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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF A LANCHESTER–TYPE MODEL

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Abstract. An ordinary differential system referred to as Lanchester-type model is studied. Asymptotic properties of solutions for such systems are considered. In particular, we examine how the limit of the solution as time tends to the infinity varies according to the initial data and we find asymptotic form of solutions that decay to $(0,0)$.

1. Introduction

The differential system we study is

$$\begin{cases} x' = -a(t)xy, \\ y' = -b(t)xy, \end{cases} \tag{S}$$

where we assume throughout the paper that $a(t)$ and $b(t)$ are positive continuous functions on $[0,\infty)$. Additional conditions will be imposed later.

System (S) is known as one of Lanchester-type model, which describes many phenomena appearing in economics, logistics, biology, and so on. It was F. W. Lanchester [6] who first proposed system (S) to describe combat situations. It is said in [1, 3, 4] that system (S) is a model of guerrilla engagements.

It seems that several scientists and technicians engaged in operational research treat such models via numerical methods; see, for example, [1, 3, 10]. However, as far as we know, there are few results treating mathematical models like system (S) rigorously. In [4, 9] differential systems similar to (S) were considered mathematically. In [2, 5, 7, 8] related results are obtained for other Lanchester-type models. Motivated by these facts, in the paper [11] one of the authors of the present paper has analyzed system (S) rigorously and proved some asymptotic properties of solutions of (S). In the present paper we will proceed further in this direction.

Let $x(0) > 0$ and $y(0) > 0$. Then we can show that the (local) solution $(x(t), y(t))$ of (S) exists globally on $[0,\infty)$, and $x(t) > 0$ and $y(t) > 0$ there, because for example, the formulas

$$x(t) = x(0)\exp\left(-\int_0^t a(s)y(s)ds\right) \quad \text{and} \quad y(t) = y(0)\exp\left(-\int_0^t b(s)x(s)ds\right) \tag{1}$$

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hold as long as $(x(t), y(t))$ exists. Therefore, $x(t)$ and $y(t)$ both decrease and $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$ exist as nonnegative numbers. In this paper, among other things, we will focus on the values of $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

Let $x_0 > 0$ and $y_0 > 0$. Throughout the paper we denote by $(x(t;x_0,y_0), y(t;x_0,y_0))$ the solution of (S) satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$. In the paper [11] the following result concerning the limits of solutions to (S) is obtained.

THEOREM 0 ([11]). *Suppose that $a(t)$ and $b(t)$ satisfy the growth conditions*

$$0 < \liminf_{t\to\infty} \frac{a(t)}{t^{\lambda_1}} \leqslant \limsup_{t\to\infty} \frac{a(t)}{t^{\lambda_2}} < \infty \quad \textit{and}$$
$$0 < \liminf_{t\to\infty} \frac{b(t)}{t^{\mu_1}} \leqslant \limsup_{t\to\infty} \frac{b(t)}{t^{\mu_2}} < \infty,$$

for some constants $\lambda_1, \lambda_2, \mu_1 > -1$ and $\mu_2 > -1$. Then for arbitrarily fixed $x_0 > 0$ there are unique numbers $\beta_1 = \beta_1(x_0) > 0$ and $\beta_2 = \beta_2(x_0) (\geqslant \beta_1)$ such that:

(i) if $0 < y_0 < \beta_1$, then $\lim_{t\to\infty} x(t;x_0,y_0) > 0$ and $\lim_{t\to\infty} y(t;x_0,y_0) = 0$;

(ii) if $\beta_1 \leqslant y_0 \leqslant \beta_2$, then $\lim_{t\to\infty} x(t;x_0,y_0) = \lim_{t\to\infty} y(t;x_0,y_0) = 0$;

(iii) if $y_0 > \beta_2$, then $\lim_{t\to\infty} x(t;x_0,y_0) = 0$ and $\lim_{t\to\infty} y(t;x_0,y_0) > 0$.

EXAMPLE 1. As a typical example of system (S), consider the case where $a(t) \equiv a_0$ and $b(t) \equiv b_0$ for some positive constants a_0 and b_0:

$$\begin{cases} x' = -a_0 xy, \\ y' = -b_0 xy. \end{cases} \tag{S$_0$}$$

We note that, for a solution $(x(t), y(t)) \equiv (x(t;x_0,y_0), y(t;x_0,y_0))$ of (S_0),

$$(b_0 x(t) - a_0 y(t))' = -a_0 b_0 x(t) y(t) + a_0 b_0 x(t) y(t) \equiv 0$$

holds and therefore $b_0 x(t) - a_0 y(t) \equiv b_0 x_0 - a_0 y_0$. Employing this property and introducing the constant $m = m(x_0, y_0) = b_0 x_0 - a_0 y_0$, we can solve system (S_0) explicitly:

$$x(t) = \frac{m x_0 e^{mt}}{b_0 x_0 (e^{mt} - 1) + m} \quad \text{and} \quad y(t) = \frac{m y_0}{m e^{mt} + a_0 y_0 (e^{mt} - 1)} \quad \text{if} \quad m \neq 0$$

and

$$x(t) = \frac{x_0}{b_0 x_0 t + 1} \quad \text{and} \quad y(t) = \frac{y_0}{a_0 y_0 t + 1} \qquad \text{if} \quad m = 0.$$

