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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR FRACTIONAL NABLA DIFFERENCE SYSTEMS WITH INITIAL CONDITIONS

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Abstract. In this paper, we give sufficient conditions to guarantee the global existence and the uniqueness of solutions of nonlinear fractional nabla difference systems and study the dependence of solutions on initial conditions and parameters.

1. Introduction

Fractional differential and difference equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of fractional differential and difference equations with and without delay have received the attention of many authors, see [1]–[10], [12]–[16], [18]–[26], [28]–[31] and the references therein.

Recently, Agarwal, Zhou and He [3] discussed the existence of solutions for the neutral fractional differential equation with bounded delay

$$\begin{cases} {}^{C}D^{\alpha}\left(u(t)-g(t,u_t)\right)=f\left(t,u_t\right),\ t\geqslant t_0, \\ u_{t_0}=\phi, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $0<\alpha<1$. By employing the Krasnoselskii's fixed point theorem, the authors obtained existence results.

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The fractional difference equation

$$\begin{cases} \Delta^{\alpha} u(t) = f(t+\alpha, u(t+\alpha)),\ t \in \mathbb{N}_{1-\alpha}, \\ \Delta^{\alpha-1} u(t)\big|_{t=0} = u_0, \end{cases}$$

has been investigated in [14], where Δ^{α} denotes Riemann-Liouville like discrete fractional difference of order $0 < \alpha < 1$. By using the Krasnoselskii's fixed point theorem and discrete Arzela-Ascoli's theorem, the asymptotic stability has been established.

In [19], Jagan Mohan, Shobanadevi and Deekshitulu investigated the global existence and the uniqueness of the solutions of the following nonlinear nabla fractional difference system

$$\begin{cases} \nabla_{-1}^{\alpha} u(t) = f(t, u(t)),\ t \in \mathbb{N}_1, \\ \nabla_{-1}^{-(1-\alpha)} u(t)|_{t=0} = u(0) = c,\ 0 < \alpha < 1, \end{cases}$$

where ∇_{-1}^{α} is the Riemann-Liouville type fractional difference operators. By employing the fixed point theorems and discrete Arzela-Ascoli's theorem, the authors obtained the global existence and the uniqueness results. Also the dependence of solutions on initial conditions and parameters has been established.

Inspired and motivated by the works mentioned above and the papers [1]–[10], [12]–[16], [18]–[26], [28]-[31] and the references therein, we concentrate on the global existence and the uniqueness of the solutions for the nonlinear nabla fractional difference system

$$\begin{cases} \nabla_{-1}^{\alpha} [u(t) - g(t, u(t))] = f(t, u(t)),\ t \in \mathbb{N}_1, \\ \nabla_{-1}^{-(1-\alpha)} u(t)|_{t=0} = u(0) = c,\ 0 < \alpha < 1, \end{cases} \tag{1.1}$$

where ∇_{-1}^{α} is the Riemann-Liouville type fractional difference operators, $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$, $u : \mathbb{N}_0 \to \mathbb{R}^n$, $c \in \mathbb{R}^n$ is a constant, $f : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}$ is continuous in u, $g : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous in u. That is, there is a positive constant $L_g \in (0,1)$ such that

$$\|g(t, u(t)) - g(t, v(t))\| \leqslant L_g \|u(t) - v(t)\|,\ g(t, 0) = 0. \tag{1.2}$$

The purpose of this paper is to use Krasnoselskii's fixed point theorem, discrete Arzela-Ascoli's theorem and generalized Banach fixed point theorem to show the global existence and the uniqueness of solutions for (1.1). To apply Krasnoselskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is compact. For details on Krasnoselskii's theorem we refer the reader to [27]. In addition, the dependence of solutions of (1.1) on initial conditions and parameters is studied.

This paper is organized as follows. Section 2 contains preliminaries on nabla discrete fractional calculus and functional analysis. In section 3, we give and prove our main results on the global existence and uniqueness of solutions for (1.1). The dependence of solutions on initial conditions and parameters is the topic of section 4.

2. Preliminaries

We shall use the following notations, definitions and known results of discrete fractional calculus [6, 16, 18, 29, 30] throughout this article. For any $a,b \in \mathbb{R}$, $\mathbb{N}_{a,b} = \{a, a+1, a+2, \dots, b\}$ where $b = a+k$ for some positive integer k.

DEFINITION 2.1. For any $\alpha, t \in \mathbb{R}$, the α rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)},\ t \in \mathbb{R} \setminus \{\dots,-2,-1,0\},\ 0^{\overline{\alpha}} = 0.$$

We observe the following properties of rising factorial function.

LEMMA 2.2. *Assume the following factorial functions are well defined.*
(1) $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
(2) *If* $t \leqslant r$ *then* $t^{\overline{\alpha}} \leqslant r^{\overline{\alpha}}$.
(3) *If* $\alpha < t \leqslant r$ *then* $r^{\overline{-\alpha}} \leqslant t^{\overline{-\alpha}}$.

DEFINITION 2.3. Let $u : \mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$.

(1) (Nabla Difference) The first order backward difference or nabla difference of u is defined by

$$\nabla u(t) = u(t) - u(t-1),\ t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\nabla^N u(t) = \nabla\left(\nabla^{N-1} u(t)\right),\ t \in \mathbb{N}_{a+N}.$$

In addition, we take ∇^0 as the identity operator.

