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MULTIPLE SOLUTIONS FOR A FOURTH ORDER EQUATION WITH NONLINEAR BOUNDARY CONDITIONS: THEORETICAL AND NUMERICAL ASPECTS

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Abstract. We consider in this work the fourth order equation with nonlinear boundary conditions. We present the result for the existence of multiple solutions based on the Avery-Peterson fixed-point theorem. This work is also a study for numerical solutions based on the Levenberg-Maquardt method with a heuristic strategy for initial points that proposes to numerically determine multiple solutions to the problem addressed.

1. Introduction

In this paper we present a study on

$$u^{(iv)}(x) - M\left(\int_0^L u'^2(x)dx\right)u''(x) = f(x,u(x),u'(x)), \qquad 0 < x < L, \tag{1}$$

with border conditions

$$u(0) = u''(0) = 0, \tag{2}$$

$$u(L) = 0 \text{ and } u''(L) = g(u'(L)), \tag{3}$$

where $f : [0,L] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $g \in C(\mathbb{R})$ and $M \in C(\mathbb{R}^+)$.

The problem defined in (1), (2) and (3) models the bending equilibrium of simply supported extensible beams on nonlinear foundations. In this model, the function f represents the force that the foundation exerts on the beam and M models the effects of the small variations in the beam measurements. More arguments about modeling can be found in references [4], [6], [7] and [8].

Several works consider the non-dependence of f on the problem defined in (1), (2) and (3) with respect to the term u', ([9], [11], [12], [13], [14]).

We noted that the problem defined in (1), (2) and (3) has a non-local M term and a nonlinear dependence on the first derivatives. As highlighted in [11], the dependence exposed here makes more painful in the study on the existence of solutions even

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using more classical ideas based on the Krasnoselskii Theorem combined with Leray-Schauder's Alternatives. Naturally, this difficulty is present when we are dealing with techniques based on Avery-Peterson Theorem. As a consequence, to the best of our knowledge, this kind of problem is little considered. Our main objective is to prove the existence and multiplicity of positive solutions to the problem defined in (1), (2) and (3), using the Avery-Peterson fixed-point theorem. Following the results of [5], [12], [11], our results are declared assuming local conditions M, f and g that generalize the results presented in [12], which uses Krasnoselskii's Theorem. In addition, we present an unprecedented strategy for the determination of numerical solutions. This new procedure is able to handle with multiple solutions by the theoretical approach of this work.

The paper is organized as follows: In Section 2 we present preliminary results, in Section 3, of a theoretical result that guarantees the existence of multiple solutions, considering certain conditions on the component functions of the problem. In Section 4, we discuss an optimization method with a heuristic strategy for the determination of approximate solutions to the problem defined in (1), (2) and (3). Finally, Section 5 presents some final considerations about the work.

Then we state hereafter the Avery-Peterson theorem that we will use for proving multiple solutions.

Now, we need to consider the convex sets

$$P(\gamma,d) = \{x \in P | \gamma(x) < d\},$$

$$P(\gamma,\alpha,b,d) = \{x \in P | b \leqslant \alpha(x) \text{ and } \gamma(x) < d\},$$

$$P(\gamma,\theta,\alpha,b,c,d) = \{x \in P | b \leqslant \alpha(x), \theta(x) \leqslant c \text{ and } \gamma(x) < d\},$$

and the closed set

$$R(\gamma,\psi,a,d) = \{x \in P | a \leqslant \psi(x) \text{ and } \gamma(x) < d\}.$$

THEOREM 1. *Let P be a cone in a real Banach space X. Let γ and θ nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leqslant \lambda\psi(x)$ for $0 \leqslant \lambda \leqslant 1$, such that for some positive numbers μ and d,*

$$\alpha(x) \leqslant \psi(x) \text{ and } \|x\| \leqslant \mu\gamma(x),$$

for all $x \in \overline{P(\gamma,d)}$. Suppose

$$T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$$

is completely continuous and there exist positive numbers a, b, c with $a < b$, such that

$$\{u \in P(\gamma,\theta,\alpha,b,c,d) | \alpha(u) > b\} \neq \emptyset \text{ and}$$

$$u \in P(\gamma,\theta,\alpha,b,c,d) \Rightarrow \alpha(Tu) > b, \tag{4}$$

$$\alpha(Tu) > b \text{ for } u \in P(\gamma,\alpha,b,d) \text{ with } \theta(Tu) > c, \tag{5}$$

$$0 \notin R(\gamma,\psi,a,d) \text{ and } \psi(Tu) < a \text{ for } u \in R(\gamma,\psi,a,d) \text{ with } \psi(u) = a. \tag{6}$$

Then T has at least three distinct fixed points in $\overline{P(\gamma,d)}$.