This shows that, for system (S_0), the two critical values β_1 and β_2 obtained by Theorem 0 are identical: $\beta_1 = \beta_2 = b_0 x_0 / a_0$. Further, we find that the decay rates of solutions decaying to $(0,0)$ are $O(t^{-1})$, whereas decaying components of all of the other solutions have exponential decay rates as $t \to \infty$. That is, solutions of (S_0)

tending to $(0,0)$ decay slower than those of (S_0) tending to non-zero vectors. By (1) we can find that generally this fact is true.

From this simple example, the following three problems arise naturally.

PROBLEM I. Do the critical numbers β_1 and β_2 (referred in Theorem 0) coincide? That is, for arbitrarily fixed $x_0 > 0$, is the solution (x,y) satisfying $x(0) = x_0$ as well as $\lim_{t\to\infty}(x(t),y(t)) = (0,0)$ unique?

PROBLEM II. How do the limit values $\lim_{t\to\infty}(x(t),y(t))$ for solutions (x,y) of (S) vary according to the initial values $(x(0),y(0))$?

PROBLEM III. How do solutions that decay to $(0,0)$ behave at $+\infty$?

The aim of the paper is to solve these problems. Under several additional assumptions we can answer them concretely. In Section 2 we consider Problems I and II. In Section 3 we consider Problem III.

2. The properties of limit points of positive solutions

In this section we consider system (S) under the following additional assumptions:

(A1) $0 < \inf_{t\geqslant 0} \frac{a(t)}{b(t)} \leqslant \sup_{t\geqslant 0} \frac{a(t)}{b(t)} < \infty$ and

(A2) $\int_0^\infty a(t)dt = \infty$.

REMARK 2. Under assumption (A1), (A2) implies that $\int_0^\infty b(t)dt = \infty$.

Let us define the set $S \subset \mathbf{R}^2$ by

$$S = \{(C,0)\,|\,C > 0\} \cup \{(0,0)\} \cup \{(0,C)\,|\,C > 0\}.$$

By assumption (A2) the limit point of every solution of (S) belongs to S; see [11, Remark 2].

For arbitrarily fixed $x_0 > 0$, we introduce the set S_{x_0} by

$$S_{x_0} = \{(x,y) \in S\,|\,x < x_0\}.$$

To consider Problems I and II we introduce the mapping $\omega_{x_0} = \omega : (0,\infty) \to [0,\infty) \times [0,\infty)$ defined by

$$\omega(y_0) = \left(\lim_{t\to\infty} x(t;x_0,y_0), \lim_{t\to\infty} y(t;x_0,y_0)\right).$$

For example, the values of ω_{x_0} associated to the simple system (S_0) in Example 1 is given explicitly by

$$\omega_{x_0}(y_0) = \begin{cases} (x_0 - a_0y_0/b_0, 0), & \text{for } y_0 \in (0, b_0x_0/a_0), \\ (0,0), & \text{for } y_0 = b_0x_0/a_0, \\ (0, y_0 - b_0x_0/a_0), & \text{for } y_0 \in (b_0x_0/a_0, \infty). \end{cases}$$

The following is an answer to Problem II.

THEOREM 3. *Let (A1) and (A2) hold, and $x_0 > 0$ be fixed arbitrarily. Then, the mapping $\omega_{x_0} = \omega$ is a continuous bijection from $(0,\infty)$ to S_{x_0}. Therefore, for any $(x_\infty, y_\infty) \in S_{x_0}$ there is one and only one solution (x,y) of (S) satisfying $x(0) = x_0$ and $\lim_{t\to\infty}(x(t),y(t)) = (x_\infty, y_\infty)$.*

By Theorem 3 we can conclude that the critical values β_1 and β_2 obtained by Theorem 0 coincide, which is an answer to Problem I.

COROLLARY 4. *Let (A1) and (A2) hold and $x_0 > 0$ be fixed arbitrarily. Then there is a unique number $\beta_0 = \beta_0(x_0) > 0$ such that:*

(i) if $0 < y_0 < \beta_0$, then $\lim_{t\to\infty} x(t;x_0,y_0) > 0$ and $\lim_{t\to\infty} y(t;x_0,y_0) = 0$;

(ii) if $y_0 = \beta_0$, then $\lim_{t\to\infty} x(t;x_0,y_0) = \lim_{t\to\infty} y(t;x_0,y_0) = 0$;

(iii) if $y_0 > \beta_0$, then $\lim_{t\to\infty} x(t;x_0,y_0) = 0$ and $\lim_{t\to\infty} y(t;x_0,y_0) > 0$.

