Fractional
Differential
Calculus
Volume 7, Number 2 (2017), 283–300 doi:10.7153/fdc-2017-07-12

POSITIVE SOLUTIONS OF A SYSTEM OF FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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(*Communicated by N. Vasylyeva*)

Abstract. We study the system of two fractional functional differential equations with the Caputo fractional derivative. Using the Guo–Krasnoselskii fixed point theorem on cones and the nonlinear Leray–Schauder alternative the existence of positive solutions to the system satisfying nonlocal boundary conditions is proved. The boundary conditions are given by linear bounded functionals. Examples are given to illustrate our results.

1. Introduction

Let $T>0$ be given, $J=[0,T]$, $\mathbb{R}_+=[0,\infty)$ and $X=C(J)\times C(J)$. Let $\|x\|=\max\{|x(t)|\colon t\in J\}$ and $\|(x_1,x_2)\|_1=\|x_1\|+\|x_2\|$ be the norm in $C(J)$ and X, respectively. Besides, $P_+=\{x\in C(J)\colon x(t)\geqslant 0 \text{ for } t\in J\}$ and $X_+=\{(x_1,x_2)\in X\colon x_1(t)\geqslant 0, x_2(t)\geqslant 0 \text{ for } t\in J\}$.

Let $\mathscr{A}$ be the set of all linear bounded functionals $\ell\colon C(J)\to\mathbb{R}$ which are nonnegative, that is,

$$x\in C(J),\ x\geqslant 0 \text{ on } J \Rightarrow \ell(x)\geqslant 0,$$

and $\|\ell\|<1$, where $\|\ell\|$ is the norm of ℓ.

REMARK 1. The Riesz representation theorem says that linear bounded functionals ℓ on $C(J)$ are given by the Riemann–Stieltjes integral as

$$\ell(x)=\int_0^T x(s)\mathrm{d}g(t),\quad x\in C(J), \tag{1}$$

and $\|\ell\|=var_0^T g$, where $var_0^T g$ denotes the total variation of g over J. Hence functionals ℓ belonging to the set $\mathscr{A}$ are represented by (1), where $g:J\to\mathbb{R}$ is nondecreasing and $g(T)-g(0)<1$.

In particular, if $v\colon J\to\mathbb{R}$ is nondecreasing, $v(T)-v(0)<1$ and $\{r_n\}\subset(0,\infty)$, $\sum_{n=1}^\infty r_n<1$, $\{t_n\}\subset J$, $t_i\neq t_j$ for $i\neq j$, then the functionals

$$\ell_1(x)=\int_0^T x(t)v'(t)\,\mathrm{d}t,\ \ \ell_2(x)=\sum_{n=1}^\infty r_n x(t_n),$$

belong to $\mathscr{A}$.

Mathematics subject classification (2010): 34A08, 26A33, 33E12, 34B18.
Keywords and phrases: Fractional functional equation, nonlocal boundary condition, Caputo fractional derivative, Guo–Krasnoselskii fixed point theorem, nonlinear Leray–Schauder alternative, positive solution.

Paper FDC-07-12

We discuss the system of fractional differential equations

$$\begin{aligned} {}^c\!D^\alpha x_1(t) + a(t)\,{}^c\!D^\beta x_2(t) &= \mathscr{L}_1(x_1,x_2)(t), \\ {}^c\!D^\gamma x_2(t) + b(t)\,{}^c\!D^\mu x_1(t) &= \mathscr{L}_2(x_1,x_2)(t), \end{aligned} \tag{2}$$

where $0 < \mu < \alpha < 1$, $0 < \beta < \gamma < 1$, $a,b \in C(J)$, $a \leqslant 0$, $b \leqslant 0$ on J and $\mathscr{L}_j : X_+ \to P_+$ is continuous, $j = 1,2$. Here, ${}^c\!D$ denotes the Caputo fractional derivative. Further conditions on $\mathscr{L}_j$ will be specified later.

Together with system (2) we investigate the boundary conditions

$$x_j(0) = \ell_j(x_j), \quad \ell_j \in \mathscr{A},\ j = 1,2. \tag{3}$$

DEFINITION 1. We say that $(x_1,x_2) \in X_+$ is *a solution of system* (2) if ${}^c\!D^\alpha x_1$, ${}^c\!D^\gamma x_2 \in C(J)$ and equality (2) holds for $t \in J$. A solution of (2) satisfying the boundary condition (3) is said to be *a solution of problem* (2), (3).

If a solution (x_1,x_2) of (2) satisfies $\|(x_1,x_2)\|_1 > 0$, then we say that it is *a positive solution of system* (2). Similarly, for *positive solutions of problem* (2), (3).

We recall the definitions and properties of the Riemann-Liouville fractional integral and the Caputo fractional derivative [8, 3].

The Riemann-Liouville fractional integral $I^\delta x$ of order $\delta > 0$ of a function $x\colon J \to \mathbb{R}$ is defined as

$$I^\delta x(t) = \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} x(s)\,\mathrm{d}s,$$

where Γ is the Euler gamma function. I^0 is the identical operator.

The Caputo fractional derivative ${}^c\!D^\delta x$ of order $\delta > 0$, $\delta \notin \mathbb{N}$, of a function $x\colon J \to \mathbb{R}$ is given as

$${}^c\!D^\delta x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\delta-1}}{\Gamma(n-\delta)} \left(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \right) \mathrm{d}s,$$

where $n = [\delta] + 1$, $[\delta]$ means the integral part of the fractional number δ.

In particular,

$${}^c\!D^\delta x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)} (x(s) - x(0))\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\delta}(x(t) - x(0)), \quad \delta \in (0,1).$$

It is well known that $I^\delta\colon C(J) \to C(J)$ for $\delta \in (0,1)$; $I^\delta I^\nu x(t) = I^{\delta+\nu} x(t)$ for $x \in C(J)$ and $\delta, \nu \in (0,\infty)$; ${}^c\!D^\delta I^\delta x(t) = x(t)$ for $x \in C(J)$ and $\delta > 0$; $\frac{\mathrm{d}}{\mathrm{d}t} I^1 x(t) = x(t)$ for $x \in C(J)$; if $\delta \in (0,1)$, $x, {}^c\!D^\delta x \in C(J)$, then $I^\delta\, {}^c\!D^\delta x(t) = x(t) - x(0)$.

REMARK 2. If (x_1,x_2) is a solution of system (2), then ${}^c\!D^\mu x_1, {}^c\!D^\beta x_2 \in C(J)$. It follows from the conditions $\alpha > \mu$, $\gamma > \beta$ and the equalities ${}^c\!D^\mu x_1 = I^{\alpha-\mu}\,{}^c\!D^\alpha x_1$, ${}^c\!D^\beta x_2 = I^{\gamma-\beta}\,{}^c\!D^\gamma x_2$

Let $f_j \in C(J \times \mathbb{R}_+^2)$ and $f_j \geqslant 0$, $j = 1,2$. Let $\mathscr{L}_j(x_1,x_2)(t) = f_j(t, x_1(t), x_2(t))$. Then $\mathscr{L}_j : X_+ \to P_+$ is continuous, and therefore the special case of (2) is the system

$$\begin{aligned} {}^c\!D^\alpha x_1(t) + a(t)\,{}^c\!D^\beta x_2(t) &= f_1(t, x_1(t), x_2(t)), \\ {}^c\!D^\gamma x_2(t) + b(t)\,{}^c\!D^\mu x_1(t) &= f_2(t, x_1(t), x_2(t)). \end{aligned} \tag{4}$$

The solvability of systems of fractional differential equations with local and nonlocal boundary conditions was extensively considered in the literature (see [1], [5], [6], [9]–[16] and references therein). In these papers associated homogeneous systems have the form either

$$^{c}D^{\alpha}x_1(t)=0, \quad {}^{c}D^{\beta}x_2(t)=0,$$

or (D^{γ} denotes the Riemann–Liouville fractional derivative of order γ)

$$D^{\alpha}x_1(t)=0, \quad D^{\beta}x_2(t)=0,$$

where $\alpha,\beta\in(1,\infty)$. This form plays the important role to obtain operators whose fixed points are solutions of considered problems. Moreover, the nonlinearities of systems are given by functions f_1,f_2.

For example Henderson and Luca [6] considered the problem

$$\begin{aligned}
&D^{\alpha}x_1(t)+\lambda f_1(t,x_1(t),x_2(t))=0, \quad n-1<\alpha\leqslant n,\\
&D^{\beta}x_2(t)+\mu f_2(t,x_1(t),x_2(t))=0, \quad m-1<\beta\leqslant m,\\
&x_1(0)=x_1'(0)=\cdots=x_1^{(n-2)}(0)=0,\ x_1(1)=\int_0^1 x_2(s)\mathrm{d}H_1(s),\\
&x_2(0)=x_2'(0)=\cdots=x_2^{(m-2)}(0)=0,\ x_2(1)=\int_0^1 x_1(s)\mathrm{d}H_2(s),
\end{aligned}$$

where $n,m\geqslant 3$, λ,μ are parameters, f_i is continuous on $J\times\mathbb{R}_+^2$ and H_i is a function of bounded variation. Existence results for positive solutions were proved by the nonlinerar alternative of the Leray–Schauder type and the Guo–Krasnoselskii fixed point theorem on cones.