(2) (Fractional Nabla Sum) The α^{th}-order fractional nabla sum of u is given by

$$\nabla_a^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} u(s),\ t \in \mathbb{N}_a \tag{2.1}$$

where $\rho(s) = s-1$. Also, we define the trivial sum by $\nabla_a^{-0} u(t) = u(t)$ for $t \in \mathbb{N}_a$.

(3) (R-L Nabla Fractional Difference) The α^{th}-order Riemann-Liouville type nabla fractional difference of u is given by

$$\nabla_a^{\alpha} u(t) = \nabla^N \left[\nabla_a^{-(N-\alpha)} u(t)\right],\ t \in \mathbb{N}_{a+N}. \tag{2.2}$$

For $\alpha = 0$, we set $\nabla_a^0 u(t) = u(t)$, $t \in \mathbb{N}_a$.

(4) (Caputo Fractional Nabla Difference) The α^{th}-order Caputo type fractional nabla difference of u is given by

$$\nabla_{a*}^{\alpha} u(t) = \nabla_a^{-(N-\alpha)} \left[\nabla^N u(t)\right],\ t \in \mathbb{N}_{a+N}. \tag{2.3}$$

For $\alpha = 0$, we set $\nabla_{a*}^0 u(t) = u(t)$, $t \in \mathbb{N}_a$.

THEOREM 2.4. (Power Rule) *Let* $\alpha > 0$ *and* $\mu > -1$. *Then,*

(1) $\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\mu+\alpha}}$, $t \in \mathbb{N}_a$.

(2) $\nabla_a^{\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a)^{\overline{\mu-\alpha}}$, $t \in \mathbb{N}_{a+N}$.

Let $f : \mathbb{N}_a \times \mathbb{R} \longrightarrow \mathbb{R}$, $u : \mathbb{N}_a \longrightarrow \mathbb{R}$ and $0 < \alpha < 1$. Consider a fractional nabla difference equation of Riemann-Liouville type together with an initial condition of the form

$$\begin{cases} \nabla_{a-1}^{\alpha} u(t) = f(t, u(t)),\ t \in \mathbb{N}_{a+1}, \\ \nabla_{a-1}^{-(1-\alpha)} u(t)|_{t=a} = u(a) = u_0. \end{cases} \tag{2.4}$$

Then, from [30], u is a solution of the initial value problem (2.4) if and only if it has the following representation

$$u(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)),\ t \in \mathbb{N}_a. \tag{2.5}$$

Now we present some important definitions and theorems of functional analysis [11, 17, 27] which will be useful in establishing main results.

DEFINITION 2.5. $\mathbb{R}^n$ is the space of all ordered n-tuples of real numbers. Clearly, $\mathbb{R}^n$ is a Banach space with respect to the supremum norm. A closed ball with radius r centered at the origin of $\mathbb{R}^n$ is defined by

$$B_0^{\infty}(r) = \{u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n : \|u\|_{\infty} \leqslant r\}.$$

DEFINITION 2.6. $l^{\infty} = l^{\infty}(\mathbb{R})$ is the space of all real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. Clearly l^{∞} is a Banach space under the supremum norm. A closed ball with radius r centered on the null sequence of l^{∞} is defined by

$$B_0^{\infty}(r) = \{u = \{u(t)\}_{t=0}^{\infty} \in l^{\infty} : \|u\|_{\infty} \leqslant r\}.$$

DEFINITION 2.7. A subset S of l^{∞} is uniformly Cauchy (or equi-Cauchy), if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}_1$ such that $|u(t_1) - u(t_2)| < \varepsilon$ whenever $t_1, t_2 \in \mathbb{N}_{k+1}$, for any $u = \{u(t)\}_{t=0}^{\infty}$ in S.

THEOREM 2.8. (Discrete Arzela-Ascoli's Theorem) *A bounded uniformly Cauchy subset S of l^{∞} is relatively compact.*

THEOREM 2.9. (Krasnoselskii's Fixed Point Theorem) *Let S be a nonempty, closed, convex and bounded subset of a Banach space X, and let $A : X \longrightarrow X$ and $B : S \longrightarrow X$ be two operators such that*

(1) *A is a contraction with constant $L < 1$,*

(2) *B is continuous, BS resides in a compact subset of X,*

(3) $[x = Ax + By;\ y \in S] \Longrightarrow x \in S$.

Then the operator equation $Ax + Bx = x$ has a solution in S.

THEOREM 2.10. (Generalized Banach Fixed Point Theorem) *Let S be a nonempty, closed subset of a Banach space $(X,\|.\|)$, and let a $\omega_n \geqslant 0$ for every $n \in \mathbb{N}_0$ and such $\sum_{n=0}^{\infty}\omega_n$ converges. Moreover, let the mapping $T: S \longrightarrow S$ satisfy the inequality*

$$\|T^n u - T^n v\| \leqslant \omega_n \|u-v\|,$$

for every $n \in \mathbb{N}_1$ and any $u,v \in S$. Then, T has a uniquely defined fixed point u^*. Furthermore, for any $u_0 \in S$, the sequence $(T^n u_0)_{n=1}^{\infty}$ converges to this fixed point u^*.