The following corollary is a simple consequence of Theorem 3 and Lemma 6 below.

COROLLARY 5. *Let (A1) and (A2) hold and $x_0 > 0$ be fixed arbitrarily. Then for the number $\beta_0 = \beta_0(x_0)$ obtained by Corollary 4, it holds that:*

(i) if $0 < y_{01} < y_{02} < \beta_0$, then $\lim_{t\to\infty} x(t;x_0,y_{01}) > \lim_{t\to\infty} x(t;x_0,y_{02})$;

(ii) if $\beta_0 < y_{01} < y_{02}$, then $\lim_{t\to\infty} y(t;x_0,y_{01}) < \lim_{t\to\infty} y(t;x_0,y_{02})$.

To establish the results mentioned above we need the following lemma which was proved in [11].

LEMMA 6. *([11], Lemma 4; Strong comparison principle) Let $(x_1(t),y_1(t))$ and $(x_2(t),y_2(t))$ be solutions of system (S). If $x_1(0) \geqslant x_2(0), y_1(0) \leqslant y_2(0)$ and $(x_1(0),y_1(0)) \neq (x_2(0),y_2(0))$, then $x_1(t) > x_2(t)$ and $y_1(t) < y_2(t)$, for $t > 0$.*

As an immediate consequence of the lemma, we obtain the following.

COROLLARY 7. *(Comparison principle) Let $(x_1(t),y_1(t))$ and $(x_2(t),y_2(t))$ be solutions of system (S). If $x_1(0) \geqslant x_2(0)$ and $y_1(0) \leqslant y_2(0)$, then $x_1(t) \geqslant x_2(t)$ and $y_1(t) \leqslant y_2(t)$, for $t > 0$.*

The next lemma is so simple, however, which plays an important role in proving Theorem 3.

LEMMA 8. *Let $(x_1(t),y_1(t))$ and $(x_2(t),y_2(t))$ be solutions of system (S) satisfying $x_1(0) \geqslant x_2(0)$ and $y_1(0) \leqslant y_2(0)$.*

(i) The function $x_2(t)/x_1(t)$ is nonincreasing, whereas the function $y_2(t)/y_1(t)$ is nondecreasing in $t > 0$.

(ii) Furthermore, let either $x_1(0) > x_2(0)$ or $y_1(0) < y_2(0)$ hold. Then the function $x_2(t)/x_1(t)$ is strictly decreasing, whereas $y_2(t)/y_1(t)$ is strictly increasing in $t > 0$.

Proof.

(i) The straightforward computations give

$$\frac{d}{dt}\left(\frac{x_2(t)}{x_1(t)}\right) = -\frac{a(t)x_2(t)\{y_2(t)-y_1(t)\}}{x_1(t)} \quad \text{and}$$
$$\frac{d}{dt}\left(\frac{y_2(t)}{y_1(t)}\right) = -\frac{b(t)y_2(t)\{x_2(t)-x_1(t)\}}{y_1(t)}.$$

Therefore by Corollary 7 we find that $(x_2(t)/x_1(t))' \leqslant 0$ and $(y_2(t)/y_1(t))' \geqslant 0$.

(ii) By Lemma 6 and the above computation we find that $(x_2(t)/x_1(t))' < 0$ and $(y_2(t)/y_1(t))' > 0$ for $t > 0$.

This completes the proof. □

Proof of Theorem 3.

Part 1: Proof of the continuity of the function $\omega = \omega_{x_0}$. By assumption (A1), there are two positive constants m and M satisfying

$$m < b(t)/a(t) < M, \quad t > 0.$$

Then, for any solution $(x(t), y(t))$ of (S) we have

$$m < y'(t)/x'(t) \equiv dy/dx < M, \quad t > 0. \tag{2}$$

(Note that y can be regarded as a function of x because the correspodences $t \mapsto x(t)$ and $t \mapsto y(t)$ are both strictly monotone.) We prove that ω is continuous at given $\beta > 0$. The proof is devided into three cases, according to the image $\omega(\beta)$.

