Fractional
Differential
Calculus
Volume 7, Number 2 (2017), 357–369 doi:10.7153/fdc-2017-07-17

REPRESENTATION OF SOLUTIONS OF LINEAR INHOMOGENEOUS CAPUTO FRACTIONAL DIFFERENTIAL EQUATION WITH CONTINUOUS VARIABLE COEFFICIENT BY GREEN FUNCTION

SUNAE PAK, HUICHOL CHOI AND KINAM SIN

(*Communicated by S. Umarov*)

Abstract. In this paper, the representation of solution to linear inhomogeneous Caputo fractional differential equation with continuous variable coefficient has been considered by using Green function. We have proved that the solution of linear inhomogeneous equations with homogeneous initial condition can be represented by using classical Green function, generalized Green function and modified Green function.

1. Introduction

A solution method by classical Duhamel principle for general linear inhomogeneous Riemann-Liouville fractional differential equation with constant coefficients was proposed [5]. The solutions of general linear inhomogeneous Riemann-Liouville fractional differential equations with constant coefficients were obtained by using the Adomain decomposition method and it was proved that these solutions are equal to those gained by Green function method [6]. The general theory on a system of linear inhomogeneous fractional differential equation was developed and a solution method by Green function was proposed in case of constant matrix coefficients [7]. A power series solution method was presented for some linear fractional differential equations with analytic coefficients [8].

Researchers proved that classical Duhamel principle does not apply for inhomogeneous Cauchy problem of fractional differential equations and generalized Duhamel's principle for fractional pseudo-differential equations [9]. Also they applied the fractional generalization of Duhamel's principle to the Cauchy problem for inhomogeneous fractional order differential equations [10].

In this paper, we focused on how the Green function method may be applied on initial value problem of linear inhomogeneous Caputo fractional differential equations of general type according to distribution of orders of derivatives and found the representation of solution to the initial value problem by Green function. This paper is organized as follows. In Section 2, some definitions of fractional calculus were introduced.

Mathematics subject classification (2010): 26A33, 34A08.
Keywords and phrases: Linear inhomogeneous equations, Caputo fractional differential equations, continuous variable coefficients, Green function.

Paper FDC-07-17

In section 3, a representation of solution to linear inhomogeneous Caputo fractional equation with continuous variable coefficients and homogeneous initial condition were provided by using classical Green function method, generalized Green function method and modified Green function method.

2. Preliminary

DEFINITION 1. ([1]) Let $R=(-\infty,+\infty)$, $R_+=(0,+\infty)$. We denote by $C_r^n[0,T]$ the space of functions f that $f:(0,T]\to R$ $(\forall T>0)$ and $t^r f^{(n)}(t)\in C[0,T]$ for $0\leqslant r<1$. In particular, denote $C_r^0[0,T]$ by $C_r[0,T]$.

DEFINITION 2. ([1,5]) Let $\alpha\in R_+$, $f\in C_r[0,T]$, $0\leqslant r<1$. Then

$$I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau,\quad t>0$$

is called the *fractional integral* of order α $(\alpha>0)$ of function f in the means of *Riemann-Liouville.* In particular, we denote as $I^0 f(t)=f(t)$.

DEFINITION 3. ([1, 5]) Let $n-1<\alpha\leqslant n$, $n\in N$, $I^{n-\alpha}f\in C_r^n[0,T]$ and $0\leqslant r<1$. Then

$$D_{0+}^{\alpha}f(t)=D^n I^{n-\alpha}f(t),\quad D^n=\frac{d^n}{dt^n}$$

is called the *fractional derivative* of order α of function f in the means of *Riemann-Liouville*.

DEFINITION 4. ([5]) Let $n-1<\alpha\leqslant n$, $n\in N$, $I^{n-\alpha}f\in C_r^n[0,T]$, $0\leqslant r<1$. Then

$${}^cD_{0+}^{\alpha}f(t)=D_{0+}^{\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k\right]$$

is called the Caputo fractional derivatives ${}^cD_{0+}^{\alpha}f$ of order α of function f.