DEFINITION 2.11. Let X be a Banach space with respect to a norm $\|.\|$. Define the

$$\mathbb{S} = \mathbb{S}(X) = \{u : u = \{u(t)\}_{t=0}^{\infty},\ u(t) \in X\}.$$

Then, S is a linear space of sequences of elements of X under obvious definition of addition and scalar multiplication. Now we employ the notation

$$u = \{u(t)\}_{t=0}^{\infty},\ \|u\|_{\infty} = \sup_{t \in \mathbb{N}_0} |u(t)|,$$

and define the set

$$\mathbb{S}^{\infty}(X) = \{u : u \in \mathbb{S}(X) \text{ with } \|u\|_{\infty} \leqslant \infty\}.$$

Clearly $\mathbb{S}^{\infty}(X)$ is a Banach space consisting of elements of $S(X)$, with respect to the supremum norm.

DEFINITION 2.12. From Definitions 2.6 and 2.11, we observe that $l^{\infty} = l^{\infty}(\mathbb{R}) = \mathbb{S}^{\infty}(\mathbb{R})$. Now we choose $X = \mathbb{R}^n$ in Definition 2.11 to define

$$\ell^{\infty} = \ell^{\infty}(\mathbb{R}^n) = \mathbb{S}^{\infty}(\mathbb{R}^n) = \{u : u = \{u(t)\}_{t=0}^{\infty},\ u(t) \in \mathbb{R}^n \text{ with } \|u\|_{\infty} \leqslant \infty\}.$$

Thus, ℓ^{∞} denotes the Banach space comprising sequences of vectors with respect to the supremum norm $\|.\|_{\infty}$ defined by

$$\|u\|_{\infty} = \sup_{t \in \mathbb{N}_0} \|u(t)\|.$$

A closed ball with radius r centered on the null sequence in ℓ^{∞} is defined by

$$B_0^{\infty}(r) = \{u = \{u(t)\}_{t=0}^{\infty} \in \ell^{\infty} : \|u\|_{\infty} \leqslant r\}.$$

3. Existence and uniqueness

In this section we prove existence and uniqueness theorems to (1.1).

Let $u: \mathbb{N}_0 \longrightarrow \ell^{\infty}$ and $f,g: \mathbb{N}_0 \times \ell^{\infty} \longrightarrow \ell^{\infty}$. Analogous to (2.5), $u = \{u(t)\}_{t=0}^{\infty} \in \ell^{\infty}$ is any solution of the initial value problem (1.1) if and only if

$$u(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}[c - g(0,c)] + g(t,u(t)) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} f(s,u(s)),\ t \in \mathbb{N}_0. \tag{3.1}$$

Define the operators

$$Tu(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}[c-g(0,c)] + g(t,u(t)) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} f(s,u(s)),\ t \in \mathbb{N}_0, \tag{3.2}$$

$$Au(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}[c-g(0,c)] + g(t,u(t)),\ t \in \mathbb{N}_0, \tag{3.3}$$

and

$$Bu(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} f(s,u(s)),\ t \in \mathbb{N}_0. \tag{3.4}$$

It is evident from (3.1)–(3.2) that u is a fixed point of T if and only if u is a solution of (1.1). First we use Krasnoselskii's fixed point theorem (Theorem 2.9) to establish global existence of solutions of (1.1).

THEOREM 3.1. (Global Existence) *Assume that (1.2) holds and there exist constants $\beta_1 \in [\alpha, 1)$ and $L_1 \geqslant 0$ such that*

$$\|f(t,u(t))\| \leqslant L_1 t^{\overline{-\beta_1}},\ t \in \mathbb{N}_1, \tag{3.5}$$

then the initial value problem (1.1) has at least one bounded solution in ℓ^∞.

Proof. To prove condition (2) of Theorem 2.9, we define a set

$$S_1 = \left\{ u : \|u(t)\| \leqslant \frac{(1+L_g)\|c\| + L_1\Gamma(1-\beta_1)}{1-L_g},\ t \in \mathbb{N}_1 \right\}.$$

Clearly S_1 is a nonempty, closed, bounded and convex subset of ℓ^∞. First, we show that B maps S_1 into S_1. Using Lemma 2.2, Theorem 2.4 and (3.5), we have

$$\begin{aligned}
\|Bu(t)\| &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,u(s))\| \\
&\leqslant \frac{L_1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} s^{\overline{-\beta_1}} = L_1 \nabla_0^{-\alpha} t^{\overline{-\beta_1}} = \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t^{\overline{-(\beta_1-\alpha)}} \\
&\leqslant \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)}(1)^{\overline{-(\beta_1-\alpha)}} = L_1\Gamma(1-\beta_1) \\
&\leqslant \frac{(1+L_g)\|c\| + L_1\Gamma(1-\beta_1)}{1-L_g},\ t \in \mathbb{N}_1,
\end{aligned}$$

implies $BS_1 \subset S_1$. Next, we show that B is continuous on S_1. Let $\varepsilon > 0$ be given. Then there exists $m \in \mathbb{N}_1$ such that, for $t \in \mathbb{N}_{m+1}$,

$$\frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t^{\overline{-(\beta_1-\alpha)}} < \frac{\varepsilon}{2}.$$