Case 1: The case where $\omega(\beta) = (C, 0)$, *for some* $C > 0$. For arbitrary $\varepsilon > 0$ satisfying $\varepsilon < C$, let l_1 be the line with slope m passing through the point $(C-\varepsilon, 0)$ and l_2 be the line with slope M passing through the point $(C+\varepsilon, 0)$. Further, let U be the open triangular set in $\mathbf{R}^2$ surrounded by the lines l_1, l_2 and the x-axis. For sufficiently large $T > 0$, we have $(x(T;x_0,\beta), y(T;x_0,\beta)) \in U$. Therefore, for sufficiently small $\delta > 0$ the property $|y_0 - \beta| < \delta$ implies that $(x(T;x_0,y_0), y(T;x_0,y_0)) \in U$.

Then, from (2), we can show that $(x(t;x_0,y_0), y(t;x_0,y_0)) \in U$, for $t \geqslant T$. In fact, if this is not true, then there is a $T_1 > T$ satisfying

$$(x(t;x_0,y_0), y(t;x_0,y_0)) \in U, \quad \text{for } t \in [T, T_1) \quad \text{and}$$
$$(x(T_1;x_0,y_0), y(T_1;x_0,y_0)) \text{ exists either on } l_1 \text{ or on } l_2.$$

Suppose that $(x(T_1;x_0,y_0), y(T_1;x_0,y_0))$ exists on l_1. Then, $dy/dx \leqslant m$ at $t = T_1$. However, this contradicts the property (2). Similarly we can get a contradiction for the case where $(x(T_1;x_0,y_0), y(T_1;x_0,y_0))$ exists on l_2.

Therefore, $(x(t;x_0,y_0), y(t;x_0,y_0)) \in U$, for $t \geqslant T$ and so $\omega(y_0) \in [C-\varepsilon, C+\varepsilon] \times \{0\}$ for y_0 satisfying $|y_0 - \beta| < \delta$. This shows the continuity of ω at $y = \beta$.

Case 2: The case where $\omega(\beta) = (0,0)$. For arbitrary $\varepsilon > 0$, let l_1 be the line with slope m passing through the point $(0, \varepsilon)$ and l_2 be the line with slope M passing through the point $(\varepsilon,0)$. Further, let U be the open set in $\mathbf{R}^2$ surrounded by the lines l_1,l_2 and the set S. Then, as in the Case 1, we can show that for sufficiently small $\delta>0$, $\omega(y_0)\in([0,\varepsilon]\times\{0\})\cup(\{0\}\times[0,\varepsilon])$, for y_0 satisfying $|y_0-\beta|<\delta$. This shows the continuity of ω at $y=\beta$.

Case 3: The case where $\omega(\beta)=(0,C)$, for some $C>0$. For arbitrary $\varepsilon>0$ satisfying $\varepsilon<C$, let l_1 be the line with slope m passing through the point $(0,C+\varepsilon)$ and l_2 be the line with slope M passing through the point $(0,C-\varepsilon)$. Further, let U be the open triangular set in $\mathbf{R}^2$ surrounded by the lines l_1,l_2, and the y-axis. Then, as in Case 1, for sufficiently small $\delta>0$, we find that $\omega(y_0)\in\{0\}\times[C-\varepsilon,C+\varepsilon]$ if $|y_0-\beta|<\delta$. This shows the continuity of ω at $y=\beta$.

Part 2: Proof of the bijectivity of the function $\omega=\omega_{x_0}$. Since $\omega:(0,\infty)\to S$ is continuous as seen above and the set $(0,\infty)$ is connected, the image $\omega((0,\infty))$ is also connected in S. Let $y_0>0$ and consider the set $V\subset\mathbf{R}^2$ given by

$$V=\{(x,y)|x>0,y>0,M(x-x_0)+y_0<y<m(x-x_0)+y_0\}.$$

Then, as in the proof of Part 1, we find that $(x(t;x_0,y_0),y(t,x_0,y_0)\in V$, for $t>0$. So it is easy to see that

$$\lim_{y_0\to+0}\left(\lim_{t\to\infty}x(t;x_0,y_0)\right)=x_0,\quad \lim_{y_0\to\infty}\left(\lim_{t\to\infty}y(t;x_0,y_0)\right)=\infty.$$

Therefore ω is surjective.

To see the injectivity of ω, suppose to the contrary that for some y_{01} and y_{02} with $y_{01}<y_{02}$ we have $\omega(y_{01})=\omega(y_{02})$; that is, $\lim_{t\to\infty}(x(t;x_0,y_{01}),y(t;x_0,y_{01}))=\lim_{t\to\infty}(x(t;x_0,y_{02}),\ y(t;x_0,y_{02}))$. By (ii) of Lemma 8 we know that

$$\frac{x(t;x_0,y_{02})}{x(t;x_0,y_{01})}\leqslant\frac{x(t_0;x_0,y_{02})}{x(t_0;x_0,y_{01})}<\frac{x(0;x_0,y_{02})}{x(0;x_0,y_{01})}=\frac{x_0}{x_0}=1,\quad\text{for}\quad t\geqslant t_0>0,$$

