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FORCED OSCILLATION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH DAMPING TERM

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Abstract. In this paper, the forced oscillation of fractional partial differential equations of the form

$$D_{+,t}^{1+\alpha}u(x,t)+p(t)D_{+,t}^{\alpha}u(x,t) = a(t)\Delta u(x,t)+\sum_{i=1}^{m}a_i(t)\Delta u(x,t-\tau_i)$$
$$-q(x,t)\int_0^t(t-\xi)^{-\alpha}u(x,\xi)d\xi+f(x,t),\ (x,t)\in\Omega\times R_+\equiv G$$

are investigated, and some examples are given to illustrate the useful of our results.

1. Introduction

Recently, the fractional differential equations have attracted much attention due to their applications being used widely in various areas of science, engineering or bioengineering, mechanics, finance, nonlinear control and so on. Based a series of fundamental theory [2–6], there are a lot of results in some aspects of fractional differential equation, such as the existence, uniqueness, boundedness, stability or oscillation of the solutions (see [5, 6, 16–19] and the references there in).

Recently, the research on the theory of fractional partial differential equations is becoming a hot topic, and some results are established, see [7–15]. In 2013, Jafari et al. [7] derived the exact and approximate analytical solutions of fractional partial differential equations by the method of iterative Laplace transform. In 2015, Chen and Jiang et al. [8] presented the techniques of Lie group analysis to solve n-order linear partial fractional differential equations.

At the same time, the oscillation of fractional partial differential equations has been studied. In [9, 10], Prakash et al. presented the oscillation of fractional partial differential equation

$$\frac{\partial}{\partial t}(r(t)D_{+,t}^{\alpha}u(x,t))+q(x,t)f(\int_0^t(t-\nu)^{-\alpha}u(x,\nu)d\nu)=a(t)\Delta u(x.t), \tag{1.1}$$

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and the forced oscillation of a nonlinear fractional partial differential equation with damping term of the form

$$D_{+,t}^{\alpha}(r(t)D_{+,t}^{\alpha}u(x,t))+p(t)D_{+,t}^{\alpha}u(x,t)+q(x,t)f(u(x,t))$$
$$=a(t)\Delta u(x,t)+g(x,t),\ \ (x,t)\in G. \tag{1.2}$$

In [11], Harikrishnan et al. studied the oscillatory behavior of fractional partial differential equation of the form

$$D_{+,t}^{\alpha}(r(t)D_{+,t}^{\alpha}u(x,t))+q(x,t)f(u(x,t))=a(t)\Delta u(x,t)+g(x,t),\ \ (x,t)\in G. \tag{1.3}$$

Li et al. and Sheng [12,13] established the oscillation of fractional partial differential equations of the form

$$\frac{\partial}{\partial t}(D_{+,t}^{\alpha}u(x,t))+p(t)D_{+,t}^{\alpha}u(x,t)=a(t)\Delta u(x,t)-q(x,t)u(x,t)+f(x.t),\ \ (x,t)\in G, \tag{1.4}$$

and

$$D_{+,t}^{1+\alpha}u(x,t)+p(t)D_{+,t}^{\alpha}u(x,t)$$
$$=a(t)\Delta u(x,t)+\sum_{i=1}^{m}a_i(t)\Delta u(x,t-\tau_i)-q(x,t)\int_0^t(t-\xi)^{-\alpha}u(x,\xi)d\xi \tag{1.5}$$

respectively. In this paper, We will study the oscillation for the fractional partial differential equation

$$D_{+,t}^{1+\alpha}u(x,t)+p(t)D_{+,t}^{\alpha}u(x,t)=a(t)\Delta u(x,t)+\sum_{i=1}^{m}a_i(t)\Delta u(x,t-\tau_i)$$
$$-q(x,t)\int_0^t(t-\xi)^{-\alpha}u(x,\xi)d\xi+f(x,t),$$
$$(x,t)\in\Omega\times R_+\equiv G, \tag{1.6}$$

with the boundary condition

$$\frac{\partial u(x,t)}{\partial N}+g(x,t)u(x,t)=0,(x,t)\in\partial\Omega\times R_+, \tag{1.7}$$

or

$$u(x,t)=0,(x,t)\in\partial\Omega\times R_+. \tag{1.8}$$

where Ω is a bounded domain in R^n with piecewise smooth boundary $\partial\Omega,\alpha\in(0,1)$ is a constant, $R_+=(0,\infty)$. $D_{+,t}^{\alpha}u(x,t)$ is the Riemann-Liouville fractional derivative of order α of u with respect to t, Δ is the Laplacian in R^n. N is the unit exterior normal vector to $\partial\Omega$, and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega\times R_+$.