Let $\{u_k\}$, $(k = 1,2,\ldots)$ be a sequence in S_1 such that $u_k \longrightarrow u$ in S_1. Then, we have $\|u_k - u\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$. Since f is continuous with respect to the second variable, we get $\|f(t,u_k) - f(t,u)\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$. For $t \leqslant m$,

$$\begin{aligned}
&\|Bu_k(t) - Bu(t)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \|f(s,u_k(s)) - f(s,u(s))\| \\
&\leqslant \left[\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}}\right] \left[\sup_{s\in\{1,2,\ldots,m\}} \|f(s,u_k(s)) - f(s,u(s))\|\right] \\
&\leqslant \frac{t^{\overline{\alpha}}}{\Gamma(\alpha+1)} \|f(s,u_k) - f(s,u)\|_\infty \longrightarrow 0 \text{ as } k \longrightarrow \infty.
\end{aligned}$$

For $t \in \mathbb{N}_{m+1}$,

$$\begin{aligned}
&\|Bu_k(t) - Bu(t)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} [\|f(s,u_k(s))\| + \|f(s,u(s))\|] \\
&\leqslant \frac{2L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t^{\overline{-(\beta_1-\alpha)}} < \varepsilon.
\end{aligned}$$

Thus we have, $\|Bu_k - Bu\|_\infty \to 0$ as $k \longrightarrow \infty$, implies B is continuous. Now, we show that BS_1 is relatively compact. Let $t_1, t_2 \in \mathbb{N}_{m+1}$ such that $t_2 > t_1$. Then, we have

$$\begin{aligned}
&\|Bu(t_1) - Bu(t_2)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t_1} (t_1-\rho(s))^{\overline{\alpha-1}} \|f(s,u(s))\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t_2} (t_2-\rho(s))^{\overline{\alpha-1}} \|f(s,u(s))\| \\
&\leqslant \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t_1^{\overline{-(\beta_1-\alpha)}} + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t_2^{\overline{-(\beta_1-\alpha)}} < \varepsilon.
\end{aligned}$$

Thus $\{Bu : u \in S_1\}$ is a bounded and uniformly Cauchy subset of ℓ^∞. Hence, by Theorem 2.8, BS_1 is relatively compact.

Now we prove condition (3) of Theorem 2.9. Let us suppose, for a fixed $v \in S_1$, $u = Au + Bv$. Using Lemma 2.2, Theorem 2.4 and (3.5), we have

$$\begin{aligned}
\|u(t)\| &\leqslant \|Au(t)\| + \|Bv(t)\| \\
&\leqslant \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \|c - g(0,c)\| + \|g(t,u(t))\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \|f(s,v(s))\| \\
&\leqslant \frac{(1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} (1+L_g)\|c\| + L_g\|u(t)\| + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t^{\overline{-(\beta_1-\alpha)}} \\
&\leqslant (1+L_g)\|c\| + L_g\|u(t)\| + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} (1)^{\overline{-(\beta_1-\alpha)}} \\
&= L_g\|u(t)\| + (1+L_g)\|c\| + L_1\Gamma(1-\beta_1),\ t \in \mathbb{N}_1.
\end{aligned}$$

Then

$$\|u(t)\| \leqslant \frac{(1+L_g)\|c\| + L_1\Gamma(1-\beta_1)}{1-L_g},\ t \in \mathbb{N}_1.$$

Thus $u \in S_1$.

Lastly, we prove that A is contraction. Let $u, v \in \ell^\infty$, we have

$$\|Au(t) - Av(t)\| = \|g(t, u(t)) - g(t, v(t))\| \leqslant L_g \|u - v\|_\infty.$$

Then

$$\|Au - Av\|_\infty \leqslant L_g \|u - v\|_\infty,$$

which means that A is a contraction by (1.2).

According to Theorem 2.9, T has a fixed point in S_1 which is a solution of (1.1). Hence the proof is complete. □

THEOREM 3.2. (Global Existence) *Assume that (1.2) holds and there exist constants $\beta_2 \in [\alpha, 1)$ and $L_2 \geqslant 0$ such that*

$$\|f(t, u(t))\| \leqslant L_2 t^{\overline{-\beta_2}} \|u(t)\|,\ t \in \mathbb{N}_1, \tag{3.6}$$

then the initial value problem (1.1) has at least one bounded solution in ℓ^∞ provided that

$$L_g + L_2\Gamma(1-\beta_2) < 1. \tag{3.7}$$

Proof. Define

$$S_2 = \left\{ u : \|u(t)\| \leqslant \frac{(1+L_g)\|c\|}{1-L_g-L_2\Gamma(1-\beta_2)},\ t \in \mathbb{N}_1 \right\}.$$

Clearly S_2 is a nonempty, closed, bounded and convex subset of ℓ^∞. First, we show that B maps S_2 into S_2. Using Lemma 2.2, Theorem 2.4 and (3.6), we have

$$\begin{aligned}
\|Bu(t)\| &\leqslant \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \|f(s, u(s))\| \\
&\leqslant \frac{L_2}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} s^{\overline{-\beta_2}} \|u(s)\| \\
&\leqslant L_2 \frac{(1+L_g)\|c\|}{1-L_g-L_2\Gamma(1-\beta_2)} \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} s^{\overline{-\beta_2}} \\
&= \frac{L_2(1+L_g)\|c\|}{1-L_g-L_2\Gamma(1-\beta_2)} \nabla_0^{-\alpha} t^{\overline{-\beta_2}} \\
&= \frac{L_2(1+L_g)\|c\|}{1-L_g-L_2\Gamma(1-\beta_2)} \frac{\Gamma(1-\beta_2)}{\Gamma(1-\beta_2+\alpha)} t^{\overline{-(\beta_2-\alpha)}} \\
&\leqslant \frac{L_2(1+L_g)\|c\|}{1-L_g-L_2\Gamma(1-\beta_2)} \frac{\Gamma(1-\beta_2)}{\Gamma(1-\beta_2+\alpha)} (1)^{\overline{-(\beta_2-\alpha)}}
\end{aligned}$$