Zhao and Liu [16] discussed the problem

$$\begin{aligned}
&{}^{c}D^{\alpha}x_1(t)+f_1(t,x_2(t))=0,\\
&{}^{c}D^{\alpha}x_2(t)+f_2(t,x_1(t))=0,\\
&x_1^{(j)}(0)=x_2^{(j)}(0)=0, \quad 0\leqslant j\leqslant n-1,\ j\neq 1,\\
&x_1'(1)=\lambda\int_0^1 x_1(s)\mathrm{d}s,\ x_2'(1)=\lambda\int_0^1 x_2(s)\mathrm{d}s,
\end{aligned}$$

where $n-1<\alpha\leqslant n$, $n\geqslant 3$, $0\leqslant\lambda<2$ and $f_i\in C(J\times\mathbb{R}_+)$ is nonnegative. Existence and uniqueness results are established by using the monotone method, fixed point index theorems on cones and the properties of Green's function.

In [1] the authors studied the problem

$$\begin{aligned}
&{}^{c}D^{\alpha}x_1(t)=f_1\left(t,x_1(t),x_2(t),{}^{c}D^{\gamma}x_2(t)\right),\\
&{}^{c}D^{\beta}x_2(t)=f_2\left(t,x_1(t),{}^{c}D^{\delta}x_1(t),x_2(t)\right),\\
&x_1(0)=h_1(x_2),\quad \int_0^T x_2(s)\,\mathrm{d}s=\mu_1x_1(\eta),\\
&x_2(0)=h_2(x_1),\quad \int_0^T x_1(s)\,\mathrm{d}s=\mu_2x_2(\xi),
\end{aligned}$$

where $\alpha,\beta\in(1,2]$, $\gamma,\delta\in(0,1)$, $\eta,\xi\in(0,T)$, $f_i\in C([0,T]\times\mathbb{R}^3)$ and $h_i\colon C[0,T]\to\mathbb{R}$ is continuous. By using the Banach fixed point theorem and the Leray–Schauder nonlinear alternative, the existence and uniqueness of solutions was proved.

Shah at al. [13] studied the problem

$$\begin{aligned}
&{}^{c}\!D^{\alpha}x_1(t) = f_1(t,x_1(t),x_2(t)),\\
&{}^{c}\!D^{\beta}x_2(t) = f_2(t,x_1(t),x_2(t)),\\
&x_1(0) = h_1(x_1),\ \ x_1(1) = \delta x_1(\eta),\\
&x_2(0) = h_2(x_2),\ \ x_2(0) = \gamma x_2(\xi),
\end{aligned}$$

where $\alpha,\beta \in (1,2]$, $0<\delta,\gamma<1$, $f_i \in C(J\times\mathbb{R}^2)$ and $h_i\colon C(J)\to\mathbb{R}$ is continuous. The existence of solutins was proved by a coincidence degree theory approach for condensing maps.

In contrast to papers [1], [5], [6] and [9]–[16], associated homogeneous system of (2) has the form

$$\begin{aligned}
&{}^{c}\!D^{\alpha}x_1(t) + a(t)\,{}^{c}\!D^{\beta}x_2(t) = 0,\\
&{}^{c}\!D^{\gamma}x_2(t) + b(t)\,{}^{c}\!D^{\mu}x_1(t) = 0,
\end{aligned}$$

where $0<\mu<\alpha<1$, $0<\beta<\gamma<1$, and the nonlinearities are operators $\mathscr{L}_1,\mathscr{L}_2$.

The aim of this paper is to discuss the existence of positive solutions to problem (2), (3). To this end we introduce an integral operator $\mathscr{H}: X_+ \to X_+$ and show that fixed points of $\mathscr{H}$ are solutions to problem (2), (3) (Lemma 5 below). Existence of a fixed point of $\mathscr{H}$ is proved by the Guo–Krasnoselskii fixed point theorem on cones (Lemma 7 below) and the nonlinear Leray–Schauder alterative (Lemma 8 below).

We work with the following conditions on $\mathscr{L}_j$:

(H_1) $\mathscr{L}_j$ takes bounded sets into bounded sets, $j=1,2$.

(H_2) There exists $\varepsilon>0$ such that either

$$\mathscr{L}_1(x_1,x_2)(t) \geqslant \varepsilon \ \text{ for } t\in J,\ (x_1,x_2)\in X_+,$$

or

$$\mathscr{L}_2(x_1,x_2)(t) \geqslant \varepsilon \ \text{ for } t\in J,\ (x_1,x_2)\in X_+.$$

(H_3) There exists a nondecreasing positive function $w\in C(\mathbb{R}_+)$ such that

$$\lim_{x\to\infty}\frac{w(x)}{x} = 0$$

and

$$\|\mathscr{L}_j(x_1,x_2)\| \leqslant w(\|(x_1,x_2)\|_1) \quad \text{for } (x_1,x_2)\in X_+,\ j=1,2.$$

(H_4) $\lim_{(x_1,x_2)\in X_+,\|(x_1,x_2)\|_1\to 0}\frac{\|\mathscr{L}_j(x_1,x_2)\|}{\|(x_1,x_2)\|_1} = 0,\ \ j=1,2.$

(H_5) Either

$$\lim_{(x_1,x_2)\in X_+,\,\|(x_1,x_2)\|_1\to\infty}\frac{\|\mathscr{L}_1(x_1,x_2)\|}{\|(x_1,x_2)\|_1} > \frac{\Gamma(\alpha+1)}{T^{\alpha}}$$

or

$$\lim_{(x_1,x_2)\in X_+,\,\|(x_1,x_2)\|_1\to\infty}\frac{\|\mathscr{L}_2(x_1,x_2)\|}{\|(x_1,x_2)\|_1} > \frac{\Gamma(\gamma+1)}{T^{\gamma}}.$$

It is obvious that $(H_3) \Rightarrow (H_1)$.

The paper is organized as follows. In Section 2 we introduce an operator $\mathscr{Q}$, prove its properties and discuss the existence of solutions to an auxiliary linear fractional system satisfying the boundary condition (3). In Section 3, a key operator $\mathscr{H}$ is introduced and its properties are given. The existence results for positive solutions of problem (2), (3) are stated and proved in Section 4. Some examples are given to illustrate the results.

2. Preliminaries

We need a generalization of the Gronwall lemma for singular kernels [7, Lemma 7.1.1].

LEMMA 1. *Let $0<\delta<1$, $d\in C(J)$ be nonnegative and V be a positive constant. Suppose that $w\in C(J)$ is nonnegative and*

$$w(t)\leqslant d(t)+V\int_0^t (t-s)^{\delta-1}w(s)\,\mathrm{d}s,\quad t\in J.$$

Then

$$w(t)\leqslant d(t)+VK\int_0^t (t-s)^{\delta-1}d(s)\,\mathrm{d}s,\quad t\in J,$$

where $K=K(\delta)$ is a positive constant.

Introduce operator $\Lambda_j : C(J)\to C(J)$, $j=1,2$, as

$$\begin{aligned}\Lambda_1 x(t) &= a(t)I^{\gamma-\beta}b(t)I^{\alpha-\mu}x(t),\\ \Lambda_2 x(t) &= b(t)I^{\alpha-\mu}a(t)I^{\gamma-\beta}x(t),\end{aligned}$$

where a,b are from (2). Since $a\leqslant 0$, $b\leqslant 0$ on J,

$$\Lambda_j \text{ maps } P_+ \text{ into } P_+,\quad j=1,2. \tag{5}$$

For $n\in\mathbb{N}$, let Λ_j^n be nth iteration of Λ_j, that is, $\Lambda_j^n=\underbrace{\Lambda_j\circ\Lambda_j\circ\cdots\circ\Lambda_j}_{n}$, and Λ_j^0 be the identical operator in $C(J)$.

Finally, introduce operator $\mathscr{Q}_j$ acting on $C(J)$ by the formula

$$\mathscr{Q}_j x(t)=\sum_{k=0}^{\infty}\Lambda_j^k x(t),\quad j=1,2.$$

For $\nu>0$, let $E_\nu:\mathbb{R}\to\mathbb{R}$,

$$E_\nu(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(k\nu+1)}$$

be the classical Mittag–Leffler function [8, 3].

