

THE NEHARI MANIFOLD FOR A p -LAPLACIAN EQUATION WITH CONCAVE-CONVEX NONLINEARITIES AND SIGN-CHANGING POTENTIAL

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Abstract. In this paper, we study the multiplicity of solutions for a class of concave-convex p -Laplacian equations with the combined effect of coefficient functions of concave-convex terms. By the Nehari method and some analysis techniques, we obtain an exact constant for the effect of coefficient functions of concave-convex terms to ensure this problem has two nonzero and nonnegative solutions and give the relation of size of the two solutions. Moreover, under some stronger conditions, we prove that the two solutions are positive. Our results generalize and improve some known results in the literature.

1. Introduction

Consider the following concave-convex p -Laplacian equation involving sign-changing potential

$$\begin{cases} -\Delta_p u = f(x)u^{q-1} + g(x)u^{r-1}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, $1 < p < N, 1 < r < p < q < p^* = \frac{pN}{N-p}$ are constants. The operator Δ_p is defined by the formula $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, which has lots of interesting applications in the dynamics of non-Newtonian fluid flows, flows through porous media and glaciology. The coefficient functions $f \in L^{\frac{p^*}{p^*-q}}(\Omega)$ and $g \in L^{\frac{p^*}{p^*-r}}(\Omega)$ satisfy the following condition:

(F_0) The sets $\{x \in \Omega : f(x) > 0\}$ and $\{x \in \Omega : g(x) > 0\}$ have positive measures.

When $p = 2$, problem (1.1) arises in the study of non-Newtonian fluids (in particular pseudoplastic fluids), boundary-layer phenomena for viscous fluids (see [19], [21]), in the Langmuir-Hinshelwood model of chemical heterogeneous catalyst kinetics (see [23]), in enzymatic kinetics models (see [2]), as well as in the theory of heat

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conduction in electrically conducting materials (see [15]) and in the study of guided modes of an electromagnetic field in nonlinear medium (see [7]). Problem (1.1) with $p \neq 2$ arises specifically in the study of turbulent flow of a gas in porous media (see [22]).

It is well known that the pioneer work is Ambrosetti, Brézis and Cerami [1] problem (1.1) was studied under the hypothesis $p = 2, f(x) \equiv 1, g(x) \equiv \lambda > 0$, that is,

$$\begin{cases} -\Delta u = u^{q-1} + \lambda u^{r-1}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < r < 2 < q \leq 2^*$. By using sub-supersolution and variational methods they proved that there exists $\lambda_0 > 0$ such that problem (1.2) has at least two positive solutions for $\lambda \in (0, \lambda_0)$, a positive solution for $\lambda = \lambda_0$ and no positive solution for $\lambda > \lambda_0$. After that, many authors considered problem (1.2), see for examples [12], [16], [18], [26], [31] and [32].

In 1995, Boccardo, Escobedo and Peral [3] studied problem (1.1) with $f(x) = 1, g(x) = \lambda$, and obtained that there exists $\lambda_0 > 0$ such that problem (1.1) has a positive solution for all $0 < \lambda \leq \lambda_0$. Moreover, assume that Ω has a smooth boundary, they proved that there exists $\lambda_* > 0$ such that problem (1.1) has no positive solution for $\lambda > \lambda_*$. Later, García Azorero, Peral Alonso and Manfredi improved the result of [3], and obtained problem (1.1) has two positive solutions for $0 < \lambda < \lambda_*$, see [6]. After that, many authors considered problem (1.1), for examples [4], [5], [9]-[11], [13], [14], [17], [20], [25], [27], [30] and [33].

Particularly, recently, Silva and Macedo [25] considered problem (1.1) with $g \equiv \lambda$ and $f \in L^\infty(\Omega)$ may change sign, that is,

$$\begin{cases} -\Delta_p u = f(x)u^{q-1} + \lambda u^{r-1}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

They obtained that there exist $\varepsilon > 0$ and $\lambda^* > 0$ such that problem (1.3) has two positive solutions for all $0 < \lambda < \lambda^* + \varepsilon$, where λ^* is defined by

$$\lambda^* = \left(\frac{q-p}{q-r} \right) \left(\frac{p-r}{q-r} \right)^{\frac{p-r}{q-p}} \inf_{u \in W_0^{1,p}} \left\{ \frac{\|u\|_{L^q}^{\frac{p(p-r)}{q-p}}}{|u|_r^r F(u)^{\frac{p-r}{q-p}}} : F(u) > 0 \right\},$$

here $F(u) = \int_\Omega f(x)|u|^q dx$, $|u|_q = (\int_\Omega |u|^q dx)^{\frac{1}{q}}$ is the usual L^q -norm and $\|u\| = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}$ is the standard norm of the Sobolev space $W_0^{1,p}(\Omega)$. The λ^* was firstly introduced in [13], where the author studied problem (1.3) with $f \in L^\gamma(\Omega) (\gamma > \frac{p^*}{p^*-q})$ and $f(x) \geq 0$ in Ω , and obtained that problem (1.3) has a positive solution $u \in C^{1,\alpha}(\Omega)$ for some $\alpha > 0$ when $\lambda < \lambda^*$, while $0 < \lambda < \lambda^*$ problem (1.3) has a second positive solution $v \in C^{1,\alpha}(\Omega)$. Moreover, [5] generalized [6] to problem (1.1) with $g(x) = \lambda h(x)$. When $h, f \in L^\infty(\Omega)$ with h has a positive bounded from below on any compact

of Ω and f has a positive bounded from below on some ball of Ω , they obtained problem (1.1) has two positive solutions for $0 < \lambda < \lambda_*$. Further, [13] and [5] also studied the critical case for problem (1.1).

Inspired by [5], [6], [13] and [25], in this article, we consider the existence of solutions for problem (1.1) with the combined effect of coefficient functions of concave-convex terms. By the Nehari method, firstly, when (F_0) holds, we obtain two nonzero and nonnegative solutions of problem (1.1) and one of the solutions is a ground state solution; secondly, we get two positive solutions of problem (1.1) under some stronger constraint conditions on f, g . When f, g are nonzero and nonnegative functions, we can confirm that the ground state solution lies a certain part of Nehari manifold.