By a solution of (1.6), (1.7) (or (1.6), (1.8)) we mean a function $u(x,t)$ satisfies (1.6) on $\overline{G}$ and the boundary condition (1.7) (or (1.8)).

We assume throughout this paper that

(A_1) $a\in C(R_+;R_+),p\in C(R_+;R),a_i\in C(R_+;R_+)$ and $\tau_i\geqslant 0$ are constants, $i=1,2,3,\ldots,m$;

(A_2) $q\in C(\overline{G};R_+)$ and $q(t)=\min_{x\in\overline{\Omega}}q(x,t)$;

(A_3) $f\in C(\overline{G};R)$.

2. Preliminaries and Lemmas

In this part, we will present some useful preliminaries and lemmas, which will be used in the following proof for our results.

DEFINITION 1. (see [3]) The Riemann-Liouville fractional integral I^{α}_{a+} of order $\alpha > 0$ of a function $y\colon R_+ \to R$ on the half-axis R_+ is given by

$$(I^{\alpha}_{a+}y)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)ds, \quad t > a, \tag{2.1}$$

provided that the right-hand side is pointwise defined on R_+, where Γ is the gamma function.

DEFINITION 2. (see [3]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y\colon R_+ \to R$ on the half-axis R_+ is given by

$$\begin{aligned}(D^{\alpha}_{+}y)(t) &:= \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} (I^{\lceil\alpha\rceil-\alpha}_{+} y)(t) \\ &= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \int_0^t (t-s)^{\lceil\alpha\rceil-\alpha-1} y(s)ds, \quad t > 0, \end{aligned} \tag{2.2}$$

provided that the right-hand side is pointwise defined on R_+, where $\lceil\alpha\rceil$ is the ceiling function of α.

DEFINITION 3. (see [3]) The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is given by

$$D^{\alpha}_{+,t} u(x,t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(x,s)ds, \tag{2.3}$$

provided the right-hand side is pointwise defined on R_+.

There are some lemmas which are very useful in the proof of our main results.

LEMMA 2.4. (see [3]) *Let $\alpha \geqslant 0$, $m \in N$ and $D = \frac{d}{dt}$. If the fractional derivatives $D^{\alpha}_{+}y(t)$ and $D^{\alpha+m}_{+}y(t)$ exist, then*

$$D^m(D^{\alpha}_{+}y(t)) = D^{\alpha+m}_{+}y(t). \tag{2.4}$$

LEMMA 2.5. (see [9]) *Let*

$$\widetilde{E}(t) =: \int_0^t (t-s)^{-\alpha} y(s)ds \quad for \quad \alpha \in (0,1) \quad and \quad t \geqslant 0, \tag{2.5}$$

Then $\widetilde{E}'(t) = \Gamma(1-\alpha) D^{\alpha}_{+}y(t)$.

LEMMA 2.6. (see [3]) *Let $\alpha \in (0,1)$ and $I^{1-\alpha}_{a+}y(t)$ is the fractional integral of order $1-\alpha$, then*

$$(I^{\alpha}_{a+}D^{\alpha}_{a+}y(t)) = y(t) - \frac{I^{1-\alpha}_{a+}y(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1}. \tag{2.6}$$

LEMMA 2.7. (see [1]) *The smallest eigenvalue* β_0 *of the Dirichlet problem*

$$\begin{aligned} \Delta\omega(x)+\beta\omega(x)=0 \quad &in \quad \Omega, \\ \omega(x)=0 \quad &on \quad \partial\Omega, \end{aligned} \tag{2.7}$$

is positive and the corresponding eigenfunction $\varphi(x)$ *is positive in* Ω.

3. The oscillation of the problem (1,6), (1.7)

THEOREM 3.1. *Suppose that* (A_1)–(A_3) *hold,* $\lim_{t\to\infty} I_+^{1-\alpha}U(t_0)=C_0$. *If*

$$\liminf_{t\to\infty}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}F(s)V(s)ds\right)d\xi<0, \tag{3.1}$$

and

$$\limsup_{t\to\infty}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}F(s)V(s)ds\right)d\xi>0, \tag{3.2}$$

where C_0, C *are two constants,* $V(t)=e^{\int_{t_0}^t p(s)ds}$, $F(t)=\int_\Omega f(x,t)dx$, *then each solution of problem (1.6), (1.7) is oscillatory.*

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(x,t)$ of (1.6), (1.7), then there exists a $t_0\geqslant 0$, such that $u(x,t)>0$ (or $u(x,t)<0$), $t\geqslant t_0$.

Case 1. Suppose that $u(x,t)>0$, $t\geqslant t_0$.