$$= \frac{(1+L_g)\|c\| L_2\Gamma(1-\beta_2)}{1-L_g-L_2\Gamma(1-\beta_2)}$$
$$\leqslant \frac{(1+L_g)\|c\|}{1-L_g-L_2\Gamma(1-\beta_2)},\ t\in\mathbb{N}_1,$$

implies $BS_2 \subset S_2$. The remaining proof of conditions (1) and (2) is similar to that of Theorem 3.1 and we omit it. Now we prove condition (3) of Theorem 2.9. Let us suppose, for a fixed $v\in S_2$, $u = Au + Bv$. Using Lemma 2.2, Theorem 2.4 and (3.6), we have

$$\begin{aligned}\|u(t)\| &\leqslant \|Au(t)\|+\|Bv(t)\| \\ &\leqslant \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(1+L_g)\|c\| + L_g\|u(t)\| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,v(s))\| \\ &\leqslant \frac{(1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(1+L_g)\|c\| + L_g\|u(t)\| + \frac{(1+L_g)\|c\| L_2\Gamma(1-\beta_2)}{1-L_g-L_2\Gamma(1-\beta_2)} \\ &= L_g\|u(t)\| + \frac{(1+L_g)\|c\|(1-L_g)}{1-L_g-L_2\Gamma(1-\beta_2)},\ t\in\mathbb{N}_1.\end{aligned}$$

Then
$$\|u(t)\| \leqslant \frac{(1+L_g)\|c\|}{1-L_g-L_2\Gamma(1-\beta_2)},\ t\in\mathbb{N}_1.$$

Thus $u\in S_2$. According to Theorem 2.9, T has a fixed point in S_2 which is a solution of (1.1). Hence the proof is complete. □

We use generalized Banach fixed point theorem (Theorem 2.10) to prove the uniqueness of solutions of (1.1).

THEOREM 3.3. (Global Uniqueness) *Assume that (1.2) holds and there exist constants* $\gamma\in[\alpha,1)$ *and* $M\geqslant 0$ *such that*

$$\|f(t,u)-f(t,v)\|_\infty \leqslant t^{\overline{-\gamma}}M\|u-v\|_\infty,\ t\in\mathbb{N}_1, \tag{3.8}$$

for any pair of elements u *and* v *in* ℓ^∞. *Then the initial value problem (1.1) has a unique bounded solution in* ℓ^∞ *provided that*

$$\rho = L_g + M\Gamma(1-\gamma) < 1. \tag{3.9}$$

Proof. Let us define the iterates of operator T as follows

$$T^1 = T,\ T^n = T\circ T^{n-1},\ n\in\mathbb{N}_1.$$

It is sufficient to prove that T^n is a contraction operator for sufficiently large n. Actually, we have
$$\|T^n u - T^n v\|_\infty \leqslant \rho^n\|u-v\|_\infty, \tag{3.10}$$

where the constant ρ depends only on L_g, M and γ. In fact, using Lemma 2.2, Theorem 2.4 and (3.8), we get

$$\begin{aligned}
&\|Tu(t)-Tv(t)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,u(s))-f(s,v(s))\|+\|g(t,u(t))-g(t,v(t))\| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\gamma}}M\|u-v\|_\infty+L_g\|u-v\|_\infty \\
&= M\nabla_0^{-\alpha}t^{\overline{-\gamma}}\|u-v\|_\infty+L_g\|u-v\|_\infty=\frac{M\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}t^{\overline{-(\gamma-\alpha)}}\|u-v\|_\infty+L_g\|u-v\|_\infty \\
&\leqslant \frac{M\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}(1)^{\overline{-(\gamma-\alpha)}}\|u-v\|_\infty+L_g\|u-v\|_\infty \\
&= (L_g+M\Gamma(1-\gamma))\|u-v\|_\infty=\rho\|u-v\|_\infty,
\end{aligned}$$

implies

$$\|Tu-Tv\|_\infty \leqslant \rho\|u-v\|_\infty. \tag{3.11}$$

Therefore (3.10) is true for $n=1$. Assuming (3.10) is valid for n, we obtain similarly

$$\begin{aligned}
&\left\|T^{n+1}u(t)-T^{n+1}v(t)\right\| \\
&=\|(T\circ T^n)u(t)-(T\circ T^n)v(t)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,T^nu(s))-f(s,T^nv(s))\|+\|g(t,T^nu(s))-g(t,T^nv(s))\| \\
&\leqslant \frac{M}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\gamma}}\|T^nu-T^nv\|_\infty+L_g\|T^nu-T^nv\|_\infty \\
&\leqslant M\rho^n\nabla_0^{-\alpha}t^{\overline{-\gamma}}\|u-v\|_\infty+\rho^nL_g\|u-v\|_\infty \\
&=\frac{M\rho^n\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}t^{\overline{-(\gamma-\alpha)}}\|u-v\|_\infty+\rho^nL_g\|u-v\|_\infty \\
&\leqslant \frac{M\rho^n\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}(1)^{\overline{-(\gamma-\alpha)}}\|u-v\|_\infty+\rho^nL_g\|u-v\|_\infty \\
&=[L_g+M\Gamma(1-\gamma)]\rho^n\|u-v\|_\infty=\rho^{n+1}\|u-v\|_\infty.
\end{aligned}$$