The properties of $\mathscr{Q}_j$ are summarize in the following two lemmas. Let

$$\rho=\alpha-\beta+\gamma-\mu,\qquad M=\|a\|\|b\|.$$

LEMMA 2. *For each $x \in C(J)$ the series $\sum_{k=0}^{\infty} \Lambda_j^k x$ is uniformly convergent on J and*

$$\sum_{k=0}^{\infty} \left|\Lambda_j^k x(t)\right| \leqslant \|x\| E_\rho\left(Mt^\rho\right), \quad t \in J,\ j = 1,2. \tag{6}$$

Proof. We first prove that

$$\left|\Lambda_j^k x(t)\right| \leqslant M^k I^{k\rho}|x(t)|, \quad t \in J,\ x \in C(J),\ k \in \mathbb{N},\ j = 1,2. \tag{7}$$

It is clear that $|\Lambda_j x| \leqslant MI^\rho |x|$ for $x \in C(J)$. We now proceed by induction. Suppose that $|\Lambda_j^n x| \leqslant M^n I^{n\rho}|x|$ for $x \in C(J)$ and some $n \in \mathbb{N}$. Then

$$\left|\Lambda_j^{n+1} x\right| = \left|\Lambda_j^n \Lambda_j x\right| \leqslant M^n I^{n\rho} \left|\Lambda_j x\right| \leqslant M^{n+1} I^{(n+1)\rho}|x|, \quad x \in C(J).$$

Therefore estimate (7) is valid.

Choose $x \in C(J)$. Since

$$I^{k\rho}|x(t)| = \int_0^t \frac{(t-s)^{k\rho-1}}{\Gamma(k\rho)}|x(s)|\,\mathrm{d}s \leqslant \|x\| \frac{t^{k\rho}}{\Gamma(k\rho+1)},$$

we have (note that $\Lambda_j^0 x = x$)

$$\sum_{k=0}^{\infty} \left|\Lambda_j^k x(t)\right| \leqslant \|x\| \sum_{k=0}^{\infty} \frac{(Mt^\rho)^k}{\Gamma(k\rho+1)} = \|x\| E_\rho(Mt^\rho) \leqslant \|x\| E_\rho(MT^\rho), \quad j = 1,2.$$

Consequently, $\sum_{k=0}^{\infty} \Lambda_j^k x$ is uniformly convergent on J and (6) follows. □

LEMMA 3. *$\mathcal{Q}_j : C(J) \to C(J)$, $\mathcal{Q}_j$ is a linear bounded operator,*

$$|\mathcal{Q}_j x(t)| \leqslant \|x\| E_\rho\left(Mt^\rho\right), \quad t \in J,\ x \in C(J),\ j = 1,2, \tag{8}$$

and for $x \in C(J)$,

$$\begin{aligned} a(t) I^{\gamma-\beta} \mathcal{Q}_2 x(t) &= \mathcal{Q}_1 a(t) I^{\gamma-\beta} x(t), \\ b(t) I^{\alpha-\mu} \mathcal{Q}_1 x(t) &= \mathcal{Q}_2 b(t) I^{\alpha-\mu} x(t). \end{aligned} \tag{9}$$

Proof. Let $x \in C(J)$ and $j = 1,2$. By Lemma 2, the series $\sum_{k=0}^{\infty} \Lambda_j^k x$ is uniformly convergent on J, and since $\Lambda_j^k x \in C(J)$ for $k \in \mathbb{N}$, we have $\mathcal{Q}_j x \in C(J)$.

Inequality (8) follows from (6). The linearity of $\mathcal{Q}_j$ and (8) imply that $\mathcal{Q}_j$ is a linear bounded operator.

In order to prove (9) it is sufficient to show that

$$\begin{aligned} a(t) I^{\gamma-\beta} \Lambda_2^k x(t) &= \Lambda_1^k a(t) I^{\gamma-\beta} x(t), \\ b(t) I^{\alpha-\mu} \Lambda_1^k x(t) &= \Lambda_2^k b(t) I^{\alpha-\mu} x(t), \end{aligned} \quad k \in \mathbb{N} \cup \{0\},\ x \in C(J). \tag{10}$$

In view of $\Lambda_j^0 x = x$, (10) holds for $k = 0$. Suppose that

$$aI^{\gamma-\beta} \Lambda_2^n x = \Lambda_1^n a I^{\gamma-\beta} x \quad \text{for some } n \in \mathbb{N} \cup \{0\}.$$

Then

$$\begin{aligned} aI^{\gamma-\beta} \Lambda_2^{n+1} x &= aI^{\gamma-\beta} \Lambda_2^n \Lambda_2 x = \Lambda_1^n a I^{\gamma-\beta} \Lambda_2 x = \Lambda_1^n a I^{\gamma-\beta} b I^{\alpha-\mu} a I^{\gamma-\beta} x \\ &= \Lambda_1^n \Lambda_1 a I^{\gamma-\beta} x = \Lambda_1^{n+1} a I^{\gamma-\beta} x. \end{aligned}$$

We have proved by induction that the first equality of (10) holds. Similarly, we can verify its second equality. □

COROLLARY 1. *$\mathscr{Q}_j$ maps P_+ into P_+ and*

$$0 \leqslant \mathscr{Q}_j x(t) \leqslant \|x\| E_\rho\left(MT^\rho\right), \quad t \in J,\ x \in P_+,\ j = 1,2. \tag{11}$$

Proof. In view of (5), we see that $\mathscr{Q}_j$ maps P_+ into P_+. Estimate (11) follows from (8). □

We now consider the auxiliary linear system of fractional differential equations

$$\begin{aligned} {}^c\mathcal{D}^\alpha x_1(t) + a(t)\,{}^c\mathcal{D}^\beta x_2(t) &= r_1(t), \\ {}^c\mathcal{D}^\gamma x_2(t) + b(t)\,{}^c\mathcal{D}^\mu x_1(t) &= r_2(t), \end{aligned} \tag{12}$$

where $r_1, r_2 \in C(J)$.

LEMMA 4. *Let $r_1, r_2 \in C(J)$ and*

$$x_1(t) = \ell_1(x_1) + I^\alpha \mathscr{Q}_1\left(r_1(t) - a(t) I^{\gamma-\beta} r_2(t)\right), \tag{13}$$

$$x_2(t) = \ell_2(x_2) + I^\gamma \mathscr{Q}_2\left(r_2(t) - b(t) I^{\alpha-\mu} r_1(t)\right). \tag{14}$$

Then $(x_1, x_2) \in X$ and it is the unique solution of problem (12), (3).

Proof. It is obvious that $(x_1, x_2) \in X$ and $x_j(0) = \ell_j(x_j)$, $j = 1,2$. Hence (x_1, x_2) satisfies the boundary condition (3).

We prove that ${}^c\mathcal{D}^\alpha x_1, {}^c\mathcal{D}^\gamma x_2 \in C(J)$. In view of

$$\begin{aligned} I^{1-\alpha}(x_1(t) - x_1(0)) &= I^1 \mathscr{Q}_1(r_1(t) - a(t) I^{\gamma-\beta} r_2(t)), \\ I^{1-\gamma}(x_2(t) - x_2(0)) &= I^1 \mathscr{Q}_2(r_2(t) - b(t) I^{\alpha-\mu} r_1(t)), \end{aligned}$$

it follows that

$$\begin{aligned} {}^c\mathcal{D}^\alpha x_1(t) &= \mathscr{Q}_1(r_1(t) - a(t) I^{\gamma-\beta} r_2(t)), \\ {}^c\mathcal{D}^\gamma x_2(t) &= \mathscr{Q}_2(r_2(t) - b(t) I^{\alpha-\mu} r_1(t)), \end{aligned}$$

and therefore ${}^c\mathcal{D}^\alpha x_1, {}^c\mathcal{D}^\gamma x_2 \in C(J)$, because $r_1 - aI^{\gamma-\beta} r_2,\ r_2 - bI^{\alpha-\mu} r_1 \in C(J)$.

We now show that (x_1, x_2) satisfies (12) for $t \in J$. Since

$$\begin{aligned} I^{1-\alpha}(x_1(t) - x_1(0)) &= I^1 \mathscr{Q}_1(r_1(t) - a(t) I^{\gamma-\beta} r_2(t)), \\ I^{1-\beta}(x_2(t) - x_2(0)) &= I^{1+\gamma-\beta} \mathscr{Q}_2(r_2(t) - b(t) I^{\alpha-\mu} r_1(t)), \end{aligned}$$

we obtain (cf. (9))

$$\begin{aligned} {}^c\mathcal{D}^\alpha x_1 + a\,{}^c\mathcal{D}^\beta x_2 &= \mathscr{Q}_1(r_1 - aI^{\gamma-\beta} r_2) + aI^{\gamma-\beta} \mathscr{Q}_2(r_2 - bI^{\alpha-\mu} r_1) \\ &= \mathscr{Q}_1 r_1 - \mathscr{Q}_1 a I^{\gamma-\beta} r_2 + aI^{\gamma-\beta} \mathscr{Q}_2 r_2 - aI^{\gamma-\beta} \mathscr{Q}_2 b I^{\alpha-\mu} r_1 \\ &= \mathscr{Q}_1 r_1 - aI^{\gamma-\beta} \mathscr{Q}_2 r_2 + aI^{\gamma-\beta} \mathscr{Q}_2 r_2 - \mathscr{Q}_1 a I^{\gamma-\beta} b I^{\alpha-\mu} r_1 \\ &= \mathscr{Q}_1 r_1 - \mathscr{Q}_1 \Lambda_1 r_1 = r_1. \end{aligned}$$