Integrating both sides of (1.6) with respect to x over the domain Ω, we have

$$\begin{aligned} &\int_\Omega D_{+,t}^{1+\alpha}u(x,t)dx+p(t)\int_\Omega D_{+,t}^{\alpha}u(x,t)dx \\ &=a(t)\int_\Omega \Delta u(x,t)dx+\sum_{i=1}^m a_i(t)\int_\Omega \Delta u(x,t-\tau_i)dx \\ &\quad -\int_\Omega q(x,t)\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi dx+\int_\Omega f(x,t)dx, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &D_+^{1+\alpha}\int_\Omega u(x,t)dx+p(t)D_+^{\alpha}\int_\Omega u(x,t)dx \\ &=a(t)\int_\Omega \Delta u(x,t)dx+\sum_{i=1}^m a_i(t)\int_\Omega \Delta u(x,t-\tau_i)dx \\ &\quad -\int_\Omega q(x,t)\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi dx+\int_\Omega f(x,t)dx. \end{aligned} \tag{3.4}$$

Remark that $\int_\Omega u(x,t)dx=U(t)$, then $U(t)>0$.

By the Green's formula and (1.7), we can get

$$\int_\Omega \Delta u(x,t)dx=\int_{\partial\Omega}\frac{\partial u}{\partial N}ds=-\int_{\partial\Omega}g(x,t)u(x,t)ds<0, \tag{3.5}$$

$$\int_\Omega \Delta u(x,t-\tau_i)dx<0, \tag{3.6}$$

and

$$\begin{aligned}\int_\Omega q(x,t)\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi dx &\geqslant q(t)\int_0^t (t-\xi)^{-\alpha}\int_\Omega u(x,\xi)dxd\xi\\ &= q(t)\int_0^t (t-\xi)^{-\alpha}U(\xi)d\xi = q(t)E(t),\end{aligned} \tag{3.7}$$

where

$$E(t)=\int_0^t (t-\xi)^{-\alpha}U(\xi)d\xi. \tag{3.8}$$

By (3.3)–(3.8), we can get

$$D_+^{1+\alpha}U(t)+p(t)D_+^{\alpha}U(t)\leqslant -q(t)E(t)+F(t)\leqslant F(t). \tag{3.9}$$

By lemma 2.4 and (3.9), we have

$$[D_+^{\alpha}U(t)V(t)]' = D_+^{1+\alpha}U(t)V(t)+p(t)D_+^{\alpha}U(t)V(t)\leqslant F(t)V(t). \tag{3.10}$$

Integrating both sides of the above inequality from t_0 to t, we get

$$D_+^{\alpha}U(t)V(t)\leqslant D_+^{\alpha}U(t_0)V(t_0)+\int_{t_0}^t F(s)V(s)ds.$$

Setting $D_+^{\alpha}U(t_0)V(t_0)=C$, we can get

$$D_+^{\alpha}U(t)\leqslant \frac{C}{V(t)}+\frac{\int_{t_0}^t F(s)V(s)ds}{V(t)}. \tag{3.11}$$

Taking the Riemann-Liouville fractional integral of order α on both sides of (3.11), by lemma 2.6 and the above inequality, we obtain

$$\begin{aligned}I_+^{\alpha}D_+^{\alpha}U(t) = U(t)-\frac{I_0^{1-\alpha}U(0)}{\Gamma(\alpha)}t^{\alpha-1} &\leqslant I_+^{\alpha}\left[\frac{C}{V(t)}+\frac{\int_{t_0}^t F(s)V(s)ds}{V(t)}\right]\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}F(s)V(s)ds\right)d\xi.\end{aligned} \tag{3.12}$$

In (3.12), set $t\to\infty$, we have

$$\liminf_{t\to\infty}U(t)\leqslant \liminf_{t\to\infty}\frac{C_0}{\Gamma(\alpha)}t^{\alpha-1}+\liminf_{t\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}F(s)V(s)ds\right)d\xi.$$

From (3.1), we can get $\liminf_{t\to\infty}U(t)\leqslant 0$. Which contradicts the assumption that $U(t)>0$.

Case 2. Suppose that $u(x,t)<0$, $t\geqslant t_0$.

Just as the case 1, we can obtain (3.3) holds and $U(t) < 0$.