2. Preliminary results

We present in this section observations and preliminary results that help and motivate the studies of existence of multiple solutions in the next section.

Let's represent the problem by an integral equation, for this will reduce the order of the problem (1)-(2)-(3), with the identification $v = u'' - M(\|u'\|_2^2)u$, we conclude that this problem is equivalent to the pair of systems:

$$\begin{cases} u'' = M(\|u'\|_2^2)u + v, \\ u(0) = u(L) = 0 \end{cases} \tag{7}$$

$$\begin{cases} v'' = f(x,u,u'), \\ v(0) = 0 \quad v(L) = g(u'(L)). \end{cases} \tag{8}$$

Let G be the Green function for the problem $-w'' = h$, with $w(0) = w(L) = 0$, defined by

$$G(x,t) = \begin{cases} \frac{x(L-t)}{L}, & 0 \leqslant x \leqslant t \leqslant L, \\ \frac{t(L-x)}{L}, & 0 \leqslant t \leqslant x \leqslant L. \end{cases} \tag{9}$$

Next

$$w(x) = \int_0^L G(x,t)h(t)dt.$$

Therefore we conclude from (7) and (8) that:

$$u(t) = \int_0^L -G(x,t)(M(\|u'\|_2^2)u(t) + v(t))dt,$$

where

$$v(t) = \int_0^L -G(t,s)f(s,u(s),u'(s))ds + \frac{t}{L}g(u'(L)).$$

By combining u, v, we can expect to find a (1) - (2) - (3) of the problem of fixed point:

$$Tu(x) = \int_0^L G(x,t)\left(\int_0^L (G(t,s)f(s,u(s),u'(s))ds - M(\|u'\|_2^2)u(t) - \frac{t}{L}g(u'(L))\right)dt. \tag{10}$$

Due to the term $M(\|u'\|_2^2)$ in the equation equation (1) we must obtain an estimate first to u' with the norm of $L^2(0,L)$. We could study the possibility of operator fixed points T in $H_0^1(0,L)$. But unfortunately, we can not estimate the edge in terms of $u'(0)$ and $u'(L)$. Two function spaces suitable for our analysis, would be $C^1[0,L]$ or $H^2(0,L)$. We prefer to work in the space of Banach:

$$E = C_0[0,L] \cap C^1[0,L] = \{u \in C^1[0,L]; u(0) = u(L) = 0\},$$

with the norm

$$\|u\|_E = \|u'\|_\infty = \max_{t\in[0,L]} |u'(t)|.$$

LEMMA 1. *If* $u \in E$*, then*

$$\|u'\|_2^2 \leqslant L\|u'\|_\infty^2. \tag{11}$$

Proof. Indeed

$$\|u'\|_2^2 = \int_0^L |u'(s)|^2 ds \leqslant \|u'\|_\infty^2 \int_0^L ds = L\|u'\|_\infty^2. \quad \square$$

LEMMA 2. *If* $u \in E$*, then*

$$\|u\|_\infty \leqslant \frac{L}{2}\|u\|_E. \tag{12}$$

Proof. We assume by contradiction that (12) does not occur, then there is $u \in E$ and $t_0 \in (0,L)$ such that $|u(t_0)| > \frac{L}{2}\|u\|_E$. The mean value theorem implies that we have $t_1 \in (0,t_0)$ and $t_2 \in (t_0,L)$ such that

$$u'(t_1) = \frac{u(t_0)-u(0)}{t_0} = \frac{u(t_0)}{t_0}, \qquad u'(t_2) = \frac{u(L)-u(t_0)}{L-t_0} = \frac{-u(t_0)}{L-t_0}.$$

Then

$$t_0 \leqslant \frac{L}{2} \text{ implies } |u'(t_1)| \geqslant \frac{|u(t_0)|}{L/2} = \frac{2|u(t_0)|}{L},$$
$$t_0 \geqslant \frac{L}{2} \text{ implies } |u'(t_2)| \geqslant \frac{|u(t_0)|}{L/2} = \frac{2|u(t_0)|}{L}.$$

Next $\max\{|u'(t_1)|,|u'(t_2)|\} > \|u'\|_\infty$, which is a contradiction, so we conclude the demonstration. $\square$

We assume that $f:[0,L]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous.