Similarly,

$$\begin{aligned} I^{1-\gamma}(x_2(t) - x_2(0)) &= I^1 \mathscr{Q}_2(r_2(t) - b(t) I^{\alpha-\mu} r_1(t)), \\ I^{1-\mu}(x_1(t) - x_1(0)) &= I^{1+\alpha-\mu} \mathscr{Q}_1(r_1(t) - a(t) I^{\gamma-\beta} r_2(t)), \end{aligned}$$

and

$$\begin{aligned} {}^c\!D^{\gamma}x_2 + b\,{}^c\!D^{\mu}x_1 &= \mathscr{Q}_2(r_2 - bI^{\alpha-\mu}r_1) + bI^{\alpha-\mu}\mathscr{Q}_1(r_1 - aI^{\gamma-\beta}r_2) \\ &= \mathscr{Q}_2r_2 - \mathscr{Q}_2bI^{\alpha-\mu}r_1 + bI^{\alpha-\mu}\mathscr{Q}_1r_1 - bI^{\alpha-\mu}\mathscr{Q}_1aI^{\gamma-\beta}r_2 \\ &= = \mathscr{Q}_2r_2 - \mathscr{Q}_2bI^{\alpha-\mu}r_1 + \mathscr{Q}_2bI^{\alpha-\mu}r_1 - \mathscr{Q}_2bI^{\alpha-\mu}aI^{\gamma-\beta}r_2 \\ &= \mathscr{Q}_2r_2 - \mathscr{Q}_2\Lambda_2r_2 = r_2. \end{aligned}$$

Consequently, (x_1,x_2) satisfies (12) for $t \in J$.

It remains to prove that (x_1,x_2) is the unique solution of problem (12), (3). Let (y_1,y_2) be another solution of this problem and let $w_j = x_j - y_j$, $j = 1,2$. Then $w_j(0) = \ell_j(w_j)$ and the equalities

$$\begin{aligned} {}^c\!D^{\alpha}w_1(t) + a(t)\,{}^c\!D^{\beta}w_2(t) &= 0, \\ {}^c\!D^{\gamma}w_2(t) + b(t)\,{}^c\!D^{\mu}w_1(t) &= 0, \end{aligned} \tag{15}$$

hold for $t \in J$. Since (cf. Remark 2) ${}^c\!D^{\mu}w_1 = I^{\alpha-\mu}\,{}^c\!D^{\alpha}w_1$, ${}^c\!D^{\beta}w_2 = I^{\gamma-\beta}\,{}^c\!D^{\gamma}w_2$, we conclude that

$$\begin{aligned} {}^c\!D^{\alpha}w_1(t) + a(t)I^{\gamma-\beta}\,{}^c\!D^{\gamma}w_2(t) &= 0, \\ {}^c\!D^{\gamma}w_2(t) + b(t)I^{\alpha-\mu}\,{}^c\!D^{\alpha}w_1(t) &= 0. \end{aligned}$$

Hence

$${}^c\!D^{\alpha}w_1(t) - a(t)I^{\gamma-\beta}b(t)I^{\alpha-\mu}\,{}^c\!D^{\alpha}w_1(t) = 0, \quad t \in J,$$

and so

$$u(t) = a(t)I^{\gamma-\beta}b(t)I^{\alpha-\mu}u(t), \quad t \in J, \tag{16}$$

where $u = {}^c\!D^{\alpha}w_1$. It follows from (16) that $|u(t)| \leqslant MI^{\rho}|u(t)|$, hence that

$$|u(t)| \leqslant M\int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)}|u(s)|\,\mathrm{d}s, \quad t \in J. \tag{17}$$

If $\rho \in (0,1)$, then (17) together with Lemma 1 yield $u = 0$. If $\rho \in [1,2)$, then

$$|u(t)| \leqslant \frac{MT^{\rho-1}}{\Gamma(\rho)}\int_0^t |u(s)|\,\mathrm{d}s, \quad t \in J,$$

and $u = 0$ now follows from the Gronwall–Bellman lemma. We have proved that $u = 0$, that is, ${}^c\!D^{\alpha}w_1 = 0$. Then w_1 is a constant function on J and it follows from $w_1 = \ell_1(w_1)$ and $\|\ell_1\| < 1$ that $w_1 = 0$. Therefore (cf. (15)) ${}^c\!D^{\gamma}w_2 = 0$ and the analyze similar to w_1 gives $w_2 = 0$. To summarize, $(w_1,w_2) = (0,0)$. □

3. Operator $\mathscr{H}$ and its properties

Keeping in mind Lemma 4, define operators $\mathscr{H}_1,\mathscr{H}_2 : X_+ \to C(J)$, $j = 1,2$, as

$$\begin{aligned} \mathscr{H}_1(x_1,x_2)(t) &= \ell_1(x_1) + I^{\alpha}\mathscr{Q}_1\mathscr{K}_1(x_1,x_2)(t), \\ \mathscr{H}_2(x_1,x_2)(t) &= \ell_2(x_2) + I^{\gamma}\mathscr{Q}_2\mathscr{K}_2(x_1,x_2)(t), \end{aligned}$$

where $\mathscr{K}_j : X_+ \to C(J)$,

$$\begin{aligned} \mathscr{K}_1(x_1,x_2)(t) &= \mathscr{L}_1(x_1,x_2)(t) - a(t)I^{\gamma-\beta}\mathscr{L}_2(x_1,x_2)(t), \\ \mathscr{K}_2(x_1,x_2)(t) &= \mathscr{L}_2(x_1,x_2)(t) - b(t)I^{\alpha-\mu}\mathscr{L}_1(x_1,x_2)(t). \end{aligned}$$

Finally, let an operator $\mathscr{H}: X_+ \to X$ be given by the formula

$$\mathscr{H}(x_1,x_2) = (\mathscr{H}_1(x_1,x_2), \mathscr{H}_2(x_1,x_2)).$$

The following two lemmas state that fixed points of $\mathscr{H}$ are positive solutions to problem (2), (3) and $\mathscr{H}$ is completely continuous.

LEMMA 5. *$\mathscr{H}: X_+ \to X_+$ and if (x_1,x_2) is a fixed point of $\mathscr{H}$, then (x_1,x_2) is a solution of problem (2), (3).*

Proof. Since $\mathscr{L}_j(x_1,x_2)(t): X_+ \to P_+$ and $a \leqslant 0$, $b \leqslant 0$ on J, we conclude that $\mathscr{K}_j: X_+ \to P_+$, which together with Corollary 1 and the non-negativity of ℓ_j give $\mathscr{H}_j: X_+ \to P_+$. Consequently, $\mathscr{H}: X_+ \to X_+$.

Let (x_1,x_2) be a fixed point of $\mathscr{H}$ and let $r_j(t) = \mathscr{L}_j(x_1,x_2)(t)$, $j = 1,2$. Then $(x_1,x_2) \in X_+$,

$$\mathscr{K}_1(x_1,x_2) = r_1 - aI^{\gamma-\beta} r_2, \quad \mathscr{K}_2(x_1,x_2) = r_2 - bI^{\alpha-\mu} r_1,$$

and equalities (13), (14) hold. Hence Lemma 4 guarantees that (x_1,x_2) is a solution of problem (2), (3). $\square$

LEMMA 6. *Let (H_1) hold. Then $\mathscr{H}$ is a completely continuous operator.*

Proof. $\mathscr{H}$ is completely continuous if and only if the operators $\mathscr{H}_1$ and $\mathscr{H}_2$ are completely continuous. We only prove that $\mathscr{H}_1$ is completely continuous, because for $\mathscr{H}_2$ the proof is similar.

Step 1. $\mathscr{H}_1$ is continuous.

Let $\{(u_n, v_n)\} \subset X_+$ be a convergent sequence and let $(u,v) \in X_+$ be its limit. Then $\lim_{n\to\infty} \|\mathscr{L}_j(u_n,v_n) - \mathscr{L}_j(u,v)\| = 0$, $j = 1,2$, and so $\lim_{n\to\infty} \|\mathscr{K}_1(u_n,v_n) - \mathscr{K}_1(u,v)\| = 0$. This together with the continuity of ℓ_1 and $\mathscr{Q}_1$ give

$$\lim_{n\to\infty} \|\mathscr{H}_1(u_n,v_n) - \mathscr{H}_1(u,v)\| = 0.$$

Step 2. $\mathscr{H}_1$ takes bounded sets into bounded sets.