from which, we have

$$\lim_{t\to\infty}\frac{x(t;x_0,y_{02})}{x(t;x_0,y_{01})}<1.$$

Since $\lim_{t\to\infty}x(t;x_0,y_{01})=\lim_{t\to\infty}x(t;x_0,y_{02})$, it follows that

$$\lim_{t\to\infty}x(t;x_0,y_{01})=\lim_{t\to\infty}x(t;x_0,y_{02})=0.$$

Similarly,

$$\lim_{t\to\infty}y(t;x_0,y_{01})=\lim_{t\to\infty}y(t;x_0,y_{02})=0.$$

Since (2) implies that $-mx'(t;x_0,y_{0i})<-y'(t;x_0,y_{0i})<-Mx'(t;x_0,y_{0i})$, for $t>0,i=1,2,$ integrations of these inequalities on $[t,\infty)$ show that

$$m<\frac{y(t;x_0,y_{01})}{x(t;x_0,y_{01})},\quad\frac{y(t;x_0,y_{02})}{x(t;x_0,y_{02})}<M,\quad t>0. \tag{3}$$

Put

$$K_x=\lim_{t\to\infty}\frac{x(t;x_0,y_{02})}{x(t;x_0,y_{01})}\quad\text{and}\quad K_y=\lim_{t\to\infty}\frac{y(t;x_0,y_{02})}{y(t;x_0,y_{01})}.$$

As seen above, we observe that $0 \leqslant K_x < 1$ and similarly $1 < K_y \leqslant \infty$. We will show that $K_x = 0$. In fact, if $0 < K_x < 1$, then L'Hospital's rule implies that

$$K_x = \lim_{t\to\infty} \frac{x'(t;x_0,y_{02})}{x'(t;x_0,y_{01})} = \lim_{t\to\infty} \frac{x(t;x_0,y_{02})y(t;x_0,y_{02})}{x(t;x_0,y_{01})y(t;x_0,y_{01})}.$$

So, $K_x = K_xK_y$ if $K_y < +\infty$ and $K_x = \infty$ if $K_y = +\infty$. Both of these cases give a contradiction. So, $K_x = 0$. Similarly we can obtain $K_y = \infty$. Therefore

$$\lim_{t\to\infty} \frac{x(t;x_0,y_{02})}{x(t;x_0,y_{01})} \cdot \frac{y(t;x_0,y_{01})}{y(t;x_0,y_{02})} = 0.$$

On the other hand, by (3) we have

$$\frac{x(t;x_0,y_{02})}{x(t;x_0,y_{01})} \cdot \frac{y(t;x_0,y_{01})}{y(t;x_0,y_{02})} > \frac{m}{M}.$$

This contradiction proves the bijectivity of ω.

Therefore the proof of Theorem 3 is complete. □

3. Asymptotic forms of solutions that decay to $(0,0)$

In this section we give answers to Problem III in the introduction. That is, we give asymptotic forms of solutions of system (S) decaying to $(0,0)$. Thoughout this section we suppose that

(A3) $\int_0^\infty a(t)dt = \int_0^\infty b(t)dt = \infty$

and let us introduce the auxiliary functions $A(t)$ and $B(t)$ by

$$A(t) \equiv \int_0^t a(s)ds \quad \text{and} \quad B(t) \equiv \int_0^t b(s)ds.$$

In what follows, "$f(t) \sim g(t)$, as $t \to \infty$" means, as usual, that $\lim_{t\to\infty} f(t)/g(t) = 1$, for positive functions f and g defined near $+\infty$. Similarly, "$(f_1(t), f_2(t)) \sim (g_1(t), g_2(t))$, as $t \to \infty$", for vector-valued functions means that $f_i(t) \sim g_i(t)$, as $t \to \infty, i = 1,2$.

The first result treats the case where $a(t)$ and $b(t)$ have the same asymptotic behavior in some sense, while the second one treats the case where $b(t)$ grows faster than $a(t)$.

THEOREM 9. *Let (A3) hold and*

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} = const > 0.$$

Then every solution $(x(t), y(t))$ of system (S) decaying to $(0,0)$ has the asymptotic form

$$(x(t), y(t)) \sim \left(\frac{1}{B(t)}, \frac{1}{A(t)}\right), \quad as \quad t \to \infty.$$

THEOREM 10. *Let (A3) hold and $a(t)$, $b(t)$ be of class C^1. Suppose that*

$$\left(\frac{a(t)}{b(t)}\right)' \leqslant 0, \quad \textit{for sufficiently large } t,$$

$$\lim_{t\to\infty} \frac{a(t)B(t)}{A(t)b(t)} = k = const > 0 \tag{4}$$

and

$$\lim_{t\to\infty} \left(\frac{a(t)B(t)}{A(t)b(t)}\right)' \frac{B(t)}{b(t)} = 0. \tag{5}$$