REMARK 1. Note that the function $h:[0,L]\longrightarrow\mathbb{R}$ defined by

$$h(t) = \int_0^L G(t,s)q(s)ds,$$

is a solution to the problem of Dirichlet $-h'' = q(s)$, with $h(0) = h(L) = 0$. For q not negative, h is a concave function with $h(t) \geqslant 0$, $\forall t \in [0,L]$.

In order to demonstrate the existence of multiple solutions we will assume the following conditions on f:

$$q(s) \leqslant f(s,u_1,u_2), \qquad \forall (s,u_1,u_2) \in [0,L]\times[0,\frac{Ld}{2}]\times[-d,d], \tag{13}$$

$$\max_{t\in[0,L]} \int_0^L G(t,s)q(s)ds = \frac{A}{R}, \qquad \text{where } A>0 \text{ and } R>0. \tag{14}$$

REMARK 2. The function h defined in the observation 1 is concave, and (14), The function h has the property $h(t) = 0$ if, and only if, $t = 0$ or $t = L$. Thus given $m \in (0, \frac{1}{2})$ exist $\omega \in (0,1)$ such that

$$h(t) \geqslant \omega \frac{A}{R}, \quad \forall t \in [Lm, L(1-m)], m \in (0, \frac{1}{2}),$$

and if $h(t) \geqslant \omega_0 \frac{A}{R}, \quad \forall t \in [Lm, L(1-m)]$ for some $\omega_0 \in (0,1)$ then $\omega_0 \leqslant \omega$.

From the above notice to $m \in (0, \frac{1}{2})$ there is ω such that:

$$\left| \int_0^L G(t,s)q(s)ds \right| \geqslant \omega \frac{A}{R} \qquad \forall t \in [Lm, (1-m)L]. \tag{15}$$

LEMMA 3. *If f meets the conditions (13), (14) and M meets*

$$M(s) \leqslant A, \qquad \forall s \in [0, Ld^2], \tag{16}$$

to $d > 0$ then to $u \in C$, with $\|u\|_E \leqslant d$, it has

$$\int_0^L G(t,s)f(s,u(s),u'(s))ds - M(\|u'\|_2^2)u(t) \geqslant 0, \; \forall t \in [0,L]. \tag{17}$$

Proof. Considering $d = \min\{\frac{\omega}{RLm}, \frac{2\omega}{LR}\}$, where $m \in (0, \frac{1}{2})$ have from equation (16) that $M(\|u'\|_2^2) \leqslant A, \quad \forall u \in E$ tal que $\|u\|_E \leqslant d$.
Let's prove (17). If $t \in [Lm, (1-m)L]$, from (13) , (14) and observation 2 we obtain

$$\int_0^L G(t,s)f(s,u(s),u'(s))ds \geqslant \omega \frac{A}{R}, \qquad \forall u \in E \text{ and } \forall t \in [Lm, (1-m)L].$$

As $\|u\|_E \leqslant d$, then $\|u\|_\infty \leqslant \frac{L\|u'\|_\infty}{2} \leqslant \frac{Ld}{2} \leqslant \frac{\omega}{R}$. Thus $\forall t \in [Lm, (1-m)L]$ we conclude:

$$\int_0^L G(t,s)f(s,u(s),u'(s))ds \geqslant \omega \frac{A}{R} \geqslant M(\|u'\|_2^2)\frac{\omega}{R} \geqslant M(\|u'\|_2^2)u(t).$$

It follows that (17) is in range $[Lm, (1-m)L]$.
Now $t \in [0, Lm]$, as $\|u'\|_\infty \leqslant \frac{\omega}{RLm}$ then it follows from the average value theorem, that $Au(t)$ is below the segment passing through $(0,0)$ and $(Lm, \omega\frac{A}{R})$. As $\int_0^L G(t,s)q(s)ds$ is a concave function, from (15), it follows that this application is above this segment, and therefore (17) occurs. Analogously, it proves to $t \in [(1-m)L, L]$. $\square$

3. Multiple solutions

We present in this section a result of the existence that motivates the numerical studies that will be presented in the next section. We demonstrate the existence of multiple solutions to the problem defined in (1)-(2)-(3) through the theorem (2).

In order to demonstrate the existence of multiple solutions, let first show that $Tu \geqslant 0$ for $\|u\|_E \leqslant d$, will need the following hypotheses:

(H1) We assume the conditions (13), (14) and (16) imposed on f and M, and that

$$g(s) \leqslant 0, \quad \forall s \in [-d,0], \text{ and } -g(s) \leqslant \frac{\lambda_1 d}{r_1}, \ \forall s \in [-d,0],$$

where, $\lambda_1 \in (0,1)$ and

$$r_1 = \max\left\{\int_0^L \frac{(L-t)t}{L^2}dt, \int_0^L \frac{t^2}{L^2}dt\right\}.$$

The signal condition imposed here on the g function was motivated by the fact that we are looking for concave solutions to the problem, so we expect its derivative second is negative, and also its derivative in L, hence the condition for the sign of g.