Let $\Omega \subset X_+$ be bounded. Then there exist positive constants L_1, L_2 such that

$$\|(u,v)\|_1 \leqslant L_1, \quad 0 \leqslant \mathscr{L}_j(u,v)(t) \leqslant L_2, \quad t \in J,\ (u,v) \in \Omega, \quad j = 1,2,$$

and so

$$|\mathscr{K}_1(u,v)(t)| \leqslant L_2 \left(1 + \frac{\|a\| t^{\gamma-\beta}}{\Gamma(\gamma-\beta+1)}\right) \leqslant W, \quad t \in J,\ (u,v) \in \Omega,$$

where

$$W = L_2 \left(1 + \frac{\|a\| T^{\gamma-\beta}}{\Gamma(\gamma-\beta+1)}\right).$$

Hence (cf. (8))

$$\begin{aligned} |\mathscr{H}_1(u,v)(t)| &\leqslant \ell_1(u) + \|\mathscr{K}_1(u,v)\| E_\rho(MT^\rho) \frac{T^\alpha}{\Gamma(\alpha+1)} \\ &\leqslant \|\ell_1\| \|u\| + \frac{W E_\rho(MT^\rho) T^\alpha}{\Gamma(\alpha+1)} \\ &< L_1 + \frac{W E_\rho(MT^\rho) T^\alpha}{\Gamma(\alpha+1)}, \quad t \in J,\ (u,v) \in \Omega, \end{aligned}$$

and therefore $\mathscr{H}_1(\Omega) = \{\mathscr{H}_1(u,v) : (u,v) \in \Omega\}$ is bounded in P_+.

Step 3. $\mathscr{H}_1$ takes bounded sets into equicontinuous sets.

Let Ω, L and W be from Step 2. Let $(u,v) \in \Omega$ and $0 \leqslant t_1 < t_2 \leqslant T$. Then

$$\begin{aligned}|\mathscr{H}_1(u,v)(t_2) - \mathscr{H}_1(u,v)(t_1)| &= |I^\alpha \mathscr{Q}_1 \mathscr{K}_1(u,v)(t_2) - I^\alpha \mathscr{Q}_1 \mathscr{K}_1(u,v)(t_1)| \\ &\leqslant \int_0^{t_1} \frac{(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |\mathscr{Q}_1 \mathscr{K}_1(u,v)(s)| \, \mathrm{d}s \\ &\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |\mathscr{Q}_1 \mathscr{K}_1(u,v)(s)| \, \mathrm{d}s \\ &\leqslant W E_\rho(MT^\rho) \frac{t_1^\alpha + 2(t_2-t_1)^\alpha - t_2^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Since the function t^α is uniformly continuous on the interval J, we see that the family $\mathscr{H}_1(\Omega)$ is equicontinuous on J.

To summarize, we conclude from Steps 1–3 and the Arzelà–Ascoli theorem that $\mathscr{H}_1$ is completely continuous. $\square$

4. Existence results for problem (2), (3)

In the first part of this section, we prove the existence of positive solutions to problem (2), (3) (Theorems 1 and 2 below) by the well known Guo–Krasnoselskii fixed point theorem on cones [4] (Lemma 7 bellow).

LEMMA 7. *Let Y be a Banach space and $\mathscr{D} \subset Y$ be a cone in Y. Assume that Ω_1, Ω_2 are open subset of Y with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $\mathscr{S} : \mathscr{D} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathscr{D}$ be a completely continuous operator such that either*

$$\|\mathscr{S}u\| \geqslant \|u\|,\ u \in \mathscr{D} \cap \partial\Omega_1 \ \ \text{and} \ \ \|\mathscr{S}u\| \leqslant \|u\|,\ u \in \mathscr{D} \cap \partial\Omega_2$$

or

$$\|\mathscr{S}u\| \leqslant \|u\|,\ u \in \mathscr{D} \cap \partial\Omega_1 \ \ \text{and} \ \ \|\mathscr{S}u\| \geqslant \|u\|,\ u \in \mathscr{D} \cap \partial\Omega_2$$

Then $\mathscr{S}$ has a fixed point in $\mathscr{D} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

We are now in the position to state and prove the existence results for problem (2), (3).

THEOREM 1. *Let (H_2) and (H_3) hold. Then there exist $0 < r_1 < r_2$ and a positive solution $(x_1,x_2) \in X_+$ of problem (2), (3) such that $r_1 \leqslant \|(x_1,x_2)\|_1 \leqslant r_2$.*

Proof. We apply Lemma 7 for $Y = X$, $\mathscr{D} = X_+$, $\mathscr{S} = \mathscr{H}$ and $\Omega_i = \{(x_1,x_2) \in X : \|(x_1,x_2)\|_1 < r_i\}$, $i = 1,2$, where r_1, r_2 will be specified later.

It is clear that X_+ is a cone in X. By Lemmas 5 and 6, $\mathscr{H} : X_+ \to X_+$ is a completely continuous operator. The next part of the proof is divided into two steps.

Step 1. There exists $r_1 > 0$ such that $\|\mathscr{H}(x_1,x_2)\|_1 \geqslant \|(x_1,x_2)\|_1$ for $(x_1,x_2) \in X_+$ and $\|(x_1,x_2)\|_1 = r_1$.

In view of (5), we have $\mathscr{Q}_j x(t) \geqslant \Lambda_j^0 x(t) = x(t)$ for $x \in P_+$ and $j = 1,2$.

Suppose that (cf. (H_2)) $\mathscr{L}_1(x_1,x_2)(t) \geqslant \varepsilon$ for $t \in J$ and $(x_1,x_2) \in X_+$. Then (note that $\mathscr{L}_2(x_1,x_2) \geqslant 0$ because $\mathscr{L}_2 \colon X_+ \to P_+$)

$$\mathscr{K}_1(x_1,x_2)(t) \geqslant \mathscr{L}_1(x_1,x_2)(t) \geqslant \varepsilon, \ \ \mathscr{K}_2(x_1,x_2)(t) \geqslant 0,$$

and therefore for $t \in J$ and $(x_1,x_2) \in X_+$

$$I^\alpha \mathscr{Q}_1 \mathscr{K}_1(x_1,x_2)(t) \geqslant I^\alpha \mathscr{K}_1(x_1,x_2)(t) \geqslant \frac{\varepsilon t^\alpha}{\Gamma(\alpha+1)}, \ \ I^\gamma \mathscr{Q}_2 \mathscr{K}_2(x_1,x_2)(t) \geqslant 0.$$

Hence (note that $\ell_j(x_j) \geqslant 0$ for $x_j \in P_+$ since $\ell_j \in \mathscr{A}$)

$$\mathscr{H}_1(x_1,x_2)(t) \geqslant I^\alpha \mathscr{Q}_1 \mathscr{K}_1(x_1,x_2)(t) \geqslant \frac{\varepsilon t^\alpha}{\Gamma(\alpha+1)},$$
$$\mathscr{H}_2(x_1,x_2)(t) \geqslant I^\gamma \mathscr{Q}_2 \mathscr{K}_2(x_1,x_2)(t) \geqslant 0$$

for $t \in J$ and $(x_1,x_2) \in X_+$. Consequently,

$$\|\mathscr{H}(x_1,x_2)\|_1 = \|\mathscr{H}_1(x_1,x_2)\| + \|\mathscr{H}_2(x_1,x_2)\| \geqslant \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)}, \quad (x_1,x_2) \in X_+.$$

Let $c_1 = \varepsilon T^\alpha / \Gamma(\alpha+1)$. Then

$$\|\mathscr{H}(x_1,x_2)\|_1 \geqslant c_1, \quad (x_1,x_2) \in X_+.$$

If $\mathscr{L}_2(x_1,x_2)(t) \geqslant \varepsilon$ for $t \in J$ and $(x_1,x_2) \in X_+$, we have in an analogous way that

$$\mathscr{H}_1(x_1,x_2)(t) \geqslant 0, \ \ \mathscr{H}_2(x_1,x_2)(t) \geqslant \frac{\varepsilon t^\gamma}{\Gamma(\gamma+1)}$$

and

$$\|\mathscr{H}(x_1,x_2)\|_1 \geqslant c_2, \quad (x_1,x_2) \in X_+,$$

where $c_2 = \varepsilon T^\gamma / \Gamma(\gamma+1)$. Let $r_1 = \min\{c_1, c_2\}$. We have proved that

$$\|\mathscr{H}(x_1,x_2)\|_1 \geqslant r_1, \quad \text{for } (x_1,x_2) \in X_+.$$

In particular,

$$\|\mathscr{H}(x_1,x_2)\|_1 \geqslant \|(x_1,x_2)\|_1 \ \text{ for } (x_1,x_2) \in X_+, \ \|(x_1,x_2)\|_1 = r_1.$$

Step 2. There exists $r_2 > r_1$ such that $\|\mathscr{H}(x_1,x_2)\|_1 \leqslant \|(x_1,x_2)\|_1$ for $(x_1,x_2) \in X_+$ and $\|(x_1,x_2)\|_1 = r_2$.