Then,

$$\int_{\Omega} \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} ds = -\int_{\partial\Omega} g(x,t)u(x,t)ds > 0, \tag{3.13}$$

$$\int_{\Omega} \Delta u(x,t-\tau_i) > 0, \tag{3.14}$$

and

$$\begin{aligned}\int_{\Omega} q(x,t)\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi dx &\leqslant q(t)\int_0^t (t-\xi)^{-\alpha}\int_{\Omega} u(x,\xi)dxd\xi \\ &= q(t)E(t). \end{aligned} \tag{3.15}$$

From the conditions (3.13)–(3.15), we get

$$D_+^{1+\alpha}U(t) + p(t)D_+^{\alpha}U(t) \geqslant -q(t)E(t) + F(t) \geqslant F(t). \tag{3.16}$$

Similarly, by lemma 2.4 and (3.16), we can obtain

$$[D_+^{\alpha}U(t)V(t)]' = D_+^{1+\alpha}U(t)V(t) + p(t)D_+^{\alpha}U(t)V(t) \geqslant F(t)V(t). \tag{3.17}$$

Then integrating both sides of the above inequality from t_0 to t, we have

$$D_+^{\alpha}U(t)V(t) \geqslant D_+^{\alpha}U(t_0)V(t_0) + \int_{t_0}^t F(s)V(s)ds = C + \int_{t_0}^t F(s)V(s)ds,$$

and

$$D_+^{\alpha}U(t) \geqslant \frac{C}{V(t)} + \frac{\int_{t_0}^t F(s)V(s)ds}{V(t)}. \tag{3.18}$$

Taking the Riemann-Liouville fractional integral of order α on both sides of (3.18), by lemma 2.6 and the above inequality, we obtain

$$\begin{aligned} I_+^{\alpha}D_+^{\alpha}U(t) = U(t) - \frac{I_+^{1-\alpha}U(0)}{\Gamma(\alpha)}t^{\alpha-1} &\geqslant I_+^{\alpha}\left[\frac{C}{V(t)} + \frac{\int_{t_0}^t F(s)V(s)ds}{V(t)}\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F(s)V(s)ds\right)d\xi. \end{aligned} \tag{3.19}$$

In (3.19), setting $t \to \infty$, we have

$$\limsup_{t\to\infty} U(t) \geqslant \limsup_{t\to\infty} \frac{C_0}{\Gamma(\alpha)}t^{\alpha-1} + \limsup_{t\to\infty} \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F(s)V(s)ds\right)d\xi.$$

By the condition (3.2), we can get $\limsup_{t\to\infty} U(t) \geqslant 0$. Which contradicts the assumption that $U(t) < 0$. This proof is completed. □

4. The oscillation of the problem (1.6), (1.8)

THEOREM 4.1. *Suppose that* (A_1)–(A_3) *hold,* $\lim_{t\to\infty} I_+^{1-\alpha}U_1(t_0) = C_1$. *If*

$$\liminf_{t\to\infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} F_1(s)V(s)ds \right) d\xi < 0, \tag{4.1}$$

and

$$\limsup_{t\to\infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} F_1(s)V(s)ds \right) d\xi > 0, \tag{4.2}$$

where C_1, C *are constants,* $V(t) = e^{\int_{t_0}^t p(s)ds}$, $F_1(s) = \int_\Omega f(x,t)\varphi(x)dx$, $U_1(t) = \int_\Omega u(x,t)\varphi(x)dx$, *then each solution* $u(x,t)$ *of problem (1.6), (1.8) is oscillatory.*

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(x,t)$ of (1.6), (1.8), then there exists a $t_0 \geqslant 0$, such that $u(x,t) > 0$ (or $u(x,t) < 0$), $t \geqslant t_0$.

Case 1. Suppose that $u(x,t) > 0$, $t \geqslant t_0$.

Multiplying both sides of (1.6) by $\varphi(x)$ and integrating with respect to x over the domain Ω, we have

$$\begin{aligned} & D_+^{1+\alpha} \int_\Omega u(x,t)\varphi(x)dx + p(t)D_+^{\alpha} \int_\Omega u(x,t)\varphi(x)dx \\ &= a(t) \int_\Omega \Delta u(x,t)\varphi(x)dx + \sum_{i=1}^m a_i(t) \int_\Omega \Delta u(x,t-\tau_i)\varphi(x)dx \\ &\quad - \int_\Omega q(x,t) \int_0^t (t-\xi)^{-\alpha} u(x,\xi)\varphi(x)d\xi dx + \int_\Omega f(x,t)\varphi(x)dx. \end{aligned} \tag{4.3}$$

By the Green's formula and Lemma 2.7, we can get

$$\int_\Omega \Delta u(x,t)\varphi(x)dx = \int_\Omega u(x,t)\Delta\varphi(x)dx = -\int_\Omega u(x,t)\beta_0\varphi(x)dx < 0, \tag{4.4}$$

$$\int_\Omega \Delta u(x,t-\tau_i)\varphi(x)dx < 0, \tag{4.5}$$

and

$$\begin{aligned} \int_\Omega q(x,t) \int_0^t (t-\xi)^{-\alpha} u(x,\xi)\varphi(x)d\xi dx &\geqslant q(t) \int_0^t (t-\xi)^{-\alpha} \int_\Omega u(x,\xi)\varphi(x)dxd\xi \\ &= q(t)E_1(t). \end{aligned} \tag{4.6}$$

Therefore, $U_1(t) > 0$, $E_1(t) = \int_0^t (t-\xi)^{-\alpha} U_1(\xi)d\xi > 0$.