(H2)

$$f(t,u,v) \leqslant \frac{(1-\lambda_1)d}{r_2}, \qquad \forall (t,u,v) \in [0,L]\times[0,\frac{Ld}{2}]\times[-d,d],$$

com

$$r_2 = \max\left\{\int_0^L\int_0^L \frac{(L-t)}{L}G(t,s)dsdt, \int_0^L\int_0^L \frac{t}{L}G(t,s)dsdt\right\}.$$

REMARK 3. From $(H1)$ and Lemma 3 such that

$$Tu(x) \geqslant 0, \quad \forall x \in [0,L], \text{ and } (Tu)'' \leqslant 0,$$

for all $u \in E$ such that $\|u\|_E \leqslant d$, therefore $T(u)$ is a concave function.

PROPOSITION 1. *If $u \in E$, and u concave then*

$$\|u'\|_\infty = \max\{|u'(0)|, |u'(L)|\}.$$

Proof. In fact being $u \in C^1[0,1]$ a concave application, so u' is decreasing in the interval $[0,L]$ and how $u(0) = u(L) = 0$, then exist $c \in [0,L]$ such that $u'(c) = 0$. Therefore

$$u'(x) \geqslant 0, \forall x \in [0,c] \text{ and}, \qquad u'(x) \leqslant 0, \forall x \in [c,L].$$

So

$$\max_{x\in[0,c]} |u'(x)| = u'(0), \qquad \max_{x\in[c,L]} |u'(x)| = |u'(L)|. \quad \square$$

We denote:

$$z(t) = \int_0^L G(t,s)f(s,u(s),u'(s))ds - M(\|u'\|_2^2)u(t) - \frac{t}{L}g(u'(L)).$$

It is immediate from (H1) and the lemma 3 that $z(t) \geqslant 0$ for all $t \in [0,L]$ and $||u||_E \leqslant d$.

LEMMA 4. *If $(H1)-(H2)$ they occur then T applies $\overline{P(\gamma,d)}$ in $\overline{P(\gamma,d)}$.*

Proof. Using $(H1)$ we have $Tu \geqslant 0$ if $\gamma(u) = \|u\|_E \leqslant d$, and obtain:

$$\|Tu\|_E = \max_{x\in[0,L]} \left| \int_0^L \partial_x G(x,t) z(t) dt \right|.$$

Now follows from Proposition 1 that

$$\|Tu\|_E = \max \left\{ \int_0^L \frac{t}{L} z(t) dt, \int_0^L \frac{L-t}{L} z(t) dt, \right\}.$$

With $u \in C$ and $M(s) \geqslant 0, \quad \forall s \in \mathbb{R}^+$ obtain

$$z(t) \leqslant \int_0^L G(t,s) f(s,u(s),u'(s)) ds - \frac{t}{L} g(u'(L)).$$

This fact, of (H1) and (H2) it has:

$$\begin{aligned}
\|Tu\|_E \leqslant \max \Bigg\{ & \int_0^L \frac{t}{L} \left(\int_0^L G(t,s) \frac{(1-\lambda_1)d}{r_2} ds + \frac{t}{L} \frac{\lambda_1 d}{r_1} \right) dt \,, \\
& \int_0^L \frac{L-t}{L} \left(\int_0^L G(t,s) \frac{(1-\lambda_1)d}{r_2} ds + \frac{t}{L} \frac{\lambda_1 d}{r_1} \right) dt \Bigg\} \\
\leqslant d \max \Bigg\{ & \frac{(1-\lambda_1)}{r_2} \int_0^L \int_0^L \frac{t}{L} G(t,s) ds dt + \frac{\lambda_1}{r_1} \int_0^L \frac{t^2}{L^2} dt, \\
& \frac{(1-\lambda_1)}{r_2} \int_0^L \int_0^L \frac{L-t}{L} G(t,s) ds dt + \frac{\lambda_1}{r_1} \int_0^L \frac{(L-t)t}{L^2} dt \Bigg\} \\
\leqslant & \, d((1-\lambda_1)+\lambda_1) \leqslant d. \quad \square
\end{aligned}$$

THEOREM 2. *Suppose that the hypothesis $(H1)-(H2)$ hold. Suppose in addition that there exist a, $0 < a < d$ such that f, M and g satisfies the following conditions:*