Thus, by the principle of mathematical induction on n, the statement (3.10) is true for each $n\in\mathbb{N}_1$. Since $\rho<1$, the geometric series $\sum_{n=o}^{\infty}\rho^n$ converges. Hence T has a uniquely defined point u^* in S_1 (or S_2). This completes the proof. □

EXAMPLE 3.4. Consider the scalar initial value problem

$$\begin{cases}\nabla_{-1}^{0.5}[u(t)-(0.32)\cos(t)\sin(u(t))]=(0.15)\overline{t^{-0.75}}\cos(u(t)),\ t\in\mathbb{N}_1,\\ \nabla_{-1}^{-0.5}u(t)|_{t=0}=u(0)=c.\end{cases} \tag{3.12}$$

Then

$$\alpha = 0.5,\ g(t,u) = (0.32)\cos(t)\sin(u),\ f(t,u) = (0.15)\overline{t^{-0.75}}\cos(u).$$

Doing straightforward computations, we obtain

$$\gamma = 0.75,\ g(t,0) = 0,\ L_g = 0.32,\ M = 0.15,\ \rho = 0.864.$$

Thus, all the assumptions of Theorem 3.3 hold and hence the initial value problem (3.12) has a unique solution in ℓ^∞.

4. Dependence of solutions on initial conditions and parameters

The initial value problem (1.1) describes a model of a physical problem in which often some parameters such as lengths, masses, temperature, etc. are involved. The values of these parameters can be measured only up to a certain degree of accuracy. Thus, in (1.1), the initial value c, the order of the difference operator and the function f, may be subject to some errors either by necessity or for convenience. Hence, it is important to know how the solution changes when these parameters are slightly altered. We shall discuss this question quantitatively in the following theorems.

THEOREM 4.1. *Assume that (1.2) holds and f satisfies (3.8). Suppose u and v are the solutions of the initial value problems*

$$\begin{cases} \nabla_{-1}^{\alpha+\varepsilon}[u(t) - g(t,u(t))] = f(t,u(t)), \\ \nabla_{-1}^{-(1-\alpha-\varepsilon)}u(t)|_{t=0} = u(0) = c,\ t \in \mathbb{N}_1, \end{cases} \tag{4.1}$$

$$\begin{cases} \nabla_{-1}^{\alpha}[v(t) - g(t,v(t))] = f(t,v(t)), \\ \nabla_{-1}^{-(1-\alpha)}v(t)|_{t=0} = v(0) = c,\ t \in \mathbb{N}_1, \end{cases} \tag{4.2}$$

respectively, where $\varepsilon > 0$ and $0 < \alpha < \alpha + \varepsilon < 1$. Then

$$\|u - v\|_\infty = O(\varepsilon), \tag{4.3}$$

provided that (3.9) holds.

Proof. We have

$$u(t) = \frac{(t+1)^{\overline{\alpha+\varepsilon-1}}}{\Gamma(\alpha+\varepsilon)}[c - g(0,c)] + g(t,u(t)) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\varepsilon-1}}f(s,u(s)),\ t \in \mathbb{N}_0,$$

$$v(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}[c - g(0,c)] + g(t,v(t)) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,v(s)),\ t \in \mathbb{N}_0.$$

Consider

$$\begin{aligned}
&\|u(t)-v(t)\| \\
&\leqslant \left|\frac{(t+1)^{\overline{\alpha+\varepsilon-1}}}{\Gamma(\alpha+\varepsilon)}-\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|(1+L_g)\|c\|+\|g(t,u(t))-g(t,v(t))\| \\
&\quad+\left\|\frac{1}{\Gamma(\alpha+\varepsilon)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\varepsilon-1}}f(s,u(s))-\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,v(s))\right\| \\
&\leqslant \left|\frac{\Gamma(\alpha)}{\Gamma(\alpha+\varepsilon)}(t+\alpha)^{\overline{\varepsilon}}-1\right|\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(1+L_g)\|c\|_\infty+L_g\|u(t)-v(t)\| \\
&\quad+\left\|\frac{1}{\Gamma(\alpha+\varepsilon)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\varepsilon-1}}[f(s,u(s))-f(s,v(s))]\right\| \\
&\quad+\left\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,v(s))\left[1-\frac{\Gamma(\alpha)}{\Gamma(\alpha+\varepsilon)}(t-s+\alpha)^{\overline{\varepsilon}}\right]\right\| \\
&\leqslant \left|\frac{\Gamma(\alpha)}{\Gamma(t+\alpha)}\frac{\Gamma(\varepsilon+t+\alpha)}{\Gamma(\varepsilon+\alpha)}-1\right|\frac{(2)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(1+L_g)\|c\|_\infty+L_g\|u(t)-v(t)\| \\
&\quad+\frac{1}{\Gamma(\alpha+\varepsilon)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\varepsilon-1}}\|f(s,u(s))-f(s,v(s))\| \\
&\quad+\frac{1}{\Gamma(\alpha)}\textstyle\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,v(s))\|\left|1-\frac{\Gamma(\alpha)}{\Gamma(t-s+\alpha)}\frac{\Gamma(\varepsilon+t-s+\alpha)}{\Gamma(\varepsilon+\alpha)}\right|,\ t\in\mathbb{N}_1.
\end{aligned} \tag{4.4}$$

