(t)\|_r^r, \end{aligned}$$

where we used the fact that $\|u_2(t)\|_3$ is uniformly bounded with respect to time by virtue of interpolation inequality, Sobolev’s inequality and the global bounds for $\|u_2(t)\|_2$ and $\|u_2(t)\|$. Since $r \geq 2$, it is easy to see that $\frac{r^2}{8(r-1)} \leq r$. Then, from these observations, (42) leads to

$$\frac{1}{r} \frac{d}{dt} \|u_1(t)\|_r^r + \frac{2(r-1)}{r^2} \| |u_1(t)|^{\frac{r}{2}} \|^2 \leq C_{15}^2 r \|u_1(t)\|_r^r + \|u_1(t)\|_r^r,$$

that is,

$$\frac{d}{dt} \|u_1(t)\|_r^r + \| |u_1(t)|^{\frac{r}{2}} \|^2 \leq C_{16} r^2 (\|u_1(t)\|_r^r + 1). \tag{43}$$

Here we used the fact that $1 \leq \frac{2(r-1)}{r}$ provided that $r \geq 2$. Then $u_1(t)$ satisfies (39) with $c_1 = 1$, $c_2 = C_{16}$, $\theta_1 = 0$ and $\theta_2 = 2$. Thus applying Lemma 3.3 to (43), we see that there exists $C_{17} > 0$ such that

$$\sup_{t \geq 0} \|u_1(t)\|_{\infty} \leq C_{17}. \tag{44}$$

Finally, applying the same argument as above for $u_2(t)$, we have

$$\frac{1}{r} \frac{d}{dt} \|u_2(t)\|_r^r + \frac{4(r-1)}{r^2} \| |u_2(t)|^{\frac{r}{2}} \|^2 \leq a \int_{\Omega} u_1 u_2^{r-1} dx + \|u_2(t)\|_r^r. \tag{45}$$

Since $\frac{r-1}{r} \leq 1$ and $\frac{1}{r} \leq 1$, due to (44) we can deduce

$$\begin{aligned} a \int_{\Omega} u_1 u_2^{r-1} dx &\leq a C_{17} \|u_2(t)\|_{r-1}^{r-1} \\ &\leq a C_{17} \left\{ \frac{r-1}{r} \|u_2(t)\|_r^r + \frac{1}{r} |\Omega| \right\} \\ &\leq a C_{17} (\|u_2(t)\|_r^r + |\Omega|), \end{aligned}$$

which implies

$$\frac{1}{r} \frac{d}{dt} \|u_2(t)\|_r^r + \frac{4(r-1)}{r^2} \| |u_2(t)|^{\frac{r}{2}} \|^2 \leq C_{18} \left(\|u_2(t)\|_r^r + 1 \right),$$

for some $C_{18} > 0$. Since $2 \leq \frac{4(r-1)}{r}$, we conclude that

$$\frac{d}{dt} \|u_2(t)\|_r^r + 2 \| |u_2(t)|^{\frac{r}{2}} \|^2 \leq C_{18} r \left(\|u_2(t)\|_r^r + 1 \right). \quad (46)$$

Then we can apply Lemma 3.3 to (46) with $c_1 = 2$, $c_2 = C_{18}$, $\theta_1 = 0$ and $\theta_2 = 1$. Thus there exists $C_{19} > 0$ such that

$$\sup_{t \geq 0} \|u_2(t)\|_\infty \leq C_{19}. \quad (47)$$

These a priori bounds (44) and (47) complete the proof. \square

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