In view of (H_3), we have

$$0 \leqslant \mathscr{K}_1(x_1,x_2)(t) \leqslant \left(1 + \frac{\|a\| T^{\gamma-\beta}}{\Gamma(\gamma-\beta+1)}\right) w(\|(x_1,x_2)\|_1),$$

$$0 \leqslant \mathscr{K}_2(x_1,x_2)(t) \leqslant \left(1 + \frac{\|b\| T^{\alpha-\mu}}{\Gamma(\alpha-\mu+1)}\right) w(\|(x_1,x_2)\|_1),$$

which together with (8) give

$$0 \leqslant \mathscr{Q}_1 \mathscr{K}_1(x_1,x_2)(t) \leqslant E_\rho\left(MT^\rho\right) \left(1 + \frac{\|a\| T^{\gamma-\beta}}{\Gamma(\gamma-\beta+1)}\right) w(\|(x_1,x_2)\|_1),$$

$$0 \leqslant \mathscr{Q}_2 \mathscr{K}_2(x_1,x_2)(t) \leqslant E_\rho\left(MT^\rho\right) \left(1 + \frac{\|b\| T^{\alpha-\mu}}{\Gamma(\alpha-\mu+1)}\right) w(\|(x_1,x_2)\|_1).$$

Therefore

$$0 \leqslant I^{\alpha}\mathscr{Q}_1\mathscr{K}_1(x_1,x_2)(t) \leqslant R_1 w(\|(x_1,x_2)\|_1),$$
$$0 \leqslant I^{\gamma}\mathscr{Q}_2\mathscr{K}_2(x_1,x_2)(t) \leqslant R_2 w(\|(x_1,x_2)\|_1),$$

where

$$R_1 = \frac{T^{\alpha}E_{\rho}(MT^{\rho})}{\Gamma(\alpha+1)}\left(1+\frac{\|a\|T^{\gamma-\beta}}{\Gamma(\gamma-\beta+1)}\right),$$
$$R_2 = \frac{T^{\gamma}E_{\rho}(MT^{\rho})}{\Gamma(\gamma+1)}\left(1+\frac{\|b\|T^{\alpha-\mu}}{\Gamma(\alpha-\mu+1)}\right).$$

Let $d = \max\{\|\ell_1\|,\|\ell_2\|\}$, $R = \max\{R_1,R_2\}$. Then $d < 1$ and

$$\|\mathscr{H}_1(x_1,x_2)\| \leqslant \|\ell_1\|\|x_1\| + R_1 w(\|(x_1,x_2)\|_1) \leqslant d\|x_1\| + Rw(\|(x_1,x_2)\|_1),$$
$$\|\mathscr{H}_2(x_1,x_2)\| \leqslant \|\ell_2\|\|x_2\| + R_2 w(\|(x_1,x_2)\|_1) \leqslant d\|x_2\| + Rw(\|(x_1,x_2)\|_1).$$

Hence

$$\|\mathscr{H}(x_1,x_2)\|_1 \leqslant d\|(x_1,x_2)\|_1 + 2Rw(\|(x_1,x_2)\|_1), \quad (x_1,x_2) \in X_+. \tag{18}$$

Since, by (H_3), $\lim_{x\to\infty} w(x)/x = 0$, we have $\lim_{x\to\infty}(d+2Rw(x)/x) = d < 1$, and therefore there exists $r_2 > r_1$ such that $d + 2Rw(x)/x < 1$ for all $x \geqslant r_2$. In particular, $dx + 2Rw(x) < x$ for $x = r_2$, that is,

$$d\|(x_1,x_2)\|_1 + 2Rw(\|(x_1,x_2)\|_1) < \|(x_1,x_2)\|_1 \ \text{ for } \|(x_1,x_2)\|_1 = r_2. \tag{19}$$

Combining (18) with (19) yields

$$\|\mathscr{H}(x_1,x_2)\|_1 < \|(x_1,x_2)\|_1 \ \text{ for } (x_1,x_2) \in X_+,\ \|(x_1,x_2)\|_1 = r_2.$$

To summarize, we conclude from Steps 1 and 2 that for the sets $\Omega_i = \{(x_1,x_2) \in X_+ \colon \|(x_1,x_2)\|_1 < r_i\}$ the inequalities

$$\|\mathscr{H}(x_1,x_2)\|_1 \geqslant \|(x_1,x_2)\|_1 \ \text{ for } (x_1,x_2) \in X_+ \cap \partial\Omega_1,$$
$$\|\mathscr{H}(x_1,x_2)\|_1 < \|(x_1,x_2)\|_1 \ \text{ for } (x_1,x_2) \in X_+ \cap \partial\Omega_2,$$

hold. Hence there exists at least one fixed point (x_1,x_2) of $\mathscr{H}$ in $X_+ \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$. Therefore $r_1 \leqslant \|(x_1,x_2)\|_1 \leqslant r_2$ and Lemma 5 gives that (x_1,x_2) is a positive solution of problem (2), (3). □

EXAMPLE 1. Let $r_j \colon J \times \mathbb{R}_+^2 \to \mathbb{R}_+$ be continuous and bounded, $j = 1,2$, $r_1 \geqslant \varepsilon > 0$ on $J \times \mathbb{R}_+^2$, $\varphi, \psi \colon J \to J$ be continuous, $\nu \in (0,1/2)$ and $\tau_i \in (0,1)$, $i = 1,2,3$. Let $\mathscr{L}_j \colon X_+ \to P_+$,

$$\mathscr{L}_1(x_1,x_2)(t) = r_1(t,x_1(t),x_2(t)) + \|x_1x_2\|^{\nu} + (x_1(\varphi(t)))^{\tau_1},$$
$$\mathscr{L}_2(x_1,x_2)(t) = \int_0^{\psi(t)} r_2(s,x_1(s),x_2(s))\Big((x_1(s))^{\tau_2} + (x_2(s))^{\tau_3}\Big)\,\mathrm{d}s.$$

Then $\mathscr{L}_j$ is continuous and satisfies (H_2) and (H_3) for $\omega(x) = K + 2(1+KT) + 2(1+KT)x^{\eta}$, where $K = \sup\{r_j(t,x_1,x_2) \colon (t,x_1,x_2) \in J \times \mathbb{R}_+^2,\ j = 1,2\}$ and $\eta =$

$\max\{2\nu,\tau_1,\tau_2,\tau_3\}$. Hence, by Theorem 1, there exists at least one positive solution (x_1,x_2) of the system

$$\begin{aligned} {}^c\mathcal{D}^{\alpha}x_1(t)+a(t)\,{}^c\mathcal{D}^{\beta}x_2(t) &= r_1(t,x_1(t),x_2(t))+\|x_1x_2\|^{\nu}+(x_1(\varphi(t)))^{\tau_1}, \\ {}^c\mathcal{D}^{\gamma}x_2(t)+b(t)\,{}^c\mathcal{D}^{\mu}x_1(t) &= \int_0^{\psi(t)} r_2(s,x_1(s),x_2(s))\Big((x_1(s))^{\tau_2}+(x_2(s))^{\tau_3}\Big)\mathrm{d}s, \end{aligned} \tag{20}$$

satisfying the boundary condition (3).

For the solvability of problem (4), (3) we have the following result.

COROLLARY 2. *Let* $f_j \in C(J\times\mathbb{R}_+^2)$ *and* $f_j \geqslant 0$, $j=1,2$. *Let*

(P_1) *there exists* $\varepsilon>0$ *such that either*

$$f_1(t,x_1,x_2)\geqslant \varepsilon \ \textit{for}\ t\in J,\ x_1,x_2\in\mathbb{R}_+$$

or

$$f_2(t,x_1,x_2)\geqslant \varepsilon \ \textit{for}\ t\in J,\ x_1,x_2\in\mathbb{R}_+,$$

(P_2) *there exists a nondecreasing* $w\in C(\mathbb{R}_+)$ *such that* $\lim_{x\to\infty} w(x)/x=0$ *and*

$$f_j(t,x_1,x_2)\leqslant w(x_1+x_2)\ \textit{for}\ t\in J,\ x_1,x_2\in\mathbb{R}_+,\ j=1,2.$$

Then there exists at least one positive solution (x_1,x_2) *of problem* (4), (3).

Proof. Let $\mathscr{L}_j(x_1,x_2)(t)=f_j(t,x_1(t),x_2(t))$ for $t\in J$, $(x_1,x_2)\in X_+$, $j=1,2$. Then system (4) can be written as (2). Conditions (P_1) and (P_2) guarantee that $\mathscr{L}_j$ satisfies (H_2), (H_3). The existence result for problem (4), (3) now follows from Theorem 1. □

EXAMPLE 2. Let $\tau_i\in(0,1)$, $i=1,2,3$. Then $f_1(t,x_1,x_2)=e^t+x_2^{\tau_1}\ln\left(1+x_1^2\right)$, $f_2(t,x_1,x_2)=|\sin t|+x_1^{\tau_2}+x_2^{\tau_3}$ satisfy conditions (P_1) and (P_2) for $\varepsilon=1$ and $w(x)=e^T+x^{\tau_1}\ln\left(1+x^2\right)+x^{\tau_2}+x^{\tau_3}$. Hence Corollary 2 guarantees that the system

$$\begin{aligned} {}^c\mathcal{D}^{\alpha}x_1+a(t)\,{}^c\mathcal{D}^{\beta}x_2 &= e^t+x_2^{\tau_1}\ln\left(1+x_1^2\right), \\ {}^c\mathcal{D}^{\gamma}x_2+b(t)\,{}^c\mathcal{D}^{\mu}x_1 &= |\sin t|+x_1^{\tau_2}+x_2^{\tau_3}, \end{aligned}$$

has at least one positive solution (x_1,x_2) satisfying the boundary condition (3).