Let S be the best Sobolev constant, Θ and $\tilde{\Theta}$ be two positive constants, separately noted by

$$\begin{aligned}
 S &:= \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} |u|^{p^*} dx\right)^{\frac{p}{p^*}}} : u \in W_0^{1,p}(\Omega), u \neq 0 \right\}, \tag{1.4} \\
 \Theta &= \left(\frac{q-p}{q-r}\right)^{\frac{1}{p-r}} \left(\frac{p-r}{q-r}\right)^{\frac{1}{q-p}} S^{\frac{q-r}{(q-p)(p-r)}}, \\
 \tilde{\Theta} &= |\Omega|^{-\frac{(p^*-q)(q-r)+(p^*-r)(q-p)}{p^*(q-p)(p-r)}} \Theta.
 \end{aligned}$$

We define

$$I(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{1}{q} \int_{\Omega} f(x)|u|^q dx - \frac{1}{r} \int_{\Omega} g(x)|u|^r dx, \quad \forall u \in W_0^{1,p}(\Omega).$$

Obviously, the functional I is of class C^1 on $W_0^{1,p}(\Omega)$. As well known that there exists a one to one correspondence between the solutions of problem (1.1) and the critical points of I on $W_0^{1,p}(\Omega)$. More precisely, we say that a function $u \in W_0^{1,p}(\Omega)$ is called a weak solution of problem (1.1), if for all $\varphi \in W_0^{1,p}(\Omega)$ there holds

$$\int_{\Omega} [|\nabla u|^{p-2}(\nabla u, \nabla \varphi) - f(x)u^{q-1}\varphi - g(x)u^{r-1}\varphi] dx = 0. \tag{1.5}$$

Notice that u is a weak solution of problem (1.1), then u satisfies the following equation

$$\int_{\Omega} |\nabla u|^p dx - \int_{\Omega} f(x)|u|^q dx - \int_{\Omega} g(x)|u|^r dx = 0.$$

So it suggests that we could define a set

$$\Lambda = \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} f(x)|u|^q dx - \int_{\Omega} g(x)|u|^r dx = 0 \right\},$$

and make the following splitting for Λ :

$$\Lambda^+ = \left\{ u \in \Lambda : (p-r) \int_{\Omega} |\nabla u|^p dx - (q-r) \int_{\Omega} f(x)|u|^q dx > 0 \right\},$$

$$\Lambda^0 = \left\{ u \in \Lambda : (p-r) \int_{\Omega} |\nabla u|^p dx - (q-r) \int_{\Omega} f(x)|u|^q dx = 0 \right\},$$

$$\Lambda^- = \left\{ u \in \Lambda : (p-r) \int_{\Omega} |\nabla u|^p dx - (q-r) \int_{\Omega} f(x)|u|^q dx < 0 \right\}.$$

Assume that $u \in W_0^{1,p}(\Omega)$ is a solution of problem (1.1), u must belong to Λ . The main results can be described as follows:

THEOREM 1. *Assume that $1 < p < N$, $1 < r < p < q < p^*$, $f \in L^{\frac{p^*}{p^*-q}}(\Omega)$ and $g \in L^{\frac{p^*}{p^*-r}}(\Omega)$ satisfy (F_0) . Then, for $|f|^{\frac{1}{q-r}} |g|^{\frac{1}{p-r}} < \Theta$, problem (1.1) has at least two nonzero and nonnegative solutions $u_* \in \Lambda^+$, $u_{**} \in \Lambda^-$ with*

$$I(u_*) < 0 \text{ and } \|u_*\| < \left[\frac{q-r}{S^{\frac{r}{p}}(q-p)} \right]^{\frac{1}{p-r}} |g|^{\frac{1}{p-r}} < \|u_{**}\|.$$

Moreover, one of the two solutions is a ground state solution.

REMARK 1. To our best knowledge, Theorem 1.1 is up to date. In [5], [6], [13] and [25], they only studied the relation between the coefficient of the concave term and the existence of solutions for problem (1.1). While we provide the exact estimate Θ for the combined action of f and g . Moreover, we give the relation of size for the two solutions in $W_0^{1,p}(\Omega)$.

THEOREM 2. *Suppose $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$. Assume that $1 < p < N$, $1 < r < p < q < p^*$, and f, g satisfy the following condition,*
 (F_1) $f \in L^\infty(\Omega)$ with the set $\{x \in \Omega : f(x) > 0\}$ of positive measures, and $g \in L^\infty(\Omega)$ with $g(x) \geq 0, g \not\equiv 0$.
 Then, for $|f|^{\frac{1}{q-r}} |g|^{\frac{1}{p-r}} < \tilde{\Theta}$, problem (1.1) has at least two positive solutions $u_* \in \Lambda^+$, $u_{**} \in \Lambda^-$ with

$$I(u_*) < 0 \text{ and } \|u_*\| < \left[\frac{(q-r)|\Omega|^{\frac{p^*}{p^*-r}}}{S^{\frac{r}{p}}(q-p)} \right]^{\frac{1}{p-r}} |g|^{\frac{1}{p-r}} < \|u_{**}\|.$$

Moreover, one of the two solutions is a ground state solution.

THEOREM 3. *Assume that $1 < p < N$, $1 < r < p < q < p^*$, and f, g satisfy the following condition,*
 (F_2) $f \in L^{\frac{p^*}{p^*-q}}(\Omega)$ and $g \in L^{\frac{p^*}{p^*-r}}(\Omega)$ are nonzero and nonnegative functions.
 Then the same conclusions of Theorem 1 hold. Moreover, the two nonzero and nonnegative solutions are positive, and $u_* \in \Lambda^+$ is the positive ground state solution.

REMARK 2. On the one hand, under the condition of (F_1) or (F_2) , by the strong maximum principle, we prove that the nonzero and nonnegative solutions are positive. According to [8](pp:158, 198), under the condition of (F_1) , in order to obtain positive solutions, the condition of the boundary of is necessary.

On the other hand, it is worth noticing that we could not confirm the ground state solution lying in Λ^+ or Λ^- when f may change sign, because Λ^+ and Λ^- may not be connected submanifolds. Under the condition of (F_2) , we obtain that the positive ground state solution lies in Λ^+ .

This paper is organized as follows: in Section 2, we give some preliminaries which will be used to prove our main result, and in Section 3, we give the proof of Theorems 1-3.

2. Preliminaries

In this section, we give some lemmas to get ready for the proof of our main result.

LEMMA 1. Suppose that $1 < p < N$, $1 < r < p < q < p^*$, $f \in L^{\frac{p^*}{p^*-q}}(\Omega)$ and $g \in L^{\frac{p^*}{p^*-r}}(\Omega)$ satisfy (F_0) . Then there exists a constant $\Theta > 0$ such that $\Lambda^\pm \neq \emptyset$ for $|f|_{\frac{1}{q-p} \frac{p^*}{p^*-q}} |g|_{\frac{1}{p-r} \frac{p^*}{p^*-r}} < \Theta$. Moreover, $\Lambda^0 = \{0\}$ and Λ^- is a closed set for $|f|_{\frac{1}{q-p} \frac{p^*}{p^*-q}} |g|_{\frac{1}{p-r} \frac{p^*}{p^*-r}} < \Theta$.