Consider (4.3)–(4.6), we can get

$$D_+^{1+\alpha} U_1(t) + p(t)D_+^{\alpha} U_1(t) \leqslant -q(t)E_1(t) + F_1(t) \leqslant F_1(t), \tag{4.7}$$

By lemma 2.4 and (4.7), we have

$$[D_+^{\alpha} U_1(t)V(t)]' = D_+^{1+\alpha} U_1(t)V(t) + p(t)D_+^{\alpha} U_1(t)V(t) \leqslant F_1(t)V(t). \tag{4.8}$$

Then integrating both sides of the above inequality from t_0 to t, we have

$$D_+^{\alpha}U_1(t)V(t) \leqslant D_+^{\alpha}U_1(t_0)V(t_0) + \int_{t_0}^{t} F_1(s)V(s)ds.$$

Setting $D_+^{\alpha}U_1(t_0)V(t_0) = C$, We can get

$$D_+^{\alpha}U_1(t) \leqslant \frac{C}{V(t)} + \frac{\int_{t_0}^{t} F_1(s)V(s)ds}{V(t)}. \tag{4.9}$$

Taking the Riemann-Liouville fractional integral of order α on both sides of (4.9), by lemma 2.6 and the above inequality, we obtain

$$\begin{aligned} I_+^{\alpha}D_+^{\alpha}U_1(t) &= U_1(t) - \frac{I_+^{1-\alpha}U_1(0)}{\Gamma(\alpha)}t^{\alpha-1} \leqslant I_+^{\alpha}\left[\frac{C}{V(t)} + \frac{\int_{t_0}^{t} F(s)V(s)ds}{V(t)}\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F_1(s)V(s)ds\right)d\xi, \end{aligned}$$

and

$$U_1(t) \leqslant \frac{I_+^{1-\alpha}U_1(0)}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F_1(s)V(s)ds\right)d\xi. \tag{4.10}$$

In (4.10), setting $t \to \infty$, we have

$$\liminf_{t\to\infty} U_1(t) \leqslant \liminf_{t\to\infty} \frac{C_1}{\Gamma(\alpha)}t^{\alpha-1} + \liminf_{t\to\infty} \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F_1(s)V(s)ds\right)d\xi.$$

From (4.1), we can get $\liminf_{t\to\infty} U_1(t) \leqslant 0$. Which contradicts the assumption that $U_1(t) > 0$.

Case 2. Suppose that $u(x,t) < 0$, $t \geqslant t_0$.

Multiplying both sides of (1.6) by $\varphi(x)$ and integrating with respect to x over the domain Ω, we have (4.3) holds.

By the Green's formula, we can get

$$\int_{\Omega} \Delta u(x,t)\varphi(x)dx = \int_{\Omega} u(x,t)\Delta\varphi(x)dx = -\int_{\Omega} u(x,t)\beta_0\varphi(x)dx > 0, \tag{4.11}$$

$$\int_{\Omega} \Delta u(x,t-\tau_i)\varphi(x)dx > 0, \tag{4.12}$$

and

$$\begin{aligned} \int_{\Omega} q(x,t)\int_0^t (t-\xi)^{-\alpha}u(x,\xi)\varphi(x)d\xi dx &\leqslant q(t)\int_0^t (t-\xi)^{-\alpha}\int_{\Omega} u(x,\xi)\varphi(x)dxd\xi \\ &= q(t)E_1(t). \end{aligned} \tag{4.13}$$

Therefore, $U_1(t) < 0$, $E_1(t) = \int_0^t (t-\xi)^{-\alpha}U_1(\xi)d\xi < 0$.

From (4.11)–(4.13), we can get

$$D_+^{1+\alpha}U_1(t)+p(t)D_+^{\alpha}U_1(t) \geqslant -q(t)E_1(t)+F_1(t) \geqslant F_1(t). \tag{4.14}$$

By lemma 2.4 and (4.14)

$$[D_+^{\alpha}U_1(t)V(t)]' = D_+^{1+\alpha}U_1(t)V(t)+p(t)D_+^{\alpha}U_1(t)V(t) \geqslant F_1(t)V(t).$$

Integrating both sides of the above inequality from t_0 to t, we have

$$D_+^{\alpha}U_1(t)V(t) \geqslant D_+^{\alpha}U_1(t_0)V(t_0)+\int_{t_0}^{t} F_1(s)V(s)ds.$$