THEOREM 2. *Let* (H_1), (H_4) *and* (H_5) *hold. Then there exist* $0<r_1<r_2$ *and a positive solution* $(x_1,x_2)\in X_+$ *of problem* (2), (3) *such that* $r_1\leqslant\|(x_1,x_2)\|_1\leqslant r_2$.

Proof. As in the proof of Theorem 1, we apply Lemma 7.

Step 1. There exists $r_1>0$ such that $\|\mathscr{H}(x_1,x_2)\|_1\leqslant\|(x_1,x_2)\|_1$ for $(x_1,x_2)\in X_+$ and $\|(x_1,x_2)\|_1=r_1$.

Let

$$R_1 = 1 + \frac{\|a\| T^{\gamma-\beta}}{\Gamma(\gamma-\beta+1)}, \quad R_2 = 1 + \frac{\|b\| T^{\alpha-\mu}}{\Gamma(\alpha-\mu+1)}$$
$$K_1 = \frac{R_1 E_\rho(MT^\rho) T^\alpha}{\Gamma(\alpha+1)}, \quad K_2 = \frac{R_2 E_\rho(MT^\rho) T^\gamma}{\Gamma(\gamma+1)},$$
$$S = \frac{1-d}{K_1+K_2}, \quad d = \max\{\|\ell_1\|, \|\ell_2\|\},$$

where ℓ_1, ℓ_2 are from (3). In view of (H_4), there exists $r_1 > 0$ such that

$$\|\mathscr{L}_j(x_1,x_2)\| \leqslant S\|(x_1,x_2)\|_1 \text{ for } (x_1,x_2) \in X_+,\ \|(x_1,x_2)\|_1 \leqslant r_1,\ j = 1,2.$$

Hence for these (x_1,x_2) and j we have

$$\|\mathscr{K}_j(x_1,x_2)\| \leqslant SR_j\|(x_1,x_2)\|_1$$

and (cf. (8))

$$\|\mathscr{Q}_j\mathscr{K}_j(x_1,x_2)\| \leqslant SR_jE_\rho(MT^\rho)\|(x_1,x_2)\|_1.$$

Therefore for $(x_1,x_2) \in X_+$, $\|(x_1,x_2)\| \leqslant r_1$ and $j = 1,2$,

$$\|\mathscr{H}_j(x_1,x_2)\| \leqslant \|\ell_j\|\|x_j\| + SK_j\|(x_1,x_2)\|_1.$$

Consequently, for $(x_1,x_2) \in X_+$, $\|(x_1,x_2)\| \leqslant r_1$,

$$\|\mathscr{H}(x_1,x_2)\|_1 \leqslant (d + S(K_1+K_2))\,\|(x_1,x_2)\|_1 = \|(x_1,x_2)\|_1.$$

In particular,

$$\|\mathscr{H}(x_1,x_2)\|_1 \leqslant \|(x_1,x_2)\|_1, \quad (x_1,x_2) \in X_+,\ \|(x_1,x_2)\|_1 = r_1.$$

Step 2. There exists $r_2 > r_1$ such that $\|\mathscr{H}(x_1,x_2)\|_1 \geqslant \|(x_1,x_2)\|_1$ for $(x_1,x_2) \in X_+$ and $\|(x_1,x_2)\|_1 = r_2$.

Let (cf. (H_5))

$$\lim_{(x_1,x_2)\in X_+,\,\|(x_1,x_2)\|_1\to\infty} \frac{\|\mathscr{L}_1(x_1,x_2)\|}{\|(x_1,x_2)\|_1} > \frac{\Gamma(\alpha+1)}{T^\alpha}.$$

Then there exists $p_1 > r_1$ such that

$$\|\mathscr{L}_1(x_1,x_2)\| \geqslant \frac{\Gamma(\alpha+1)}{T^\alpha}\|(x_1,x_2)\|_1 \text{ for } (x_1,x_2) \in X_+,\ \|(x_1,x_2)\|_1 \geqslant p_1.$$

Hence for these (x_1,x_2) the estimates

$$\|\mathscr{K}_1(x_1,x_2)\| \geqslant \frac{\Gamma(\alpha+1)}{T^\alpha}\|(x_1,x_2)\|_1,$$
$$\mathscr{H}_1(x_1,x_2)(t) \geqslant I^\alpha\mathscr{Q}_1\mathscr{K}_1(x_1,x_2)(t), \quad t \in J,$$

hold. Since $\mathscr{Q}_1x(t) \geqslant x(t)$ for $x \in P_+$, we have

$$\mathscr{H}_1(x_1,x_2)(t) \geqslant I^\alpha\mathscr{K}_1(x_1,x_2)(t) \geqslant \frac{\Gamma(\alpha+1)t^\alpha}{T^\alpha\Gamma(\alpha+1)}\|(x_1,x_2)\|_1, \quad t \in J,$$

and so

$$\|\mathscr{H}_1(x_1,x_2)\| \geqslant \|(x_1,x_2\|_1, \quad (x_1,x_2) \in X_+,\ \|(x_1,x_2)\|_1 \geqslant p_1.$$

Consequently,

$$\|\mathscr{H}(x_1,x_2)\|_1 \geqslant \|\mathscr{H}_1(x_1,x_2)\| \geqslant \|(x_1,x_2)\|_1, \ \ (x_1,x_2)\in X_+, \ \|(x_1,x_2)\|_1 \geqslant p_1. \quad (21)$$

If

$$\lim_{(x_1,x_1)\in X_+,\, \|(x_1,x_2)\|_1\to\infty} \frac{\|\mathscr{L}_2(x_1,x_2)\|}{\|(x_1,x_2)\|_1} > \frac{\Gamma(\gamma+1)}{T^\gamma},$$

then there exists $p_2 > r_1$ such that

$$\|\mathscr{L}_2(x_1,x_2)\| \geqslant \frac{\Gamma(\gamma+1)}{T^\gamma}\|(x_1,x_2)\|_1 \ \text{ for } (x_1,x_2)\in X_+, \ \|(x_1,x_2)\|_1 \geqslant p_2.$$

We can now proceed analogously to the above part of this step and have

$$\|\mathscr{H}_2(x_1,x_2)\| \geqslant \|(x_1,x_2)\|_1 \ \text{ for } (x_1,x_2)\in X_+, \ \|(x_1,x_2)\|_1 \geqslant p_2. \quad (22)$$

Let $r_2 = \max\{p_1,p_2\}$. Then it follows from (21) and (22) that

$$\|\mathscr{H}(x_1,x_2)\|_1 \geqslant \|(x_1,x_2)\|_1 \ \text{ for } (x_1,x_2)\in X_+, \ \|(x_1,x_2)\|_1 = r_2.$$

As a result, we conclude from Steps 1 and 2 and Lemma 7 that $\mathscr{H}(x_1,x_2) = (x_1,x_2)$ for some $(x_1,x_2)\in X_+$, $r_1 \leqslant \|(x_1,x_2)\|_1 \leqslant r_2$. This fact together with Lemma 5 give that (x_1,x_2) is a positive solution of problem (2), (3). □

EXAMPLE 3. Let $r_1, r_2 \in C(J)$, $r_1 > 0$, $r_2 \geqslant 0$ on J, $\varphi\colon J\to J$ be continuous and $\tau_i \in (1,\infty)$, $i = 1,2,3,4$. Let $\mathscr{L}_j\colon X_+ \to P_+$,

$$\mathscr{L}_1(x_1,x_2)(t) = r_1(t)\,(\|x_1\|^{\tau_1} + \|x_2\|^{\tau_2}),$$

$$\mathscr{L}_2(x_1,x_2)(t) = \int_0^t r_2(s)\,(x_1(s))^{\tau_3}\,\mathrm{d}s + (x_2(\varphi(t)))^{\tau_4}.$$

Then $\mathscr{L}_j$ is continuous, satisfies condition (H_1) and it follows from the estimates

$$\mathscr{L}_1(x_1,x_2)(t) \leqslant \|r_1\|\,(\|(x_1,x_2)\|^{\tau_1} + \|(x_1,x_2)\|^{\tau_2})$$
$$\mathscr{L}_2(x_1,x_2)(t) \leqslant \|r_2\|T\|(x_1,x_2)\|^{\tau_3} + \|(x_1,x_2)\|^{\tau_4},$$

that $\mathscr{L}_j$ fulfils condition (H_4).