Proof. According to the assumptions of f there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\int_{\Omega} f(x)|u|^q dx > 0.$$

In fact, let $E = \{x \in \Omega : f(x) > 0\}$, one obtains that E is a positive measure set. Then for any $\varepsilon > 0$ there exist a closed set F and a open set G such that $F \subset E \subset G$ and $meas(G - F) < \varepsilon$. From the arbitrariness of ε , we have $meas F > 0$. We choose $\tilde{u} \in C_0^1(\Omega)$ with $0 \leq \tilde{u} \leq 1$ such that $\tilde{u} = 1$ in F and $\tilde{u} = 0$ in $\Omega \setminus G$. Obviously, $\tilde{u} \in W_0^{1,p}(\Omega)$. By Hölder’s inequality, one has

$$\begin{aligned} \int_{\Omega} f(x)|\tilde{u}|^q dx &\geq \int_F f(x) dx - \int_{G-F} |f(x)| |\tilde{u}|^q dx \\ &\geq \int_F f(x) dx - (meas(G - F))^{\frac{q}{p^*}} \left(\int_{G-F} |f|^{\frac{p^*}{p^*-q}} dx \right)^{\frac{p^*-q}{p^*}} \\ &\geq \int_F f(x) dx - \varepsilon^{\frac{q}{p^*}} |f|_{\frac{p^*}{p^*-q}} \geq \frac{1}{2} \int_F f(x) dx > 0, \end{aligned}$$

where we choose $\varepsilon = \min \left\{ \left(\frac{\int_F f(x) dx}{2|f|_{\frac{p^*}{p^*-q}}} \right)^{\frac{p^*}{q}}, \frac{meas G}{2} \right\}$ such that $meas F \geq \frac{meas G}{2} > 0$

and $\varepsilon^{\frac{q}{p^*}} |f|_{\frac{p^*}{p^*-q}} \leq \frac{1}{2} \int_F f(x) dx$. Similarly, we can prove that there exists $u \in W_0^{1,p}(\Omega)$ such that $\int_{\Omega} g(x)|u|^r dx > 0$.

Case one. For any $u \in W_0^{1,p}(\Omega)$ such that $\int_{\Omega} f(x)|u|^q dx > 0$, we define $\phi \in C(\mathbb{R}^+, \mathbb{R})$ by

$$\phi(t) = t^{p-r} \|u\|^p - t^{q-r} \int_{\Omega} f(x)|u|^q dx.$$

Since

$$\phi'(t) = (p-r)t^{p-r-1} \|u\|^p - (q-r)t^{q-r-1} \int_{\Omega} f(x)|u|^q dx,$$

the function ϕ has a global maximum at

$$t_{\max} = \left[\frac{(p-r)\|u\|^p}{(q-r) \int_{\Omega} f(x)|u|^q dx} \right]^{\frac{1}{q-p}}.$$

By calculation, we have

$$\phi(t_{\max}) = \frac{q-p}{q-r} \left(\frac{p-r}{q-r} \right)^{\frac{p-r}{q-p}} \frac{\|u\|^{\frac{p(q-r)}{q-p}}}{\left(\int_{\Omega} f(x)|u|^q dx \right)^{\frac{p-r}{q-p}}}.$$

By the Hölder inequality and (1.4), we have

$$\int_{\Omega} g(x)|u|^r dx \leq |g|_{\frac{p^*}{p^*-r}} |u|_{p^*}^r \leq |g|_{\frac{p^*}{p^*-r}} S^{-\frac{r}{p}} \|u\|^r, \tag{2.1}$$

$$\int_{\Omega} f(x)|u|^q dx \leq |f|_{\frac{p^*}{p^*-q}} |u|_{p^*}^q \leq |f|_{\frac{p^*}{p^*-q}} S^{-\frac{q}{p}} \|u\|^q, \tag{2.2}$$

Then from (2.1)-(2.2), one has

$$\begin{aligned} \phi(t_{\max}) - \int_{\Omega} g(x)|u|^r dx &> \frac{q-p}{q-r} \left(\frac{p-r}{q-r} \right)^{\frac{p-r}{q-p}} \frac{\|u\|^{\frac{p(q-r)}{q-p}}}{\left(|f|_{\frac{p^*}{p^*-q}} |u|_{p^*}^q \right)^{\frac{p-r}{q-p}}} - |g|_{\frac{p^*}{p^*-r}} |u|_{p^*}^r \\ &= \left\{ \frac{q-p}{q-r} \left(\frac{p-r}{q-r} \right)^{\frac{p-r}{q-p}} \left(\frac{\|u\|^p}{|u|_{p^*}^p} \right)^{\frac{q-r}{q-p}} \frac{1}{|f|_{\frac{p^*}{p^*-q}}^{\frac{p-r}{q-p}} - |g|_{\frac{p^*}{p^*-r}}} \right\} |u|_{p^*}^r \\ &\geq \left\{ \frac{q-p}{q-r} \left(\frac{p-r}{q-r} \right)^{\frac{p-r}{q-p}} S^{\frac{q-r}{q-p}} \frac{1}{|f|_{\frac{p^*}{p^*-q}}^{\frac{p-r}{q-p}} - |g|_{\frac{p^*}{p^*-r}}} \right\} |u|_{p^*}^r > 0, \end{aligned} \tag{2.3}$$

provided that $|f|_{\frac{p^*}{p^*-q}}^{\frac{1}{q-p}} |g|_{\frac{p^*}{p^*-r}}^{\frac{1}{p-r}} < \Theta$, where

$$\Theta = \left(\frac{q-p}{q-r} \right)^{\frac{1}{p-r}} \left(\frac{p-r}{q-r} \right)^{\frac{1}{q-p}} S^{\frac{q-r}{(q-p)(p-r)}}.$$

On the one hand, assume that $\int_{\Omega} g(x)|u|^r dx > 0$. From (2.3), we obtain that there exist unique positive numbers $t^+ = t^+(u) < t_{max} < t^- = t^-(u)$ such that

$$\phi(t^+) = \int_{\Omega} g(x)|u|^r dx = \phi(t^-)$$

and

$$\phi'(t^+) > 0, \quad \phi'(t^-) < 0.$$

That is, $t^+u \in \Lambda^+$ and $t^-u \in \Lambda^-$. On the other hand, assume that $\int_{\Omega} g(x)|u|^r dx \leq 0$. Since $\phi'(t) > 0$ for all $0 < t < t_{max}$ and $\phi'(t) < 0$ for all $t > t_{max}$, and $\phi(0) = 0, \phi(t_{max}) > 0$ and $\phi(t) \rightarrow -\infty$ as $t \rightarrow +\infty$, it follows from (2.3) that there exists a unique $t^- > 0$ satisfying $t^- > t_{max}$ such that

$$\int_{\Omega} g(x)|u|^r dx = \phi(t^-), \quad \phi'(t^-) < 0,$$

which implies that $t^-u \in \Lambda^-$. Thus $\Lambda^{\pm} \neq \emptyset$ for all $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p^*-r}} < \Theta$.