$(H3)$ $f(t,u,v) < \dfrac{\lambda_2 a}{r_3}$, $\forall (t,u,v) \in [0,L] \times [0,a] \times [-d,d]$ *and*
$-g(y) < \min\left\{ \frac{(1-\lambda_2)a}{r_4}, \; \frac{\lambda_1 d}{r_1} \right\}$ *with* $y \in [-d,0]$,
where $r_3 = \max_{x\in[0,L]} \left\{ \int_0^L G(x,t) \int_0^L G(t,s) ds dt \right\}$,
$r_4 = \max_{x\in[0,L]} \left\{ \int_0^L G(x,t) \frac{t}{L} dt \right\}$ *and* $\lambda_2 \in (0,1)$.

$(H4)$ $f(t,u,v) > \dfrac{b\lambda_3}{r_5}$, $\forall (t,u,v) \in [0,1] \times [2a, 8\sqrt{2}a] \times [-d,d]$, *and*
$M(s) < \min\left\{ \frac{(1-\lambda_3)b}{cr_6}, A \right\}$, $\forall s \in [0, Ld^2]$, *where*
$r_5 = \min\left\{ \int_0^L G(\frac{L}{4},t) \int_0^L G(t,s) ds dt, \int_0^L G(\frac{3L}{4},t) \int_0^L G(t,s) ds dt \right\}$,
$r_6 = \min\left\{ \int_0^L G(\frac{L}{4},t) dt, \int_0^L G(\frac{3L}{4},t) dt \right\}$ *and* $\lambda_3 \in (0,1)$.

Then the problem (1) has at least three positive solutions.

Proof. We apply Avery-Peterson theorem. Thus, we consider T and P defined as before. Furthermore, we need to define the following functionals

$$\gamma(u) = \|u\|_E,\ \psi(u) = \max_{t\in[0,L]} |u(t)|,\ \theta(u) = \left[\int_{\frac{L}{4}}^{\frac{3L}{4}} [u(t)]^2 dt\right]^{\frac{1}{2}},\ \alpha(u) = \min_{t\in[\frac{L}{4},\frac{3L}{4}]} |u(t)|.$$

Using $(H1)$ we have $Tu \geqslant 0$ if $\gamma(u) = \|u\|_E \leqslant d$, we obtain: From Lemma 4 T applies $\overline{P(\gamma,d)}$ in $\overline{P(\gamma,d)}$.
Now, we consider $b = 2a$ and $c = 8\sqrt{2}a$. Clearly, we have $\{u \in P(\gamma,\theta,\alpha,b,c,d) | \alpha(u) > b\} \neq \emptyset$. Let us verify (4). Using (H4) we can get

$$\begin{aligned}
\alpha(Tu) &= \min_{x\in[\frac{L}{4},\frac{3L}{4}]} (Tu)x \\
&\geqslant \min_{x\in[\frac{L}{4},\frac{3L}{4}]} \int_0^L G(x,t)\left[int_0^L G(t,s)f(s,u(s),u'(s))ds - M(\|u'\|_2^2)u(t)\right] dt \\
&\geqslant \min\left\{\int_0^L G(\frac{L}{4},t)\left[int_0^L G(t,s)f(s,u(s),u'(s))ds - M(\|u'\|_2^2)u(t)\right] dt,\right. \\
&\qquad \left.\int_0^L G(\frac{3L}{4},t)\left[\int_0^L G(t,s)f(s,u(s),u'(s))ds - M(\|u'\|_2^2)u(t)\right] dt\right\} \\
&\geqslant \min\left\{\int_0^L G(\frac{L}{4},t)\left[\frac{b\lambda_3}{r_5}\int_0^L G(t,s)ds - \frac{(1-\lambda_3)b}{cr_6}u(t)\right] dt,\right. \\
&\qquad \left.\int_0^L G(\frac{3L}{4},t)\left[\frac{b\lambda_3}{r_5}\int_0^L G(t,s)ds - \frac{(1-\lambda_3)b}{cr_6}u(t)\right] dt\right\} \\
&\geqslant \min\left\{\int_0^L G(\frac{L}{4},t)\left[\frac{b\lambda_3}{r_5}\int_0^L G(t,s)ds - \frac{(1-\lambda_3)b}{r_6}\right] dt,\right. \\
&\qquad \left.\int_0^L G(\frac{3L}{4},t)\left[\frac{b\lambda_3}{r_5}\int_0^L G(t,s)ds - \frac{(1-\lambda_3)b}{r_6}\right] dt\right\} \\
&\geqslant b\lambda_3 + (1-\lambda_3)b = b.
\end{aligned}$$