Then every solution $(x(t), y(t))$ of system (S) decaying to $(0,0)$ has the asymptotic form

$$(x(t),\, y(t)) \sim \left(\frac{k}{B(t)},\, \frac{1}{kA(t)}\right), \quad as \quad t \to \infty.$$

COROLLARY 11. *Let $a(t)$ and $b(t)$ be of class C^1. Suppose that*

$$a(t) \sim a_0 t^{\lambda} \quad and \quad b(t) \sim b_0 t^{\mu}, \quad as \quad t \to \infty$$

and

$$\lim_{t\to\infty} t\left(\frac{a(t)}{t^{\lambda}}\right)' = \lim_{t\to\infty} t\left(\frac{b(t)}{t^{\mu}}\right)' = 0,$$

where $a_0 > 0$ and $b_0 > 0$ are constants and λ and μ are constants satisfying $-1 < \lambda < \mu$. Then every solution $(x(t), y(t))$ of system (S) decaying to $(0,0)$ has the asymptotic form

$$(x(t),\, y(t)) \sim \left(\frac{\lambda+1}{b_0 t^{\mu+1}},\, \frac{\mu+1}{a_0 t^{\lambda+1}}\right), \quad as \quad t \to \infty.$$

Proof of Theorem 9. Put $L = \lim_{t\to\infty} a(t)/b(t)$. Then, by L'Hospital' s rule we have

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = \lim_{t\to\infty} \frac{x'(t)}{y'(t)} = \lim_{t\to\infty} \frac{a(t)}{b(t)} = L.$$

Therefore, again by L'Hospital' s rule, we obtain

$$\lim_{t\to\infty} A(t)y(t) = \lim_{t\to\infty} \frac{A(t)}{y(t)^{-1}} = \lim_{t\to\infty} \frac{a(t)}{-y(t)^{-2}y'(t)} = \lim_{t\to\infty} \frac{a(t)y(t)}{b(t)x(t)} = L\cdot\frac{1}{L} = 1.$$

Similarly, $\lim_{t\to\infty} B(t)x(t) = 1$. This completes the proof. □

To see Theorem 10, we need the following lemma.

LEMMA 12. *Under the assumptions of Theorem 10, every solution* $(x(t),y(t))$ *of system (S) decaying to* $(0,0)$ *satisfies* $x(t)=O(1/B(t))$ *and* $y(t)=O(1/A(t))$*, as* $t\to\infty$*.*

Proof. From system (S) we have $-x'(t)=(a(t)/b(t))[-y'(t)]$. So an integration gives

$$x(t)=\int_t^\infty[-x'(s)]\,ds=\int_t^\infty\frac{a(s)}{b(s)}[-y'(s)]\,ds=\frac{a(t)}{b(t)}y(t)+\int_t^\infty\left(\frac{a(s)}{b(s)}\right)'y(s)ds$$
$$\leqslant\frac{a(t)}{b(t)}y(t),$$

that is $y(t)\geqslant(b(t)/a(t))x(t)$. Substituting this inequality to (S), we find that $-x'(t)\geqslant b(t)x(t)^2$. Solving this differential inequality, we have $x(t)\leqslant 1/B(t)$; therefore $x(t)=O(1/B(t))$.

Next we estimate $y(t)$. Since $(b(t)/a(t))x(t)\leqslant y(t)\to 0$, as $t\to\infty$, by the same manner as above, we have

$$y(t)=\int_t^\infty[-y'(s)]\,ds=\int_t^\infty\frac{b(s)}{a(s)}[-x'(s)]\,ds=\frac{b(t)}{a(t)}x(t)+\int_t^\infty\left(\frac{b(s)}{a(s)}\right)'x(s)ds. \quad (6)$$

Assumption (4) implies that $b(t)/(a(t)B(t))\leqslant C/A(t)$, for some constant $C>0$. So from the fact that $x(t)\leqslant 1/B(t)$, we get

$$\frac{b(t)}{a(t)}x(t)\leqslant\frac{b(t)}{a(t)B(t)}\leqslant\frac{C}{A(t)}\to 0,\quad \text{as } t\to\infty.$$