Next, we prove that $\Lambda^0 = \{0\}$ for all $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p^*-r}} < \Theta$. By contradiction, suppose there exists some $u_0 \in \Lambda^0 \setminus \{0\}$, such that

$$(p-r)\|u_0\|^p - (q-r) \int_{\Omega} f(x)|u_0|^q dx = 0. \tag{2.4}$$

Since $u \in \Lambda$, that is

$$\|u_0\|^p = \int_{\Omega} f(x)|u_0|^q + \int_{\Omega} g(x)|u_0|^r dx.$$

Therefore,

$$\frac{q-p}{q-r} \|u_0\|^p - \int_{\Omega} g(x)|u_0|^r dx = 0. \tag{2.5}$$

Then, according to (2.3)-(2.5), for $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p^*-r}} < \Theta$, we have

$$\begin{aligned} 0 &< \frac{q-p}{q-r} \left(\frac{p-r}{q-r} \right)^{\frac{p-r}{q-p}} \frac{\|u_0\|^{\frac{p(q-r)}{q-p}}}{\left(\int_{\Omega} f(x)|u_0|^q dx \right)^{\frac{p-r}{q-p}}} - \int_{\Omega} g(x)|u_0|^r dx \\ &= \frac{q-p}{q-r} \left(\frac{p-r}{q-r} \right)^{\frac{p-r}{q-p}} \frac{\|u_0\|^{\frac{p(q-r)}{q-p}}}{\left(\frac{p-r}{q-r} \|u_0\|^p \right)^{\frac{p-r}{q-p}}} - \frac{q-p}{q-r} \|u_0\|^p \\ &= \frac{q-p}{q-r} \|u_0\|^p - \frac{q-p}{q-r} \|u_0\|^p = 0, \end{aligned}$$

which is a contradiction. Therefore, $\Lambda^0 = \{0\}$ for all $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p^*-r}} < \Theta$.

Finally, we claim that Λ^- is a closed set for $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p^*-r}} < \Theta$. Suppose that $\{u_n\} \subset \Lambda^-$ such that $u_n \rightarrow u$ as $n \rightarrow \infty$ in $W_0^{1,p}(\Omega)$, we need prove $u \in \Lambda^-$ for $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p^*-r}} < \Theta$. Since $u_n \in \Lambda^-$, one has

$$\|u_n\|^p - \int_{\Omega} f(x)|u_n|^q dx - \int_{\Omega} g(x)|u_n|^r dx = 0 \tag{2.6}$$

and

$$(p-r)\|u_n\|^p - (q-r) \int_{\Omega} f(x)|u_n|^q dx < 0. \tag{2.7}$$

Since $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$ as $n \rightarrow \infty$, it follows that

$$\|u\|^p - \int_{\Omega} f(x)|u|^q dx - \int_{\Omega} g(x)|u|^r dx = 0$$

and

$$(p-r)\|u\|^p - (q-r) \int_{\Omega} f(x)|u|^q dx \leq 0,$$

thus $u \in \Lambda^- \cup \Lambda^0$. If $u \in \Lambda^0$, since $\Lambda^0 = \{0\}$ for $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p^*-r}} < \Theta$, one has $u = 0$. However, from (2.6) and (2.7), for all $u_n \in \Lambda^-$, we obtain

$$(p-r)\|u_n\|^p < (q-r) \int_{\Omega} f(x)|u_n|^q dx,$$

consequently, by the Hölder inequality and (1.4), one has

$$\|u_n\| > \left[\frac{(p-r)S^{\frac{q}{p}}}{(q-r)|f|^{\frac{p^*}{p^*-q}}} \right]^{\frac{1}{q-p}} > 0, \quad \forall u_n \in \Lambda^-,$$

which contradicts $u = 0$. Thus $u \in \Lambda^-$ for $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p^*-r}} < \Theta$. Thus our claim is proved to be true.

Case two. For any $u \in W_0^{1,p}(\Omega)$ such that $\int_{\Omega} g(x)|u|^r dx > 0$. we define $\Phi \in C(\mathbb{R}^+, \mathbb{R})$ by

$$\Phi(t) = t^{p-q}\|u\|^p - t^{r-q} \int_{\Omega} g(x)|u|^r dx.$$

Since $1 < r < p < q$, one has $\Phi(t) \rightarrow -\infty$ as $t \rightarrow 0^+$ and $\Phi(t) \rightarrow 0$ as $t \rightarrow +\infty$. Moreover, one has

$$\Phi'(t) = -t^{r-q-1} \left[(q-p)t^{p-r}\|u\|^p - (q-r) \int_{\Omega} g(x)|u|^r dx \right].$$

Let $\Phi'(t) = 0$, one has

$$\hat{t} = \left[\frac{(q-r) \int_{\Omega} g(x)|u|^r dx}{(q-p)\|u\|^p} \right]^{\frac{1}{p-r}},$$

$$\Phi(\hat{t}) = \frac{p-r}{q-p} \left(\frac{q-p}{q-r} \right)^{\frac{q-r}{p-r}} \frac{\|u\|^{\frac{p(q-r)}{p-r}}}{\left(\int_{\Omega} g(x)|u|^r dx \right)^{\frac{q-p}{p-r}}},$$

and $\Phi'(t) > 0$ for all $0 < t < \hat{t}$, $\Phi'(t) < 0$ for all $t > \hat{t}$. Then from (2.1)-(2.2), one has

$$\begin{aligned} \Phi(\hat{t}) - \int_{\Omega} f(x)|u|^q dx &> \frac{p-r}{q-p} \left(\frac{q-p}{q-r} \right)^{\frac{q-r}{p-r}} \frac{\|u\|^{\frac{p(q-r)}{p-r}}}{\left(\int_{\Omega} g(x)|u|^r dx \right)^{\frac{q-p}{p-r}}} - |f|_{\frac{p^*}{p^*-q}} |u|_{p^*}^q \\ &\geq \left\{ \frac{p-r}{q-p} \left(\frac{q-p}{q-r} \right)^{\frac{q-r}{p-r}} \left(\frac{\|u\|^p}{|u|_{p^*}^p} \right)^{\frac{q-r}{p-r}} \frac{1}{|g|_{\frac{p^*}{p^*-r}}} - |f|_{\frac{p^*}{p^*-q}} \right\} |u|_{p^*}^q \\ &\geq \left\{ \frac{p-r}{q-p} \left(\frac{q-p}{q-r} \right)^{\frac{q-r}{p-r}} S^{\frac{q-r}{p-r}} \frac{1}{|g|_{\frac{p^*}{p^*-r}}} - |f|_{\frac{p^*}{p^*-q}} \right\} |u|_{p^*}^q > 0, \end{aligned} \tag{2.8}$$