Let us demonstrate (5). Let $u \in P(\gamma,\alpha,b,d)$ with $\theta(Tu) > c$. Then

$$\begin{aligned}
\alpha(Tu) &= \min_{x\in[\frac{L}{4},\frac{3L}{4}]}\left[\int_0^L G(x,t)z(t)dt\right] \geqslant \frac{1}{4}\int_0^L G(t,t)z(t)dt \\
&\geqslant \frac{1}{4\sqrt{2L}}\theta\left(\max_{x\in[0,1]}\int_0^L G(x,t)z(t)dt\right) \geqslant \frac{1}{4\sqrt{2L}}\theta\left(\int_0^L G(x,t)z(t)dt\right) \\
&\geqslant \frac{1}{4\sqrt{2L}}\theta\left(\int_0^L G(x,t)z(t)dt\right) \geqslant \frac{1}{4\sqrt{2L}}\theta(Tu) > \frac{1}{4\sqrt{2L}}c = b.
\end{aligned}$$

Now, let us demonstrate (6). Thus, let $u \in R(\gamma,\psi,a,d)$ with $\psi(u) = a$. From $(H3)-(H4)$ we have,

$$\psi(Tu) = \max_{x\in[0,L]} |Tu(x)|$$

$$\leqslant \max_{x\in[0,L]} \int_0^L G(x,t)\left[\int_0^L G(t,s)f(s,u(s),u'(s))ds - \frac{t}{L}g(u'(L))\right]dt.$$

Then, from $(H4)$ we get

$$\psi(Tu) < \max_{x\in[0,L]} \int_0^L G(x,t)\left[\int_0^L G(t,s)\frac{\lambda_2 a}{r_3}ds + \frac{t}{L}\frac{(1-\lambda_2)a}{r_4}\right]dt$$
$$\leqslant \max_{x\in[0,L]}\left[\frac{\lambda_2 a}{r_3}\int_0^L G(x,t)\int_0^L G(t,s)dsdt + \frac{(1-\lambda_2)a}{r_4}\int_0^L G(x,t)\frac{t}{L}dt\right] \leqslant a.$$

Applying Avery-Peterson theorem we have the result. □

In order to show that the previously imposed hypotheses ((H1), (H2), (H3) and (H4)) and the conditions (1)-(2)-(3) assumed to guarantee the existence of a solution to the problem are possible give the following example:

EXAMPLE 1. Consider the following functions

$$f(t,u,v) = \begin{cases} 7+\frac{u^2}{4}+\left(\frac{v}{40}\right)^4, & 0 \leqslant u < 2,\\ 8+\left(\frac{v}{40}\right)^4, & u \geqslant 2,\end{cases}$$

$$g(s) = \frac{-s}{7},$$

$$M(t) = \frac{1}{70}+\frac{t}{10000},$$

$$q(s) = 7,\ L = 1 \quad \text{and} \quad d = 20.$$

We obtain the following values for the constants:

$$r_1 = \frac{1}{3},\ r_2 = \frac{7}{40},\ r_3 = 0.215,\ r_4 = 0.065,\ r_5 = \frac{19}{2048},\ r_6 = \frac{3}{32},$$
$$A = 0.0546875,\ R = 0.015625,\ \omega = 0.15625.$$

From the above values it is immediate that the equations (13), (14) and (16) are satisfied, now considering $\lambda_1 = 0.5$ the hypotheses H1 and H2 are satisfied. Now considering $a = 2$ and $\lambda_2 = 0.9$ the hypothesis H3 is satisfied. Considering $b = 2a$ and $c = 8\sqrt{2a}$ and $\lambda_3 = 0.018$ the hypothesis H4 is satisfied.

4. Numerical solutions

In most studies, numerical solutions are obtained by fixed-point methods, according to [12]. More specifically, an iterative sequence based on operator given by equation 10 define the method. However, the convergence of this method depends on the operator being a contraction in the neighborhood of the solution and consequently depends on the quality of the initial points. In order not to depend on these characteristics and look for multiple solutions our basic idea of the proposed method is to use the Levenberg-Maquardt method [15]. An algorithm of this proposed method to solve Problem (1), (2), (3) equivalent representation in two second order problems presented in (7) and (8).

Algorithm 1

1. Define an uniformly spaced mesh $\{t_j\},\ j = 1,\dots,n$, in $[0,L]$.
2. Choose initial approximation $u_j^0 = u^0(t_j)$.
3. Discretize the Problem (1) by finite difference.