Therefore

$$y(t)=O(1/A(t))+\int_t^\infty\left(\frac{b(s)}{a(s)}\right)'x(s)ds.$$

By noting that $(b(t)/a(t))'\geqslant 0$ and $x(t)\leqslant 1/B(t)$, the second term of the right-hand side is estimated as follows:

$$0\leqslant\int_t^\infty\left(\frac{b(s)}{a(s)}\right)'x(s)ds\leqslant\int_t^\infty\left(\frac{b(s)}{a(s)}\right)'B(s)^{-1}ds=-\frac{b(t)}{a(t)B(t)}+\int_t^\infty\frac{b(s)}{a(s)}\frac{b(s)}{B(s)^2}ds$$
$$\leqslant-\frac{b(t)}{a(t)B(t)}+C^2\int_t^\infty\frac{b(s)}{a(s)}b(s)\frac{a(s)^2}{A(s)^2b(s)^2}ds=-\frac{b(t)}{a(t)B(t)}+C^2\int_t^\infty\frac{a(s)}{A(s)^2}ds$$
$$=-\frac{b(t)}{a(t)B(t)}+\frac{C^2}{A(t)}\leqslant\frac{C^2}{A(t)}.$$

Here we have employed assumption (4). Hence $y(t)=O(1/A(t))$. This completes the proof. □

Proof of Theorem 10. Let us perform the change of variables $(t,x,y)\mapsto(s,X,Y)$ defined by

$$B(t)x\equiv X,\quad A(t)y\equiv Y\quad\text{and}\quad s=\log B(t).$$

Then system (S) is transformed into the new system

$$\begin{cases} X_s = X\left(1 - \dfrac{\tilde{a}(s)\tilde{B}(s)}{\tilde{A}(s)\tilde{b}(s)}Y\right), \\ Y_s = Y\left(\dfrac{\tilde{a}(s)\tilde{B}(s)}{\tilde{A}(s)\tilde{b}(s)} - X\right), \end{cases} \tag{S$'$}$$

where $(\cdot)_s = d/ds$, and $\tilde{a}(s) = a\left(B^{-1}(e^s)\right), \tilde{A}(s) = A\left(B^{-1}(e^s)\right)$ and so on. (Here, B^{-1} denotes of course the inverse function of B.) By assumption (A3), $s \to \infty$ if $t \to \infty$. From Lemma 12, we know that

$$X(s), Y(s) = O(1), \quad \text{as} \quad s \to \infty. \tag{7}$$

For simplicity below we rewrite system (S') as

$$\begin{cases} X_s = X(1 - f(s)Y), \\ Y_s = Y(f(s) - X), \end{cases} \tag{S$''$}$$

where $f(s) = \left(\tilde{a}(s)\tilde{B}(s)\right) / \left(\tilde{A}(s)\tilde{b}(s)\right)$. Assumptions (4) and (5) imply that $\lim_{s\to\infty} f(s) = k$ and $\lim_{s\to\infty} f_s(s) = 0$.

First we will show that $x(t) \sim k/B(t)$, as $t \to \infty$, that is $X(s) \sim k$, as $s \to \infty$. From system (S'') we can obtain the single equation of $X(s)$:

$$X_{ss} = -\frac{f_s}{f}X + \frac{f_s}{f}X_s + \frac{(X_s)^2}{X} - fX + X^2 + fX_s - XX_s. \tag{8}$$

We claim that $\limsup_{s\to\infty} X(s) \geqslant k$. To see this by contradiction, suppose to the contrary that $\limsup_{s\to\infty} X(s) < k$. Then, by virtue of the fact that $\lim_{s\to\infty} f(s) = k$, the second equation of (S'') implies that $Y_s \geqslant cY$, for sufficiently large $s > 0$, with some constant $c > 0$. So $\lim_{s\to\infty} Y(s) = \infty$, which contradicts to (7). So $\limsup_{s\to\infty} X(s) \geqslant k$.

We claim next that $\liminf_{s\to\infty} X(s) \leqslant k$. Suppose to the contrary that $\liminf_{s\to\infty} X(s) > k$. Then, the second equation of (S'') implies that $Y_s \leqslant -cY$, for sufficiently large $s > 0$, with some constant $c > 0$. So $\lim_{s\to\infty} Y(s) = 0$. Accordingly by the first equation of (S'') we find that $X_s \geqslant \tilde{c}X$, for sufficiently large $s > 0$, with some constant $\tilde{c} > 0$. So $\lim_{s\to\infty} X(s) = \infty$, which contradicts to (7). So $\liminf_{s\to\infty} X(s) \leqslant k$.

Since we have established

$$0 \leqslant \liminf_{s\to\infty} X(s) \leqslant k \leqslant \limsup_{s\to\infty} X(s) < \infty, \tag{9}$$

to prove $X(s) \sim k$, it suffices to show the existence of $\lim_{s\to\infty} X(s) \in [0, \infty)$.

Suppose the contrary that $0 \leqslant \liminf_{s\to\infty} X(s) < \limsup_{s\to\infty} X(s) < \infty$. Then, by virtue of (9) there are three possibilities:

Case (a) : $\liminf_{s\to\infty} X(s) < k < \limsup_{s\to\infty} X(s)$;

Case (b) : $\liminf_{s\to\infty} X(s) = k < \limsup_{s\to\infty} X(s)$;

Case (c) : $\liminf_{s\to\infty} X(s) < k = \limsup_{s\to\infty} X(s)$.