Setting $D_+^{\alpha}U_1(t_0)V(t_0)=C$, We can get

$$D_+^{\alpha}U_1(t) \geqslant \frac{C}{V(t)}+\frac{\int_{t_0}^{t} F_1(s)V(s)ds}{V(t)}. \tag{4.15}$$

Taking the Riemann-Liouville fractional integral of order α on both sides of (4.15), by lemma 2.6 and the above inequality, we obtain

$$\begin{aligned} I_+^{\alpha}D_+^{\alpha}U_1(t) &= U_1(t)-\frac{I_+^{1-\alpha}U_1(0)}{\Gamma(\alpha)}t^{\alpha-1} \geqslant I_+^{\alpha}\left[\frac{C}{V(t)}+\frac{\int_{t_0}^{t} F(s)V(s)ds}{V(t)}\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi} F_1(s)V(s)ds\right)d\xi, \end{aligned}$$

then

$$U_1(t) \geqslant \frac{I_+^{1-\alpha}U_1(0)}{\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}(C+\int_{t_0}^{\xi} F_1(s)V(s)ds)d\xi. \tag{4.16}$$

Therefore, we have

$$\limsup_{t\to\infty} U_1(t) \geqslant \limsup_{t\to\infty}\frac{C_1}{\Gamma(\alpha)}t^{\alpha-1}+\limsup_{t\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi} F_1(s)V(s)ds\right)d\xi.$$

From (4.2), we can get $\liminf_{t\to\infty} U_1(t) \geqslant 0$. Which contradicts to $U_1(t)<0$. The proof is completed. □

5. Examples

In this part, we will give two examples to illustrate the usefulness of our conclusions.

EXAMPLE 1. Consider the oscillation of the fractional partial differential equation

$$\begin{aligned} D_{+,t}^{1+\frac{1}{2}}u(x,t)-D_{+,t}^{\frac{1}{2}}u(x,t) &= \frac{1}{\pi}\Delta u(x,t)+2t\Delta u(x,t-1) \\ &\quad -\left(x^2+\frac{1}{t^2}\right)\int_0^t (t-\xi)^{-\frac{1}{2}}u(x,\xi)d\xi+e^{2t}\sin t\sin x, \\ &\quad (x,t)\in(0,\pi)\times R_+ \end{aligned} \tag{5.1}$$

with the boundary condition,

$$u_x(0,t) = u_x(\pi,t) = 0. \tag{5.2}$$

Proof. Compare with (1.6) and (1.7), where $\alpha = \frac{1}{2}$, $\Omega = (0,\pi)$, $n = 1$, $m = 1$, $p(t) = -1$, $a(t) = \frac{1}{\pi}$, $a_1(t) = 2t$, $\tau_1 = 1$, $q(x,t) = x^2 + \frac{1}{t^2}$, $q(t) = \min q(x,t) = \frac{1}{t^2}$, $f(x,t) = e^{2t}\sin t \sin x$.

Then

$$\begin{aligned} F(t) &= \int_0^\pi f(x,t)dx = 2e^{2t}\sin t, \\ V(t) &= e^{\int_{t_0}^t p(s)ds} = e^{t_0 - t}, \end{aligned}$$

and

$$\begin{aligned} \int_{t_0}^{\xi} F(s)V(s)ds &= \int_{t_0}^{\xi} 2e^{2s}\sin s e^{t_0 - s} ds \\ &= e^{t_0}(e^{\xi}\sin\xi - e^{\xi}\cos\xi + e^{t_0}\cos t_0 - e^{t_0}\sin t_0). \end{aligned}$$

Letting $t_0 = \frac{\pi}{2}$, we obtain

$$\begin{aligned} &\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F(s)V(s)ds\right)d\xi \\ &= \int_0^t (t-\xi)^{-\frac{1}{2}} e^{\xi - t_0}\left[C + e^{t_0}(e^{\xi}\sin\xi - e^{\xi}\cos\xi + e^{t_0}\cos t_0 - e^{t_0}\sin t_0)\right]d\xi \\ &= \int_0^t (t-\xi)^{-\frac{1}{2}} e^{\xi - \frac{\pi}{2}}\left[C + e^{\frac{\pi}{2}}e^{\xi}\sin\xi - e^{\frac{\pi}{2}}e^{\xi}\cos\xi - e^{\pi}\right]d\xi. \end{aligned} \tag{5.3}$$