provided that $|f|_{\frac{p^*}{p^*-q}} |g|_{\frac{p^*}{p^*-r}} < \Theta$, where Θ is defined in Case one. On the one hand, when $\int_{\Omega} f(x)|u|^q dx > 0$, From (2.8), we obtain that there exist unique positive numbers $t^+ = t^+(u) < \hat{t} < t^- = t^-(u)$ such that

$$\Phi(t^+) = \int_{\Omega} f(x)|u|^q dx = \Phi(t^-)$$

and

$$\Phi'(t^+) > 0, \quad \Phi'(t^-) < 0.$$

That is, $t^+u \in \Lambda^+$ and $t^-u \in \Lambda^-$. On the other hand, assume that $\int_{\Omega} f(x)|u|^q dx \leq 0$. From the property of Φ , there exists a unique $t^+ > 0$ satisfying $t^+ < \hat{t}$ such that

$$\int_{\Omega} f(x)|u|^q dx = \Phi(t^+), \quad \Phi'(t^+) > 0,$$

which implies that $t^+u \in \Lambda^+$. Thus $\Lambda^{\pm} \neq \emptyset$ for all $|f|_{\frac{p^*}{p^*-q}} |g|_{\frac{p^*}{p^*-r}} < \Theta$. Similar to Case one, we can also prove that $\Lambda^0 = \{0\}$ and Λ^- is a closed set for $|f|_{\frac{p^*}{p^*-q}} |g|_{\frac{p^*}{p^*-r}} < \Theta$. The proof of Lemma 1 is completed. \square

LEMMA 2. *The functional I is coercive and bounded from below on Λ .*

Proof. For any $u \in \Lambda$, we have

$$\int_{\Omega} |\nabla u|^p dx - \int_{\Omega} f(x)|u|^q dx - \int_{\Omega} g(x)|u|^r dx = 0,$$

consequently, it follows from (2.1) and $1 < r < p < q$ that

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \frac{1}{q}\int_{\Omega} f(x)|u|^q dx - \frac{1}{r}\int_{\Omega} g(x)|u|^r dx \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|^p - \left(\frac{1}{r} - \frac{1}{q}\right)\int_{\Omega} g(x)|u|^r dx \geq \frac{q-p}{pq}\|u\|^p - \frac{q-r}{qr}|g|_{\frac{p^*}{p^*-r}} S^{-\frac{r}{p}}\|u\|^r, \end{aligned}$$

which implies that I is coercive and bounded from below on Λ . This completes the proof of Lemma 2. \square

From Lemma 1 and Lemma 2, for $|f|_{\frac{q-p}{p^*-q}} |g|_{\frac{p-r}{p^*-r}} < \Theta$, the following definitions are well defined

$$m = \inf_{u \in \Lambda} I(u), \quad m^+ = \inf_{u \in \Lambda^+} I(u), \quad m^- = \inf_{u \in \Lambda^-} I(u).$$

Moreover, we can claim that $m^+ < 0$. In fact, for all $u \in \Lambda^+$, we have

$$(p-r)\|u\|^p > (q-r)\int_{\Omega} f(x)|u|^q dx,$$

consequently, since $1 < r < p < q$ and $u \neq 0$, it follows that

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \frac{1}{q}\int_{\Omega} f(x)|u|^q dx - \frac{1}{r}\int_{\Omega} g(x)|u|^r dx = \frac{q-r}{qr}\int_{\Omega} f(x)|u|^q dx - \frac{p-r}{pr}\|u\|^p \\ &< \frac{p-r}{qr}\|u\|^p - \frac{p-r}{pr}\|u\|^p = \left(\frac{1}{q} - \frac{1}{p}\right)\frac{p-r}{r}\|u\|^p < 0, \end{aligned}$$

this implies that $m^+ < 0$. Thus, one has $m \leq m^+ < 0$.

LEMMA 3. Give $u \in \Lambda$ (respectively Λ^\pm) and $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi > 0$, then there exist $\varepsilon > 0$ and a continue and differentiable function $t = t(w) > 0, w \in \mathbb{R}, |w| < \varepsilon$ satisfying that

$$t(0) = 1, \quad t(w)(u + w\varphi) \in \Lambda \text{ (respectively } \Lambda^\pm), \quad \forall w \in \mathbb{R}, |w| < \varepsilon.$$

Proof. For all $u \in \Lambda, \varphi \in W_0^{1,p}(\Omega)$ with $u, \varphi > 0$, define $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by:

$$F(t, w) = t^{p-r}\|(u + w\varphi)\|^p - t^{q-r}\int_{\Omega} f(x)|u + w\varphi|^q dx - \int_{\Omega} g(x)|u + w\varphi|^r dx.$$

Consequently, one has

$$F_t(t, w) = (p-r)t^{p-r-1}\|(u + w\varphi)\|^p - (q-r)t^{q-r-1}\int_{\Omega} f(x)|u + w\varphi|^q dx$$

Since $u \in \Lambda$ with $u > 0$, one has $F(1, 0) = 0$. Moreover, by Lemma 2.1, one obtains

$$F_t(1, 0) = (p - r) \int_{\Omega} |\nabla u|^p dx - (q - r) \int_{\Omega} |u|^q dx \neq 0.$$

Thus, applying the implicit function theorem at the point $(1, 0)$, we can obtain $\varepsilon > 0$ and a continuous and differentiable $t : B(0, \varepsilon) \rightarrow \mathbb{R}$ satisfying that

$$t(0) = 1, \quad t(w) > 0, \quad t(w)(u + w\varphi) \in \Lambda, \quad \forall w \in \mathbb{R}, |w| < \varepsilon.$$

Similarly, we can prove that the conclusion of the case $u \in \Lambda^\pm$ is true. This completes the proof of Lemma 3. \square

3. Proof of theorems

In this part, we prove that problem (1.1) has a nonnegative local minimizer solution in Λ^+ and Λ^- , respectively. Now, we give the proof of Theorem 1.

Proof of Theorem 1. Let $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p-r}} < \Theta$. The proof of Theorem 1.1 will be divided into two steps.

Step 1. We prove that there exists a nonzero and nonnegative solution of problem (1.1) in Λ^+ .

Obviously, by Lemma 1, one has that $\Lambda^+ \cup \Lambda^0$ is a non-empty closed set in $W_0^{1,p}(\Omega)$ for $|f|^{\frac{1}{q-p}} |g|^{\frac{1}{p-r}} < \Theta$. Moreover, from Lemma 2.2, $\inf_{u \in \Lambda^+ \cup \Lambda^0} I(u)$ is well defined. Applying Ekeland’s variational principle to the minimization problem

$\inf_{u \in \Lambda^+ \cup \Lambda^0} I(u)$, there exists a sequence $\{u_n\} \subset \Lambda^+ \cup \Lambda^0$ with the following properties:

- (i) $I(u_n) < \inf_{u \in \Lambda^+ \cup \Lambda^0} I(u) + \frac{1}{n}$;
- (ii) $I(u) \geq I(u_n) - \frac{1}{n} \|u - u_n\|, \quad \forall u \in \Lambda^+ \cup \Lambda^0$.