   For $j = 2,\dots,n-1$

   - $u''^k(t_j) = \frac{u^k(t_{j+1}) - 2*u^k(t_j) + u^k(t_{j-1})}{h^2}$;
   - $u'^k(t_j) = \frac{u^k(t_{j+1}) - u^k(t_{j-1})}{2*h}$.

4. Choose initial approximation $v_j^0 = u''^k(t_j) - M(\|u'\|_2^2)$, where approaching $\|u'\|_2^2$ by using trapezoidal rule.

   - $v''^k(t_j) = \frac{v^k(t_{j+1}) - 2*v^k(t_j) + v^k(t_{j-1})}{h^2}$;
   - $v'^k(t_j) = \frac{v^k(t_{j+1}) - v^k(t_{j-1})}{2*h}$.

   Thus we have the following linear system

   $$r(u^k, v^k) = 0,$$

   where

   $$r(u^k, v^k) = \begin{cases} u''^k(t_j) - M(\|u'\|_2^2)u^k(t_j) - v^k(t_j) = 0;\ j = 2,\dots,n-1. \\ u^k(t_1) = 0 \\ u^k(t_n) = 0 \\ v''^k(t_j) - f(t_j, u^k(t_j), u'^k(t_j)) = 0; \qquad j = 2,\dots,n-1. \\ v^k(t_1) = 0 \\ v^k(t_n) - g(u'^k(t_n)) = 0 \end{cases}$$

5. For $k = 1,2,3,\dots$ (Gauss-Newton)

   (a) Compute $r^k = (r_1, r_2, \dots, r_{2n})^T$ and $A_k = (a_{ij})_{2n\times 2n}$;

   $$r_i = r_i(u^k),\ \ a_{ij} = \nabla r_i(u^k).$$

   (b) Find Δ_k such that:

   $$(A_k^T A_k)\Delta^k = -A_k^T r^k.$$

   (c) Determines α_k such that the Armijo's condition is satisfied.

(d) Compute

$$u^{k+1} = u^k + \alpha_k \Delta_k.$$

6. Test convergence.

The motivation for the Algorithm 1 is the fact that fixed point methods are tendentious to find solutions in which the operator T is a contraction and consequently, chosen an initial approximation u^0 we have generally two possibilities: the method converges to solution given by Banach's Theorem or the method diverges. Anyway, if we have multiple solutions (as in Theorem 2), we can try to find these solutions. For this reason, our proposed algorithm, using an appropriated initial, allows to find others solutions. So the development of an heuristic for find a better initial approximations is relevant.

4.1. A heuristic procedure for initial guesses

We know that the solutions that we are looking must be concave or convex and shall satisfy the condition $u(0) = u(L) = 0$. Thus, approaches by parables are reasonable ways of approaching the solution. In this sense, our heuristic procedure is to generate parables about initial points as follows:

$$u^0(x) = \alpha x^\beta (L - x)^\gamma,$$

where the constants α, β, γ is a random numbers in $[-\frac{d}{2}, \frac{d}{2}] \times (0, 10] \times (0, 10]$. For practical purposes, the proposed procedure is defined by Algorithm 2.

Algorithm 2

1. Choose a vector $(\alpha, \beta, \gamma) \in [-\frac{d}{2}, \frac{d}{2}]^N \times (0, 10]^N \times (0, 10]^N$.

2. For $k = 1, ..., N$ do:

   (a) Compute $u^0_{k,i} = u^0_k(x_i) = \alpha_k x_i^{\beta_k} (1 - x_i)^{\gamma_k}$, $i = 1, ..., n$

   (b) Run the Algorithm 1 with initial guess u^0_k.

Naturally, this procedure returns multiple responses. So we need to establish a way to compare solutions in order to distinguish them. Note that the magnitude of the solutions may be different. In this sense we say that the numerical solutions u^* and u^{**} are *equivalent* if

$$\|u^* - u^{**}\| \leqslant \max\{10^{-3}, 10^{-2} \min\{\|u^*\|, \|u^{**}\|\}\} \tag{18}$$

is satisfied.

4.2. Numerical examples

The examples in sequence show how the Algorithm 2 can be promissor in order to find multiple solutions. We run the Algorithm 2 with $N = 50$ and $n = 10$. For Algorithm 2 we consider as the criterion of stop $\|u^{k+1} - u^k\| < 10^{-2}$.

EXAMPLE 2.