Let $t - s^2 = \xi$, then

$$\begin{aligned} &\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F(s)V(s)ds\right)d\xi \\ &= \int_{\sqrt{t}}^0 \frac{1}{s} e^{t-s^2-\frac{\pi}{2}}\left[C + e^{\frac{\pi}{2}}e^{t-s^2}\sin(t-s^2) - e^{\frac{\pi}{2}}e^{t-s^2}cos(t-s^2) - e^{\pi}\right](-2s)ds \\ &= e^t\left\{2(C-e^{\pi})e^{-\frac{\pi}{2}}\int_0^{\sqrt{t}} e^{-s^2}ds + 2\sqrt{2}e^t\left[\sin\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}\cos s^2 ds \right.\right. \\ &\quad \left.\left. -\cos\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}\sin s^2 ds\right]\right\}. \end{aligned}$$

Noting that

$$|e^{-2s^2}\cos s^2| \leqslant e^{-2s^2}, \qquad |e^{-2s^2}\sin s^2| \leqslant e^{-2s^2},$$

and

$$\lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2s^2}ds = \frac{\sqrt{2\pi}}{4},$$

we can get that $\lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2s^2}\cos s^2 ds$ and $\lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2s^2} sins^2 ds$ are convergent, then $\lim_{t\to\infty}\left(\sin(t-\frac{\pi}{4})\int_0^{\sqrt{t}} e^{-2s^2}\cos s^2 ds - \cos(t-\frac{\pi}{4})\int_0^{\sqrt{t}} e^{-2s^2}\sin s^2 ds\right)$ is convergent.

Setting

$$\lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2s^2}\cos s^2 ds = A,$$
$$\lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2s^2}\sin s^2 ds = B,$$

and

$$t_k = \frac{3\pi}{2}+\frac{\pi}{4}+2k\pi+\arccos\frac{A}{\sqrt{A^2+B^2}}.$$

Then

$$\begin{aligned}
&\lim_{t\to\infty}\left[\sin\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}\cos s^2 ds-\cos\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}\sin s^2 ds\right]\\
&=\lim_{k\to\infty}\left[\sin\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\cos s^2 ds-\cos\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\sin s^2 ds\right]\\
&=\sqrt{A^2+B^2}\sin\left(\frac{3\pi}{2}+\frac{\pi}{4}+2k\pi+\arccos\frac{A}{\sqrt{A^2+B^2}}-\frac{\pi}{4}-\arccos\frac{A}{\sqrt{A^2+B^2}}\right)\\
&=\sqrt{A^2+B^2}\sin\left(\frac{3\pi}{2}+2k\pi\right)\\
&=-\sqrt{A^2+B^2}.
\end{aligned}$$

We have

$$\begin{aligned}
&\liminf_{t\to\infty}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}F(s)V(s)ds\right)d\xi\\
&=\liminf_{t\to\infty} e^t\left\{2(C-e^{\pi})e^{-\frac{\pi}{2}}\int_0^{\sqrt{t}} e^{-s^2}ds+2\sqrt{2}e^t\left[\sin\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}\cos s^2 ds\right.\right.\\
&\quad\left.\left.-\cos\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}\sin s^2 ds\right]\right\}\\
&=\liminf_{k\to\infty} e^{t_k}\left\{2(C-e^{\pi})e^{-\frac{\pi}{2}}\int_0^{\sqrt{t_k}} e^{-s^2}ds+2\sqrt{2}e^{t_k}\left[\sin\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\cos s^2 ds\right.\right.\\
&\quad\left.\left.-\cos\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\sin s^2 ds\right]\right\}\\
&=-\infty,
\end{aligned}$$

then

$$\liminf_{t\to\infty}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}F(s)V(s)ds\right)d\xi=-\infty<0. \tag{5.4}$$

Similarly, taking $t_j=\frac{\pi}{2}+\frac{\pi}{4}+2j\pi+\arccos\frac{A}{\sqrt{A^2+B^2}}$, we can get that

$$\begin{aligned}
&\lim_{t\to\infty}\left[\sin\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}}e^{-2s^2}\cos s^2ds-\cos\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}}e^{-2s^2}\sin s^2ds\right]\\
&=\lim_{j\to\infty}\left[\sin\left(t_j-\frac{\pi}{4}\right)\int_0^{\sqrt{t_j}}e^{-2s^2}\cos s^2ds-\cos\left(t_j-\frac{\pi}{4}\right)\int_0^{\sqrt{t_j}}e^{-2s^2}\sin s^2ds\right]\\
&=\sqrt{A^2+B^2}\sin\left(\frac{\pi}{2}+\frac{\pi}{4}+2j\pi+\arccos\frac{A}{\sqrt{A^2+B^2}}-\frac{\pi}{4}-\arccos\frac{A}{\sqrt{A^2+B^2}}\right)\\
&=\sqrt{A^2+B^2}\sin\left(\frac{\pi}{2}+2j\pi\right)\\
&=\sqrt{A^2+B^2}.
\end{aligned}$$