Since $I(u) = I(|u|)$, we can assume from the beginning that $u_n(x) \geq 0$ for all $x \in \Omega$. Obviously, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $u_* \geq 0$ such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \rightarrow u_*, & \text{strongly in } L^q(\Omega), \quad 2 \leq q < p^*, \\ u_n(x) \rightarrow u_*(x), & \text{a.e. in } \Omega, \end{cases} \tag{3.1}$$

as $n \rightarrow \infty$. Since $m \leq m^+ < 0$, we claim that $u_* \not\equiv 0$ in Ω . By the Vitali theorem (see [24] pp:133), we can prove that

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x) |u_n|^q dx = \int_{\Omega} f(x) |u_*|^q dx. \tag{3.2}$$

Indeed, we only need prove that $\{\int_{\Omega} f(x)|u_n|^q dx, n \in N\}$ is equi-absolutely-continuous. Note that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, by the Sobolev embedding theorem, then exists a constant $C_0 > 0$ such that $|u_n|_{p^*} \leq C_0 < \infty$. From (2.2), for every $\varepsilon > 0$, setting $\delta > 0$, when $E \subset \Omega$ with $meas E < \delta$, we have

$$\int_E f(x)|u_n|^q dx \leq |u_n|_{p^*}^q \left(\int_E |f|^{\frac{p^*}{p^*-q}} dx \right)^{\frac{p^*-q}{p^*}} < \varepsilon,$$

where the last inequality is from the absolutely-continuity of $\int_{\Omega} |f|^{\frac{p^*}{p^*-q}} dx$. Thus, our claim is true. Similarly,

$$\lim_{n \rightarrow \infty} \int_{\Omega} g(x)|u_n|^r dx = \int_{\Omega} g(x)|u_*|^r dx. \tag{3.3}$$

By the weakly lower semicontinuity of the norm, and combining with (3.2) and (3.3), it follows that

$$\begin{aligned} I(u_*) &= \frac{1}{p} \|u_*\|^p - \frac{1}{q} \int_{\Omega} f(x)|u_*|^q dx - \frac{1}{r} \int_{\Omega} g(x)|u_*|^r dx \\ &\leq \liminf_{n \rightarrow \infty} \left[\frac{1}{p} \|u_n\|^p - \frac{1}{q} \int_{\Omega} f(x)|u_n|^q dx - \frac{1}{r} \int_{\Omega} g(x)|u_n|^r dx \right] = \liminf_{n \rightarrow \infty} I(u_n) \\ &= \inf_{u \in \Lambda^+ \cup \Lambda^0} I(u) < 0, \end{aligned} \tag{3.4}$$

which implies that $u_* \not\equiv 0$ in Ω . Thus $u_* \geq 0, u_* \not\equiv 0$ in Ω . Since $m^+ = \inf_{u \in \Lambda^+} I(u) < 0$, one has

$$\inf_{u \in \Lambda^+ \cup \Lambda^0} I(u) = \inf_{u \in \Lambda^+} I(u) = m^+ < 0.$$

Consequently, combining with (i), for all n large enough we have $I(u_n) < 0$. Thus, the sequence $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$, such that $\{u_n\} \subset \Lambda^+$. From now on, we could assume that $\{u_n\} \subset \Lambda^+$. Now, we will prove that $u_* \in \Lambda^+$ is a solution of problem (1.1).

First, we prove that u_* is a solution of problem (1.1). Applying Lemma 3 with $u = u_n$, and $\varphi \in W_0^{1,p}(\Omega)$, $w > 0$ small enough, we find a sequence of continuous functions $t_n = t_n(w)$ such that $t_n(0) = 1$ and $t_n(w)(u_n + w\varphi) \in \Lambda^+$. Obviously, $u_n \in \Lambda$, one has

$$\|u_n\|^p - \int_{\Omega} f(x)|u_n|^q dx - \int_{\Omega} g(x)|u_n|^r dx = 0. \tag{3.5}$$

By the subadditivity of the norm, it follows from (ii) that

$$\begin{aligned} \frac{|t_n(w) - 1| \cdot \|u_n\| + wt_n(w)\|\varphi\|}{n} &\geq \frac{1}{n} \|t_n(w)(u_n + w\varphi) - u_n\| \\ &\geq I(u_n) - I[t_n(w)(u_n + w\varphi)] \\ &= -\frac{t_n^p(w) - 1}{p} \|u_n\|^p + \frac{t_n^q(w) - 1}{q} \int_{\Omega} f(x)|u_n + w\varphi|^q dx \\ &\quad + \frac{t_n^r(w) - 1}{r} \int_{\Omega} g(x)|u_n + w\varphi|^r dx \\ &\quad + \frac{t_n^p(w)}{p} (\|u_n\|^p - \|u_n + w\varphi\|^p) \\ &\quad + \frac{1}{q} \int_{\Omega} f(x)(|u_n + w\varphi|^q - |u_n|^q) dx \\ &\quad + \frac{1}{r} \int_{\Omega} g(x)(|u_n + w\varphi|^r - |u_n|^r) dx, \end{aligned}$$

consequently, dividing by $w > 0$ and let $w \rightarrow 0^+$, combining (3.5), we have

$$\begin{aligned} \frac{1}{n} (|t'(0)| \cdot \|u_n\| + \|\varphi\|) &\geq -\left[\|u_n\|^p - \int_{\Omega} f(x)|u_n|^q dx - \int_{\Omega} g(x)|u_n|^r dx \right] t'(0) \\ &\quad - \int_{\Omega} |\nabla u_n|^{p-2} (\nabla u_n, \nabla \varphi) + \int_{\Omega} f(x)|u_n|^{q-2} u_n \varphi dx \\ &\quad + \int_{\Omega} g(x)|u_n|^{r-2} u_n \varphi dx \tag{3.6} \\ &= - \int_{\Omega} |\nabla u_n|^{p-2} (\nabla u_n, \nabla \varphi) dx + \int_{\Omega} f(x)|u_n|^{q-2} u_n \varphi dx \\ &\quad + \int_{\Omega} g(x)|u_n|^{r-2} u_n \varphi dx. \end{aligned}$$