Since

$$\lim_{\varepsilon\longrightarrow 0}\frac{1}{\varepsilon}\left[\frac{\Gamma(\alpha)}{\Gamma(t+\alpha)}\frac{\Gamma(\varepsilon+t+\alpha)}{\Gamma(\varepsilon+\alpha)}-1\right]=C_1 \text{ (a constant independent of } \varepsilon),$$

and

$$\lim_{\varepsilon\longrightarrow 0}\frac{1}{\varepsilon}\left[1-\frac{\Gamma(\alpha)}{\Gamma(t-s+\alpha)}\frac{\Gamma(\varepsilon+t-s+\alpha)}{\Gamma(\varepsilon+\alpha)}\right]=C_2 \text{ (a constant independent of } \varepsilon),$$

we have

$$\left[\frac{\Gamma(\alpha)}{\Gamma(t+\alpha)}\frac{\Gamma(\varepsilon+t+\alpha)}{\Gamma(\varepsilon+\alpha)}-1\right]=O(\varepsilon), \tag{4.5}$$

$$\left[1-\frac{\Gamma(\alpha)}{\Gamma(t-s+\alpha)}\frac{\Gamma(\varepsilon+t-s+\alpha)}{\Gamma(\varepsilon+\alpha)}\right]=O(\varepsilon). \tag{4.6}$$

Using (4.5) and (4.6) in (4.4), we get

$$\begin{aligned}
&\|u(t)-v(t)\| \\
&\leqslant O(\varepsilon)\alpha(1+L_g)\|c\|_\infty + L_g\|u-v\|_\infty + M\|u-v\|_\infty \frac{1}{\Gamma(\alpha+\varepsilon)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\varepsilon-1}}s^{\overline{-\gamma}} \\
&\quad + O(\varepsilon)\|f\|_\infty \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\gamma}} \\
&= O(\varepsilon)\alpha(1+L_g)\|c\|_\infty + L_g\|u-v\|_\infty + M\|u-v\|_\infty \nabla_0^{-(\alpha+\varepsilon)}t^{\overline{-\gamma}} + O(\varepsilon)\|f\|_\infty \nabla_0^{-\alpha}t^{\overline{-\gamma}} \\
&= O(\varepsilon)\alpha(1+L_g)\|c\|_\infty + L_g\|u-v\|_\infty + M\|u-v\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha+\varepsilon-\gamma)}t^{\overline{\alpha+\varepsilon-\gamma}} \\
&\quad + O(\varepsilon)\|f\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}t^{\overline{\alpha-\gamma}} \\
&\leqslant O(\varepsilon)\alpha(1+L_g)\|c\|_\infty + L_g\|u-v\|_\infty + M\|u-v\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha+\varepsilon-\gamma)}(1)^{\overline{\alpha+\varepsilon-\gamma}} \\
&\quad + O(\varepsilon)\|f\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}(1)^{\overline{\alpha-\gamma}} \\
&= O(\varepsilon)\alpha(1+L_g)\|c\|_\infty + L_g\|u-v\|_\infty + M\|u-v\|_\infty\Gamma(1-\gamma) + O(\varepsilon)\|f\|_\infty\Gamma(1-\gamma),\ t\in\mathbb{N}_1.
\end{aligned}$$

Then, we have the relation

$$\|u-v\|_\infty \leqslant \frac{[\alpha(1+L_g)\|c\|_\infty + \|f\|_\infty\Gamma(1-\gamma)]}{1-L_g-M\Gamma(1-\gamma)}O(\varepsilon),$$

implies

$$\|u-v\|_\infty = O(\varepsilon). \quad \square$$

THEOREM 4.2. *Assume that (1.2) holds and f satisfies (3.8). Suppose u and v are the solutions of the initial value problems*

$$\begin{cases} \nabla_{-1}^{\alpha}[u(t)-g(t,u(t))] = f(t,u(t)), \\ \nabla_{-1}^{-(1-\alpha)}u(t)|_{t=0} = u(0) = c,\ t\in\mathbb{N}_1, \end{cases} \tag{4.7}$$

$$\begin{cases} \nabla_{-1}^{\alpha}[v(t)-g(t,v(t))] = f(t,v(t)), \\ \nabla_{-1}^{-(1-\alpha)}v(t)|_{t=0} = v(0) = d,\ t\in\mathbb{N}_1, \end{cases} \tag{4.8}$$

respectively, where $0<\alpha<1$. Then

$$\|u-v\|_\infty = O(\|c-d\|_\infty), \tag{4.9}$$

provided that (3.9) holds.