If $\|x_1\| \geqslant \|x_2\|$, then

$$\|x_1\|^{\tau_1} = \left(\frac{\|x_1\|}{2} + \frac{\|x_1\|}{2}\right)^{\tau_1} \geqslant \left(\frac{\|x_1\|}{2} + \frac{\|x_2\|}{2}\right)^{\tau_1} = \left(\frac{\|(x_1,x_2)\|_1}{2}\right)^{\tau_1}.$$

Similarly, if $\|x_2\| \geqslant \|x_1\|$, then $\|x_1\|^{\tau_2} \geqslant (\|(x_1,x_2)\|_1/2)^{\tau_2}$. Hence (for $(x_1,x_2)\in X_+$, $\|(x_1,x_2)\|_1 \geqslant 1$)

$$\|\mathscr{L}_1(x_1,x_2)\| \geqslant \min\{r_1(t)\colon t\in J\}\min\left\{\frac{1}{2^{\tau_1}}, \frac{1}{2^{\tau_2}}\right\}\|(x_1,x_2)\|_1^\eta,$$

where $\eta = \min\{\tau_1,\tau_2\}$, and therefore $\mathscr{L}_1$ satisfies condition (H_5). By Theorem 2, there exists at least one positive solution of the system

$$\begin{aligned} {}^c\mathcal{D}^\alpha x_1(t) + a(t)\,{}^c\mathcal{D}^\beta x_2(t) &= r_1(t)\,(\|x_1\|^{\tau_1} + \|x_2\|^{\tau_2}),\\ {}^c\mathcal{D}^\gamma x_2(t) + b(t)\,{}^c\mathcal{D}^\mu x_1(t) &= \int_0^t r_2(s)\,(x_1(s))^{\tau_3}\,\mathrm{d}s + (x_2(\varphi(t)))^{\tau_4}, \end{aligned} \quad (23)$$

satisfying the boundary condition (3). We note that problem (23), (3) has also the trivial solution $(x_1,x_2) = (0,0)$.

In the second part of this section, the existence of positive solutions (Theorem 3 below) is proved by the following nonlinear Leray–Schauder alternative [2, Corollary 8.1].

LEMMA 8. *Let Y be a Banach space and let $\mathscr{S}: Y \to Y$ be a completely continuous operator. Then the following alternative holds: Either the equation $x = \lambda \mathscr{S} x$ has a solution for every $\lambda \in [0,1]$ or the set $\{x \in Y : x = \lambda \mathscr{S} x \text{ for some } \lambda \in (0,1)\}$ is unbounded.*

THEOREM 3. *Let (H_3) hold and let*

$$\mathscr{L}_1(0,0)(t_0) + \mathscr{L}_2(0,0)(t_0) > 0 \text{ for some } t_0 \in J. \tag{24}$$

Then problem (2), (3) has at least one positive solution.

Proof. Let $\mathscr{L}_j^*, \mathscr{K}_j^* : X \to P_+$ and $\mathscr{H}_j^* : X \to C(J)$, $j = 1,2$, be defined by the formulas

$$\mathscr{L}_j^*(x_1,x_2)(t) = \mathscr{L}_j(|x_1|,|x_2|)(t), \quad j = 1,2,$$

$$\begin{aligned}
\mathscr{K}_1^*(x_1,x_2)(t) &= \mathscr{L}_1^*(x_1,x_1)(t) - a(t) I^{\gamma-\beta} \mathscr{L}_2^*(x_1,x_2)(t),\\
\mathscr{K}_2^*(x_1,x_2)(t) &= \mathscr{L}_2^*(x_1,x_1)(t) - b(t) I^{\alpha-\mu} \mathscr{L}_1^*(x_1,x_2)(t),\\
\mathscr{H}_1^*(x_1,x_2)(t) &= \ell_1(x_1) + I^{\alpha} \mathscr{Q}_1 \mathscr{K}_1^*(x_1,x_2)(t),\\
\mathscr{H}_2^*(x_1,x_2)(t) &= \ell_2(x_2) + I^{\gamma} \mathscr{Q}_2 \mathscr{K}_2^*(x_1,x_2)(t).
\end{aligned}$$

Finally, let $\mathscr{H}^* : X \to X$,

$$\mathscr{H}^*(x_1,x_2) = \left(\mathscr{H}_1^*(x_1,x_2), \mathscr{H}_1^*(x_1,x_2)\right).$$

We can proceed analogously to the proof of Lemma 6 and show that $\mathscr{H}^*$ is a completely continuous operator.

Suppose that $(x_1,x_2) = \lambda \mathscr{H}^*(x_1,x_2)$ for some $(x_1,x_2) \in X$ and $\lambda \in (0,1]$. Then

$$\begin{aligned}
x_1(t) &= \lambda \Big(\ell_1(x_1) + I^{\alpha} \mathscr{Q}_1 \mathscr{K}_1^*(x_1,x_2)(t)\Big),\\
x_2(t) &= \lambda \Big(\ell_2(x_1) + I^{\gamma} \mathscr{Q}_2 \mathscr{K}_2^*(x_1,x_2)(t)\Big).
\end{aligned}$$

In view of $I^{\alpha} \mathscr{Q}_1 \mathscr{K}_1^*(x_1,x_2), I^{\gamma} \mathscr{Q}_2 \mathscr{K}_2^*(x_1,x_2) \in P_+$, we have $x_j(t) \geqslant \lambda \ell_j(x_j)$, $j = 1,2$. Applying ℓ_j to both sides of the last relation we get $\ell_j(x_j)(1 - \lambda \ell_j(1)) \geqslant 0$ and since $\lambda \ell_j(1) \in (0,1)$, $\ell_j(x_j) \geqslant 0$. Consequently, $x_j \geqslant 0$ on J and $\mathscr{H}^*(x_1,x_2) = \mathscr{H}(x_1,x_2)$. We now argue as in Step 2 of the proof of Theorem 1 and show that condition (H_3) guarantees the estimate

$$\|\mathscr{H}(y_1,y_2)\|_1 \leqslant (d + R_1) w(\|(y_1,y_2)\|_1), \quad (y_1,y_2) \in X_+,$$

where $d = \max\{\|\ell_1\|, \|\ell_2\|\}$ and R_1 is a positive constant. Hence

$$\|(x_1,x_2)\|_1 \leqslant \|\mathscr{H}^*(x_1,x_2)\|_1 = \|\mathscr{H}(x_1,x_2)\|_1 \leqslant (d + R_1) w(\|(x_1,x_2)\|_1). \tag{25}$$

Since $\lim_{x\to\infty} w(x)/x = 0$, we have $\lim_{x\to\infty} (d + R_1) w(x)/x = 0$, and therefore there exists $r > 0$ such that $(d + R_1) w(x) < x$ for all $x \geqslant r$. The last relation together with (25) give $\|(x_1,x_2)\|_1 < r$.

We have proved that every fixed point (x_1,x_2) of the operator $\lambda\mathscr{H}^*$, $\lambda\in(0,1]$, belongs to the set X_+ and that the set $\{(x_1,x_2)\in X\colon (x_1,x_2)=\lambda\mathscr{H}^*(x_1,x_2)\}$ is bounded in $C(J)$. By Lemma 8 (for $Y=X$ and $\mathscr{S}=\mathscr{H}^*$), there exists a fixed point (x_1,x_2) of $\mathscr{H}^*$. Hence $(x_1,x_2)\in X_+$, and so (x_1,x_2) is a fixed point of $\mathscr{H}$. In view of Lemma 5, (x_1,x_2) is a solution of problem (2), (3). It remains to prove that (x_1,x_2) is a positive solution of this problem. Suppose that $(x_1,x_2)=(0,0)$. Then $I^{\alpha}\mathscr{Q}_1\mathscr{K}_1(0,0)(t)+I^{\gamma}\mathscr{Q}_2\mathscr{K}_2(0,0)(t)=0$ for $t\in J$. Hence $\mathscr{K}_1(0,0)(t)+\mathscr{K}_2(0,0)(t)=0$ on J, which contradicts $\mathscr{K}_1(0,0)(t)+\mathscr{K}_2(0,0)(t)\geqslant\mathscr{L}_1(0,0)(t)+\mathscr{L}_2(0,0)(t)$ for $t\in J$ and (24). $\square$

EXAMPLE 4. Let $\varphi,\psi,\nu,\tau_i,\mathscr{L}_j$ be as in Example 1, where $r_j\colon J\times\mathbb{R}_+^2\to\mathbb{R}_+$ is continuous and bounded, $j=1,2$, and $r_1(t_0,0,0)>0$ for some $t_0\in J$. Then $\mathscr{L}_j\colon X_+\to P_+$ is continuous, satisfies condition (H_3) and $\mathscr{L}_1(0,0)(t_0)+\mathscr{L}_2(0,0)(t_0)=\mathscr{L}_1(0,0)(t_0)>0$. By Theorem 3 there exists at least one positive solution (x_1,x_2) of system (20) satisfying the boundary condition (3).

Acknowledgement. Supported by the grant No. 14-06958S of the Grant Agency of the Czech Republic.

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(Received October 7, 2016)

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