Moreover, Lemma 3 suggests that there exists a constant $C > 0$, such that $|t'_n(0)| \leq C$ for all $n \in \mathbf{N}^+$. Therefore, from the boundedness of $\{u_n\}$, we can choose a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$, passing to the limit as $n \rightarrow \infty$ in (3.6), we get

$$\int_{\Omega} |\nabla u_*|^{p-2} (\nabla u_*, \nabla \varphi) dx - \int_{\Omega} f(x)|u_*|^{q-2} u_* \varphi dx - \int_{\Omega} g(x)|u_*|^{r-2} u_* \varphi dx \geq 0, \tag{3.7}$$

for any $\varphi \in W_0^{1,p}(\Omega)$. Since (3.7) also holds for $-\varphi$, one has

$$\int_{\Omega} |\nabla u_*|^{p-2} (\nabla u_*, \nabla \varphi) dx - \int_{\Omega} f(x)|u_*|^{q-2} u_* \varphi dx - \int_{\Omega} g(x)|u_*|^{r-2} u_* \varphi dx = 0, \tag{3.8}$$

for any $\varphi \in W_0^{1,p}(\Omega)$. Thus, u_* is a solution of problem (1.1).

Secondly, we prove that $u_* \in \Lambda^+$. Choosing $\varphi = u_*$ in (3.8), one has

$$\|u_*\|^p = \int_{\Omega} f(x)u_*^q dx + \int_{\Omega} g(x)u_*^r dx, \tag{3.9}$$

and this implies $u_* \in \Lambda$. By (3.2) and (3.3), let $n \rightarrow \infty$, it follows from (3.5) that

$$\lim_{n \rightarrow \infty} \|u_n\|^p = \int_{\Omega} f(x)u_*^q dx + \int_{\Omega} g(x)u_*^r dx,$$

combining with (3.9), one has

$$\lim_{n \rightarrow \infty} \|u_n\|^P = \|u_*\|^P.$$

Combining with (3.1), $u_n \rightharpoonup u_*$ in $W_0^{1,p}(\Omega)$, one has $u_n \rightarrow u_*$ in $W_0^{1,p}(\Omega)$ as $n \rightarrow \infty$. Thus, we can obtain that $\lim_{n \rightarrow \infty} I(u_n) = I(u_*) = m^+ < 0$ and $u_* \in \Lambda^+ \cup \Lambda^0$. According to Lemma 1, $\Lambda^0 = \{0\}$ for $|f| \frac{\frac{1}{q-p}}{p^*} |g| \frac{\frac{1}{p-r}}{p^*} < \Theta$, one has $u_* \in \Lambda^+$. Therefore, u_* is a nonzero and nonnegative solution of problem (1.1) in Λ^+ with $I(u_*) < 0$. Moreover, since $u_* \in \Lambda^+$, from (2.1), one has

$$(q - p)\|u_*\|^p < (q - r) \int_{\Omega} g(x)|u_*|^r dx \leq (q - r) |g| \frac{p^*}{p^* - r} S^{-\frac{r}{p}} \|u_*\|^r,$$

which implies that

$$\|u_*\| < \left[\frac{q - r}{S^{\frac{r}{p}}(q - p)} \right]^{\frac{1}{p-r}} |g| \frac{\frac{1}{p-r}}{\frac{p^*}{p^* - r}}.$$

Step 2. We prove that there exists a nonzero and nonnegative solution of problem (1.1) in Λ^- .

By Lemma 1, Λ^- is close in $W_0^{1,p}(\Omega)$. Apply Ekeland’s variational principle to the minimization problem $m^- = \inf_{u \in \Lambda^-} I(u)$, there exists a sequence $\{v_n\} \subset \Lambda^-$ with the following properties:

- (i) $I(v_n) < m^- + \frac{1}{n}$;
- (ii) $I(v) \geq I(v_n) - \frac{1}{n} \|v - v_n\|, \forall v \in \Lambda^-$.

Since $I(u) = I(|u|)$, we could suppose that $v_n(x) > 0$ for all $x \in \Omega$. From Lemma 2, one has $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, there exist a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, and $u_{**} \in W_0^{1,p}(\Omega)$ with $u_{**} \geq 0$ such that

$$\begin{cases} v_n \rightharpoonup u_{**}, & \text{weakly in } W_0^{1,p}(\Omega), \\ v_n \rightarrow u_{**}, & \text{strongly in } L^q(\Omega), \ 2 \leq q < p^*, \\ v_n(x) \rightarrow u_{**}(x), & \text{a.e. in } \Omega, \end{cases} \tag{3.10}$$

as $n \rightarrow \infty$. Similar to (3.2), we also have

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x)|v_n|^q dx = \int_{\Omega} f(x)|u_{**}|^q dx, \tag{3.11}$$

$$\lim_{n \rightarrow \infty} \int_{\Omega} g(x)|v_n|^r dx = \int_{\Omega} g(x)|u_{**}|^r dx. \tag{3.12}$$

Since $v_n \in \Lambda^-$, one has

$$(p - r)\|v_n\|^p < (q - r) \int_{\Omega} f(x)|v_n|^q dx,$$

consequently, by the Hölder inequality and (1.4), it follows that

$$\|v_n\| \geq \left[\frac{S^{\frac{q}{p}}(p-r)}{|f|_{\frac{p^*}{p^*-q}}(q-r)} \right]^{\frac{1}{q-p}} > 0,$$

which implies that $u_{**}(x) \geq 0$ in Ω and $u_{**} \not\equiv 0$.

Next, we can repeat *Step 1* to prove that u_{**} is a solution of problem (1.1).

Finally, we prove $u_{**} \in \Lambda^-$. Since Λ^- is closed, we only need to prove $v_n \rightarrow u_{**}$ as $n \rightarrow \infty$ in $W_0^{1,p}(\Omega)$. As well known, from (3.10), one has

$$\int_{\Omega} |\nabla v_n|^p dx \geq \int_{\Omega} |\nabla(v_n - u_{**})|^p dx + \int_{\Omega} |\nabla u_{**}|^p dx.$$

Consequently, since $v_n \in \Lambda^-$ and u_{**} is a positive solution of problem (1.1), it follows from (3.11) and (3.12) that

$$\begin{aligned} 0 &= \int_{\Omega} |\nabla v_n|^p dx - \int_{\Omega} f(x)|v_n|^q dx - \int_{\Omega} g(x)|v_n|^r dx \\ &= \lim_{n \rightarrow \infty} \left(\int_{\Omega} |\nabla v_n|^p dx - \int_{\Omega} f(x)|v_n|^q dx - \int_{\Omega} g(x)|v_n|^r dx \right) \\ &\geq \lim_{k \rightarrow \infty} \|v_n - u_{**}\|^p + \|u_{**}\|^p - \int_{\Omega} f(x)|u_{**}|^q dx - \int_{\Omega} g(x)|u_{**}|^r dx = \lim_{k \rightarrow \infty} \|v_n - u_{**}\|^p \\ &\geq 0, \end{aligned}$$

which implies that $\lim_{n \rightarrow \infty} \|v_n - u_{**}\| = 0$. Thus, we obtain $u_{**} \in \Lambda^-$ and $I(u_{**}) = m^-$.