Proof. We have

$$u(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}[c-g(0,c)] + g(t,u(t)) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,u(s)),\ t\in\mathbb{N}_0,$$

$$v(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}[d - g(0,d)] + g(t,v(t)) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} f(s,v(s)),\ t \in \mathbb{N}_0.$$

Consider

$$\begin{aligned}
&\|u(t)-v(t)\| \\
&\leqslant \|c-d+g(0,c)-g(0,d)\| \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} + \|g(t,u(t))-g(t,v(t))\| \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,u(s))-f(s,v(s))\| \\
&\leqslant (1+L_g)\|c-d\|_\infty \frac{(2)^{\overline{\alpha-1}}}{\Gamma(\alpha)} + L_g\|u-v\|_\infty + M\|u-v\|_\infty \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} s^{\overline{-\gamma}} \\
&= \alpha(1+L_g)\|c-d\|_\infty + L_g\|u-v\|_\infty + M\|u-v\|_\infty \nabla_0^{-\alpha} t^{\overline{-\gamma}} \\
&= \alpha(1+L_g)\|c-d\|_\infty + L_g\|u-v\|_\infty + M\|u-v\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} t^{\overline{\alpha-\gamma}} \\
&\leqslant \alpha(1+L_g)\|c-d\|_\infty + L_g\|u-v\|_\infty + M\|u-v\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}(1)^{\overline{\alpha-\gamma}} \\
&= \alpha(1+L_g)\|c-d\|_\infty + L_g\|u-v\|_\infty + M\|u-v\|_\infty \Gamma(1-\gamma),\ t \in \mathbb{N}_1.
\end{aligned}$$

Then, we have the relation

$$\|u-v\|_\infty \leqslant \frac{\alpha(1+L_g)}{1-L_g-M\Gamma(1-\gamma)}\|c-d\|_\infty,$$

implies

$$\|u-v\|_\infty = O(\|c-d\|_\infty). \quad \square$$

THEOREM 4.3. *Let (1.2) holds for g and G with $g(0,c) = G(0,c)$. Assume that f and F are continuous and satisfy (3.8). Suppose u and v are the solutions of the initial value problems*

$$\begin{cases} \nabla_{-1}^{\alpha}[u(t) - g(t,u(t)))] = f(t,u(t)), \\ \nabla_{-1}^{-(1-\alpha)} u(t)|_{t=0} = u(0) = c,\ t \in \mathbb{N}_1, \end{cases} \tag{4.10}$$

$$\begin{cases} \nabla_{-1}^{\alpha}[v(t) - G(t,v(t))] = F(t,v(t)), \\ \nabla_{-1}^{-(1-\alpha)} v(t)|_{t=0} = v(0) = c,\ t \in \mathbb{N}_1, \end{cases} \tag{4.11}$$

respectively, where $0 < \alpha < 1$. Then

$$\|u-v\|_\infty = O(\|f-F\|_\infty + \|g-G\|_\infty) \tag{4.12}$$

provided that (3.9) holds.

Proof. We have

$$u(t)=\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}[c-g(0,c)]+g(t,u(t))+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,u(s)),\ t\in\mathbb{N}_0,$$

$$v(t)=\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}[c-G(0,c)]+G(t,v(t))+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}F(s,v(s)),\ t\in\mathbb{N}_0.$$

Consider

$$\begin{aligned}
&\|u(t)-v(t)\|\\
&\leqslant\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,u(s))-F(s,v(s))\|+\|g(t,u(t))-G(t,v(t))\|\\
&=\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,u(s))-f(s,v(s))+f(s,v(s))-F(s,v(s))\|\\
&\quad+\|g(t,u(t))-g(t,v(t))+g(t,v(t))-G(t,v(t))\|\\
&\leqslant\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,u(s))-f(s,v(s))\|\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|f(s,v(s))-F(s,v(s))\|+L_g\|u-v\|_\infty+\|g-G\|_\infty\\
&\leqslant[M\|u-v\|_\infty+\|f-F\|_\infty]\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\gamma}}+L_g\|u-v\|_\infty+\|g-G\|_\infty\\
&=[M\|u-v\|_\infty+\|f-F\|_\infty]\nabla_0^{-\alpha}s^{\overline{-\gamma}}+L_g\|u-v\|_\infty+\|g-G\|_\infty\\
&=[M\|u-v\|_\infty+\|f-F\|_\infty]\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}t^{\overline{\alpha-\gamma}}+L_g\|u-v\|_\infty+\|g-G\|_\infty\\
&\leqslant[M\|u-v\|_\infty+\|f-F\|_\infty]\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}(1)^{\overline{\alpha-\gamma}}+L_g\|u-v\|_\infty+\|g-G\|_\infty\\
&=[M\|u-v\|_\infty+\|f-F\|_\infty]\Gamma(1-\gamma)+L_g\|u-v\|_\infty+\|g-G\|_\infty,\ t\in\mathbb{N}_1.
\end{aligned}$$

Then, we have the relation

$$\|u-v\|_\infty\leqslant\frac{\Gamma(1-\gamma)\|f-F\|_\infty+\|g-G\|_\infty}{1-L_g-M\Gamma(1-\gamma)},$$

implies

$$\|u-v\|_\infty=O(\|f-F\|_\infty+\|g-G\|_\infty).\quad\square$$

DEFINITION 4.4. A solution $\tilde{u}\in\ell^\infty$ is said to be stable, if given $\varepsilon>0$ and $t_0\geqslant 0$, there exists $\delta=\delta(\varepsilon,t_0)$ such that $\|u(t_0)-\tilde{u}(t_0)\|_\infty<\delta\Longrightarrow\|u-\tilde{u}\|_\infty<\varepsilon$ for all $t\geqslant t_0$.

THEOREM 4.5. *Assume that (1.2) holds and f satisfies (3.8). Then the solutions of (1.1) are stable provided that (3.9) holds.*

Proof. The proof is a direct consequence of Theorem 2.9. □

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