Let Case (a) occur. Then, we can find a sequence $\{s_n\}$ satisfying

$$\lim_{n\to\infty} s_n = \infty, \quad X_s(s_n) = 0, \quad X_{ss}(s_n) \leqslant 0 \quad \text{and} \quad \lim_{n\to\infty} X(s_n) = \limsup X(s) (> k).$$

Letting $s = s_n$ in (8), we have $X_{ss}(s_n) = X(s_n)\left[X(s_n) - \left(f(s_n) + \frac{f_s(s_n)}{f(s_n)}\right)\right]$. Since $\lim_{s\to\infty} f_s(s) = 0$, this means that $X_{ss}(s_n) > 0$, for sufficiently large n. However, this is a contradiction to the property $X_{ss}(s_n) \leqslant 0$.

When Cases (b) and (c) occur, we can easily get contradictions similarly. So we find that $X(s) \sim k$.

Next, we will show that $y(t) \sim 1/(kA(t))$. Once we got $x(t) \sim k/B(t)$, as $t \to \infty$, this fact can be shown merely by computing $\lim_{t\to\infty} A(t)y(t)$.

By (6) we have

$$A(t)y(t) = \frac{b(t)}{a(t)}A(t)x(t) + A(t)\int_t^\infty \left(\frac{b(s)}{a(s)}\right)' x(s)ds = \frac{b(t)A(t)}{a(t)B(t)}B(t)x(t) + \frac{\int_t^\infty \left(\frac{b(s)}{a(s)}\right)' x(s)ds}{A(t)^{-1}}.$$

Using assumption (4) and the result $x(t) \sim k/B(t)$, we obtain

$$\lim_{t\to\infty} A(t)y(t) = 1 + \lim_{t\to\infty} \frac{\int_t^\infty \left(\frac{b(s)}{a(s)}\right)' x(s)ds}{A(t)^{-1}}. \tag{10}$$

On the other hand, a direct computation shows that

$$\left(\frac{a(t)B(t)}{A(t)b(t)}\right)' \frac{B(t)}{b(t)} = \left(\frac{a(t)}{b(t)}\right)' \frac{B(t)^2}{A(t)b(t)} + \frac{a(t)B(t)}{A(t)b(t)} - \left(\frac{a(t)B(t)}{A(t)b(t)}\right)^2$$

and so, by assumptions (4) and (5), it follows that

$$\lim_{t\to\infty} \left(\frac{a(t)}{b(t)}\right)' \frac{B(t)^2}{A(t)b(t)} = k^2 - k.$$

Then, employing L'Hospital's rule, we get from (10)

$$\begin{aligned}
\lim_{t\to\infty} A(t)y(t) &= 1 + \lim_{t\to\infty} \frac{\left(\frac{b(t)}{a(t)}\right)' x(t)}{A(t)^{-2}a(t)} = 1 + \lim_{t\to\infty} \frac{A(t)^2}{a(t)B(t)} \left(\frac{b(t)}{a(t)}\right)' \cdot B(t)x(t) \\
&= 1 + k \lim_{t\to\infty} \frac{A(t)^2}{a(t)B(t)} \left(\frac{b(t)}{a(t)}\right)' = 1 - k \lim_{t\to\infty} \left(\frac{a(t)}{b(t)}\right)' \frac{B(t)^2}{A(t)b(t)} \cdot \left(\frac{A(t)b(t)}{a(t)B(t)}\right)^3 \\
&= 1 - k(k^2 - k)k^{-3} = 1/k.
\end{aligned}$$

Hence $y(t) \sim 1/(kA(t))$, as $t \to \infty$. This completes the proof of Theorem 10. □

Proof of Corollary 11. To prove this corollary it suffices to notice that the following expressions are valid: $a(t) = a_0 t^\lambda(1 + \varepsilon(t))$, $b(t) = b_0 t^\mu(1 + \delta(t))$, $A(t) =$

$\frac{a_0}{\lambda+1} t^{\lambda+1}(1+\hat{\varepsilon}(t))$ and $B(t)=\frac{b_0}{\mu+1} t^{\mu+1}\left(1+\hat{\delta}(t)\right)$, where $\varepsilon, \delta, \hat{\varepsilon}$ and $\hat{\delta}$ are C^1-functions satisfying $\varepsilon(t), t\varepsilon'(t), \delta(t), t\delta'(t) \to 0$, as $t \to \infty$ and $\hat{\varepsilon}(t), t\hat{\varepsilon}'(t), \hat{\delta}(t), t\hat{\delta}'(t) \to 0$, as $t \to \infty$. $\square$

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