Therefore, u_{**} is a nonzero and nonnegative solution of problem (1.1) in Λ^+ . Since $u_{**} \in \Lambda^-$, similar to u_* , from (1.1), we can easy obtain

$$\|u_{**}\| > \left[\frac{q-r}{S^{\frac{r}{p}}(q-p)} \right]^{\frac{1}{p-r}} |g|_{\frac{1}{p^*-r}}.$$

Since $I(u_*) = m^+$ and $I(u_{**}) = m^-$, one of u_* and u_{**} is a nonzero and nonnegative ground state solution of problem (1.1). Then we complete the proof of Theorem 1. \square

Now, we give the proof of Theorem 2.

Proof of Theorem 2. Since f, g satisfy (F_1) , by the Hölder inequality and (1.4), we have

$$\int_{\Omega} g(x)|u|^r dx \leq |g|_{\infty} |\Omega|^{\frac{p^*-r}{p^*}} |u|_{p^*}^r \leq |g|_{\infty} |\Omega|^{\frac{p^*-r}{p^*}} S^{-\frac{r}{p}} \|u\|^r, \tag{3.13}$$

$$\int_{\Omega} f(x)|u|^q dx \leq |f|_{\infty} |\Omega|^{\frac{p^*-q}{p^*}} |u|_{p^*}^q \leq |f|_{\infty} |\Omega|^{\frac{p^*-q}{p^*}} S^{-\frac{q}{p}} \|u\|^q.$$

Similar to Lemma 1, we can obtain the exact estimate $\tilde{\Theta} = |\Omega|^{-\frac{(p^*-q)(q-r)+(p^*-r)(q-p)}{p^*(q-p)(p-r)}} \Theta$,

where Θ is defined in Lemma 1. Similar to the proof of Theorem 1, for $|f|_{\infty}^{\frac{1}{q-p}} |g|_{\infty}^{\frac{1}{p-r}} <$

$\tilde{\Theta}$, we can prove that u_* and u_{**} are two nonzero and nonnegative solutions of problem (1.1) and one of the two solutions is a ground state solution. Moreover, since $u_* \in \Lambda^+$ and $u_{**} \in \Lambda^-$, from (3.13), we can get

$$I(u_*) < 0 \text{ and } \|u_*\| < \left[\frac{(q-r)|\Omega|^{\frac{p^*}{p^*-r}}}{S^{\frac{r}{p}}(q-p)} \right]^{\frac{1}{p-r}} \|g\|_{\infty}^{\frac{1}{p-r}} < \|u_{**}\|.$$

Thus, we only need prove that $u_*, u_{**} > 0$ in Ω . Assume that (F_1) holds. Since $u_*, u_{**} \in W_0^{1,p}(\Omega)$, then by the embedding theorem we have $u_*, u_{**} \in L^{p^*}(\Omega)$. Since $f, g \in L^\infty(\Omega)$, by the regularity of weak solutions, we have $u_*, u_{**} \in W^{2, \frac{p^*}{q}}(\Omega)$. Then, by the classical bootstrap argument, one has $u_* \in W^{2,s}(\Omega)$ for all $1 \leq s < \infty$. Using the embedding theorem again, we have $u_* \in C^{1,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$. Since $u_* \geq 0, u_* \not\equiv 0$, by the Harnack inequality(see [28]), one has $u_* > 0$ in Ω . Similarly, we can obtain that $u_{**} > 0$ in Ω . Then the proof of Theorem 2 is completed. \square

Finally, we prove Theorem 3.

Proof of Theorem 3. Since $m = \inf_{u \in \Lambda} I(u) \leq m^+ < 0$, in order to prove the ground state solution belongs to Λ^+ , we need prove that $m = m^+$. Obviously, Λ is a non-empty closed set in $W_0^{1,p}(\Omega)$ for $|f|^{\frac{1}{p^*-q}} |g|^{\frac{1}{p^*-r}} < \Theta$. Applying Ekeland’s variational principle to the minimization problem $m = \inf_{u \in \Lambda} I(u)$, similarly to Step 1 of the proof of Theorem 1, we can prove that there exists a nonzero and nonnegative solution of problem (1.1) in Λ for $|f|^{\frac{1}{p^*-q}} |g|^{\frac{1}{p^*-r}} < \Theta$. Without loss of generality, we also denote this solution by u_* , then $I(u_*) = m$ and $u_* \in \Lambda$.

Now, we claim that $u_* \in \Lambda^+$. On the contrary, assume that $u_* \in \Lambda^- (\Lambda^0 = \{0\})$ for $|f|^{\frac{1}{p^*-q}} |g|^{\frac{1}{p^*-r}} < \Theta$. From (F_2) , one has

$$\int_{\Omega} f(x)|u|^{p+1} dx > 0 \text{ and } \int_{\Omega} g(x)|u|^{q+1} dx > 0.$$

By Lemma 1, then there exist positive numbers $t_0^+ < t_{max} < t_0^- = 1$ such that $t_0^+ u_* \in \Lambda^+$ and $t_0^- u_* \in \Lambda^-$ and

$$I(t_0^+ u_*) \leq I(t_0^- u_*) = I(u_*) = m,$$

which is a contradiction. Hence, $I(u_*) = m = m^-$ and $u_* \in \Lambda^+$. Obviously, u_* is a nonzero and nonnegative ground state solution of problem (1.1) with $I(u_*) = m = m^- < 0$. Similarly to Step 2 of the proof of Theorem 1, one obtains that there exists a nonzero and nonnegative solution $u_{**} \in \Lambda^-$. And we can also have

$$\|u_*\| < \left[\frac{q-r}{S^{\frac{r}{p}}(q-p)} \right]^{\frac{1}{p-r}} \|g\|_{\infty}^{\frac{1}{p-r}} < \|u_{**}\|.$$

Finally, we only need prove that $u_*, u_{**} > 0$ in Ω . Assume that (F_2) holds. From (3.7), it follows that

$$\int_{\Omega} |\nabla u_*|^{p-2} (\nabla u_*, \nabla \varphi) dx \geq 0, \quad \text{for } \varphi \in W_0^{1,p}(\Omega),$$

which means that u_* satisfies

$$-\Delta_p u_* \geq 0, \quad \text{in } \Omega.$$

Since $u_* \geq 0, u_* \not\equiv 0$, by the strong maximum principle (see [29]), one has $u_* > 0$ in Ω . Similarly, we can obtain that $u_{**} > 0$ in Ω . This completes the proof of Theorem 3. \square

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