Then

$$\begin{aligned}
&\liminf_{t\to\infty}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}F(s)V(s)ds\right)d\xi\\
&=\liminf_{t\to\infty}e^t\left\{2(C-e^{\pi})e^{-\frac{\pi}{2}}\int_0^{\sqrt{t}}e^{-s^2}ds+2\sqrt{2}e^t\left[\sin\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}}e^{-2s^2}\cos s^2ds\right.\right.\\
&\qquad\left.\left.-\cos\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}}e^{-2s^2}\sin s^2ds\right]\right\}\\
&=\limsup_{j\to\infty}e^{t_j}\left\{2(C-e^{\pi})e^{-\frac{\pi}{2}}\int_0^{\sqrt{t_j}}e^{-s^2}ds+2\sqrt{2}e^{t_j}\left[\sin\left(t_j-\frac{\pi}{4}\right)\int_0^{\sqrt{t_j}}e^{-2s^2}\cos s^2ds\right.\right.\\
&\qquad\left.\left.-\cos\left(t_j-\frac{\pi}{4}\right)\int_0^{\sqrt{t_j}}e^{-2s^2}\sin s^2ds\right]\right\}\\
&=\infty,
\end{aligned}$$

$$\limsup_{t\to\infty}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}F(s)V(s)ds\right)d\xi=\infty>0. \tag{5.5}$$

Combine (5.4) with (5.5), by Theorem 3.1, all solutions of (5.1), (5.2) are oscillatory.

EXAMPLE 2. Consider the oscillation of the fractional partial differential equation

$$\begin{aligned}
D_{+,t}^{1+\frac{1}{2}}u(x,t)-D_{+,t}^{\frac{1}{2}}u(x,t)&=e^{-t}\Delta u(x,t)+2t\Delta u(x,t-1)\\
&\quad-\left(x^2+\frac{1}{t^2}\right)\int_0^t(t-\xi)^{-\frac{1}{2}}u(x,\xi)d\xi+e^{2t}\sin t\sin x,
\end{aligned}$$

$$(x,t) \in (0,\pi) \times R_+ \tag{5.6}$$

with the boundary condition,

$$u(0,t) = u(\pi,t) = 0. \tag{5.7}$$

Proof. Compare with (1.6), (1.8), where $\alpha = \frac{1}{2}$, $\Omega = (0,\pi)$, $n = 1$, $m = 1$, $p(t) = -1$, $a(t) = e^{-t}$, $a_1(t) = 2t$, $\tau_1 = 1$, $q(x,t) = x^2 + \frac{1}{t^2}$, $q(t) = \min q(x,t) = \frac{1}{t^2}$, $f(x,t) = e^{2t}\sin t \sin x$.

So we can obtain the smallest eigenvalue β_0 of the Dirichlet problem above and the corresponding eigenfunction $\varphi(x)$, where $\beta_0 = 1$ and $\varphi(x) = \sin x$. As a result,

$$F_1(t) = \int_\Omega f(x,t)\varphi(x)dx = \int_0^\pi e^{2t}\sin t \sin x \sin x dx = \frac{\pi}{2}e^{2t}\sin t,$$
$$V(t) = e^{\int_{t_0}^t -1dx} = e^{t_0 - t}.$$

We can get

$$\begin{aligned}\int_{t_0}^{\xi} F_1(s)V(s)ds &= \int_{t_0}^{\xi} \frac{\pi}{2}e^{2s}\sin s e^{t_0-s}ds \\ &= \frac{\pi}{4}e^{t_0}(e^{\xi}\sin\xi - e^{\xi}\cos\xi + e^{t_0}\cos t_0 - e^{t_0}\sin t_0)\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F_1(s)V(s)ds\right)d\xi \\ &= \int_0^t (t-\xi)^{-\frac{1}{2}}e^{\xi - t_0}\left[C + \frac{\pi}{4}e^{t_0}(e^{\xi}\sin\xi - e^{\xi}\cos\xi + e^{t_0}\cos t_0 - e^{t_0}\sin t_0)\right]d\xi.\end{aligned}$$

Using a similar way in example 1,we have

$$\liminf_{t\to\infty}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F_1(s)V(s)ds\right)d\xi = -\infty < 0, \tag{5.8}$$

$$\limsup_{t\to\infty}\int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C + \int_{t_0}^{\xi} F_1(s)V(s)ds\right)d\xi = \infty > 0. \tag{5.9}$$

Therefore, by theorem 4.1, it is easy to see that every solution of (5.6), (5.7) is oscillatory in $(0,\pi) \times R_+$. □

REMARK 3. We note that our results obtained here can give sufficient conditions to guarantee the oscillatory of Eq. (5.1), (5.2) and Eq. (5.6), (5.7). So we can get the forced oscillation easily. However, the results in [13] can't solve problem which with forced term.

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