Consider the example presented in the article [12], where (1), (2), (3) defined by

$$f(t,u,u') = t^5 - t^4 - 21t^3 + 12t^2 + 127t - 24 - u,$$
$$M(y) = \frac{1}{2} + \frac{2025}{1352}y^2 \quad \text{and} \quad g(v) = -2v.$$

The solution is $u(x) = t^5 - t^4 - t^3 + t$. In this example we are considering $d = 10$ and $n = 10$ points for the spaced mesh. The numerical result of comparison with the exacts solutions. Applying Algorithm 2, of the 50 times that algorithm 1 was called it obtained convergence in 43 times. All converged to the solution u after 12 iterations (on average) in Algorithm 1 the precision obtained was of $\max|u^{12} - u| = 0.00193$.

EXAMPLE 3.

Consider the problem (1), (2), (3) defined by

$$f(t,u,u') = \frac{3}{2}u(u^2 + v^2), \quad g(v) = -s(1+s) \quad \text{and} \quad M(y) = \frac{1}{8\pi}y + \frac{1}{4}.$$

The solutions are $u(x) = \sin(t)$ and $\overline{u}(x) = 0$. In this example we are considering $d = 10$ and $n = 10$ points for the spaced mesh. The numerical result of comparison with the exacts solutions. Applying Algorithm 2 we obtain in the 50 times that the algorithm 1 was called it obtained convergence in 17 times. 14 initializations converged to the solution u after 20 iterations (on average) in Algorithm 1 the precision obtained was $\max|u^{12} - u| = 0.01073$, three initializations converged to the solution $\overline{u}$ after of 15 iterations (in average) in Algorithm 1 the precision obtained was of $\max|u^{15} - \overline{u}| = 0.00074$.

We introduce an additional test. Let's run the Algorithm 2 using the functions defined in Example 1. In this example we are defining $n = 10$ and $d = 20$. Using the criterion established in (18) we obtain tree solutions are different. These results illustrate the result of existence given in Theorem 2. In Figure 1 we have a graphical representation of these solutions.

5. Final remarks

We have proved, by using the of the Avery-Peterson theorem, that the problem (1), (2), (3) can have multiple solutions if the functions f, g and M meet certain conditions. With regard to the numerical aspects of this work, we present a new algorithm and a new heuristic that allows us to obtain multiple solutions for the problem addressed.

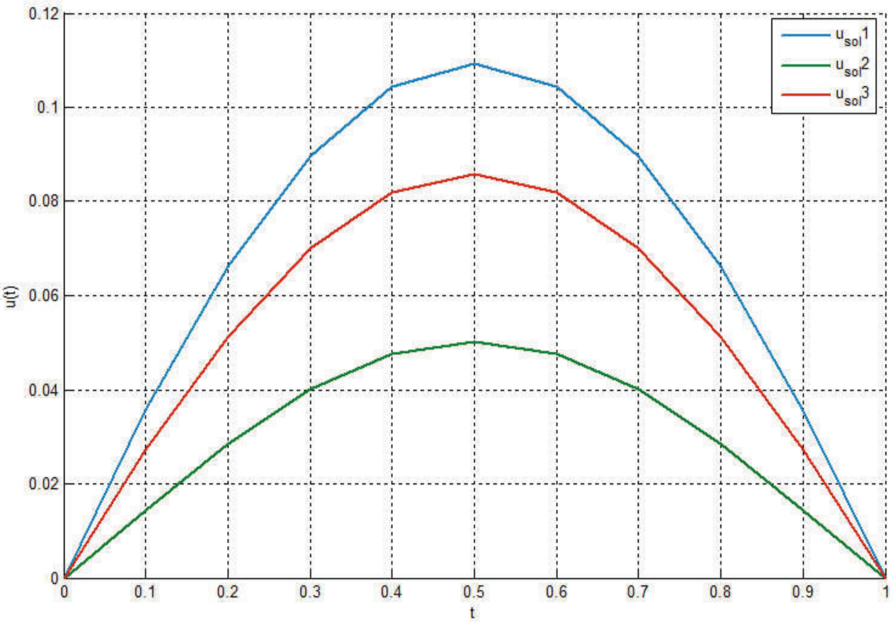

Figure 1: Solutions found by algorithm in Example 1.

The proposed method proved to be robust in solving the problem. However, the cost of this robustness is a slightly higher cost of computer processing, especially when compared to the classical method based on the contraction principle (using the operator defined in (10)), but this strategy is generally not able (when it converges, it tends to converge to the solution with the lowest norm), its convergence depends on properties that cannot be fulfilled by the integral operator, that is, it must be a contraction in a neighborhood of the solution. The level of processing is not absurd, even considering more refined meshes. In addition, the method can be adapted for parallel programming and, consequently, new features can be explored in the computational field.

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