REMARK 1. When $\alpha=n$, we have ${}^cD_{0+}^{\alpha}f(t)=D_{0+}^{\alpha}f(t)=D^n f(t)$.

DEFINITION 5. ([1, 5]) Let $n\in N$. We define the set

$$AC^n[a,b]:=\left\{f:[a,b]\to R|D^{n-1}f\in AC[a,b],\, D=\frac{d}{dt}\right\}.$$

Here $AC^1[a,b]=AC[a,b]$ is the set of absolutely continuous functions on $[a,b]$.

DEFINITION 6. ([1]) We denote by $I^{\alpha}(L_1)$ the set of functions f which is represented as integral of order $\alpha>0$ of some integrable function $\varphi\in L_1(0,T)$, that is $f=I^{\alpha}\varphi$.

3. Main result

By using Green function method, the solution of the initial value problem of linear inhomogeneous Caputo fractional differential equation with continuous variable coefficients can be obtained.

$$ {}^{c}D_{0+}^{\alpha_0}y(t)+\sum_{i=1}^{m}a_i(t)\,{}^{c}D_{0+}^{\alpha_i}y(t)=g(t),\quad t\in[0,T], \tag{1}$$

with

$$D^k y(t)\mid_{t=0+}=0,\quad k=0,1,\cdots n_0-1. \tag{2}$$

Here, Riemann-Liouville fractional differential operator

$$ {}^{R}L(D_{\tau+})=D_{\tau+}^{\alpha_0}+\sum_{i=1}^{m}a_i(t)D_{\tau+}^{\alpha_i} \tag{3}$$

and Caputo fractional differential operator

$$ {}^{c}L(D_{\tau+})={}^{c}D_{0+}^{\alpha_0}+\sum_{i=1}^{m}a_i(t)^{c}D_{0+}^{\alpha_i}, \tag{4}$$

were considered. Also $\alpha_0,\alpha_i\in R_+,\ i=1,\cdots,m$ usually satisfied the condition of $\alpha_0>0,\ \alpha_0>\alpha_1>\cdots>\alpha_m\geqslant 0$ and $n_0,\ n_i$ are natural numbers which satisfy the condition of $n_0-1<\alpha_0\leqslant n_0,\ n_i-1<\alpha_i\leqslant n_i,\ i=1,\cdots,m$. The solution of the initial value problem

$$ {}^{R}L(D_{\tau+})G(t;\tau)=0,\quad t>\tau, \tag{5}$$

$$D_{\tau+}^{\alpha_0-j}G(t;\tau)\mid_{t=\tau+}=\begin{cases}1,\ j=1,\\ 0,\ j=2,\cdots,n_0,\end{cases} \tag{6}$$

$$I_{\tau+}^{n_0-\alpha_0}G(t;\tau)\in AC^{n_0}[\tau,T],\ \ \forall T>\tau$$

can be obtained as [3]

$$G_R(t;\tau)=\Phi_{\alpha_0}(t-\tau)+\sum_{k=0}^{\infty}(-1)^{k+1}I_{\tau+}^{\alpha_0}\left[\sum_{i=1}^{m}a_i(t)I_{\tau+}^{\alpha_0-\alpha_i}\right]^k\sum_{i=1}^{m}a_i(t)\Phi_{\alpha_0-\alpha_i}(t-\tau). \tag{7}$$

DEFINITION 7. $G_R(t;\tau)$ is called the Green function of the Riemann-Liouville fractional differential operator (1).

On the other hand, the solution $G_c(t;\tau)$ of the initial value problem

$$ {}^{c}L(D_{\tau+})G\equiv{}^{c}D_{\tau+}^{\alpha_0}G+\sum_{i=1}^{m}a_i(t)\,{}^{c}D_{\tau+}^{\alpha_i}G=0,\quad t>\tau,\quad \tau>0, \tag{8}$$

$$D^kG\mid_{t=\tau}=\begin{cases}1,\ k=n_0-1,\\ 0,\ k=0,1,\cdots,n_0-2,\end{cases} \tag{9}$$

$$G\in C^{n_0}(\tau,T),\ \ \forall T>\tau$$

can be obtained as follows by [3]:

When $n_0 = n_1$,

$$G_c(t;\tau) = \Phi_{n_0}(t-\tau) + \sum_{k=0}^{\infty}(-1)^{k+1} I_{\tau+}^{\alpha_0}\left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i}\right]^k \sum_{i=h_{n_0-1}}^{m} a_i(t)\Phi_{n_0-\alpha_i}(t-\tau), \tag{10}$$

and when $n_0 > n_1$,

$$G_c(t;\tau) = \Phi_{n_0}(t-\tau) + \sum_{k=0}^{\infty}(-1)^{k+1} I_{\tau+}^{\alpha_0}\left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i}\right]^k \sum_{i=1}^{m} a_i(t)\Phi_{n_0-\alpha_i}(t-\tau). \tag{11}$$

DEFINITION 8. $G_c(t;\tau)$ is called the Green function of the Caputo fractional differential operator ${}^cL(D_{\tau+})$.

LEMMA 1. ([5]) *Let $g(t)$, $a_i(t) \in C[0,T]$, $i = 1,\cdots,m$. Then there exists the unique solution $y(t) \in C^{\alpha_0, n_0-1}[0,T]$ to the initial value problem of the inhomogeneous equation (1) with homogeneous initial condition (2) and it is represented as*

$$y(t) = \sum_{k=0}^{\infty}(-1)^k I_{0+}^{\alpha_0}\left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i}\right]^k g(t). \tag{12}$$

3.1. Representation of solution of the initial value problem of non-homogeneous equation by classical Green function me1thod

Some previous researches (e.g. [9]) had indicated that classical Green function method may not be applied for Caputo fractional differential equations and used generalized Green function method. But even for Caputo differential equations classical Green function method can be used to find out solutions of non-homogeneous differential equations equipped with homogeneous initial conditions.

THEOREM 1. *Assume that $g(t)$, $a_i(t) \in C[0,T]$, $i = 1,\cdots,m$ and $\alpha_0 = n_0$, $n_0 > n_1$. Then there exists the unique solution $y(t) \in C^{n_0}[0,T]$ the initial value problem of non-homogeneous equation (1) equipped with homogeneous initial condition (2) and it is represented as*

$$y(t) = \int_0^t G_c(t;\tau)g(\tau)d\tau = \sum_{k=0}^{\infty}(-1)^k I_{0+}^{n_0}\left[\sum_{i=1}^{m} a_i(t) I_{0+}^{n_0-\alpha_i}\right]^k g(t), \tag{13}$$

where $G_c(t;\tau)$ is the Green function of the Caputo fractional differential operator ${}^cL(D_{\tau+})$ when $n_0 > n_1$.

Proof. From the assumption, Lemma 1 shows the existence and the uniqueness of the solution of the initial value problems (1) and (2) and it is sufficient to show (13) by calculating the first part of (13) because of in the case $\alpha_0 = n_0$, the second part of (13) is equal to the expression (12) which is the solution of the initial value problem (1), (2) from Lemma 1.

So the first part of (13) can be calculated:

$$\begin{aligned}
y(t) &= \int_0^t G_c(t;\tau)g(\tau)d\tau \\
&= \int_0^t \Phi_{n_0}(t-\tau)g(\tau)d\tau + \sum_{k=0}^{\infty}(-1)^{k+1}\int_0^t I_{\tau+}^{n_0}\left[\sum_{i=1}^m a_i(t)I_{\tau+}^{n_0-\alpha_i}\right]^k \\
&\quad \times \sum_{i=1}^m a_i(t)\Phi_{n_0-\alpha_i}(t-\tau)g(\tau)d\tau \\
&\quad \text{(by using } \int_0^t \Phi_{n_0}(t-\tau)g(\tau)d\tau = I_{0+}^{n_0}g(t) \text{ from the definition of fractional integral)} \\
&= I_{0+}^{n_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}\int_0^t \left[\frac{1}{\Gamma(n_0)}\int_\tau^t (t-\xi)^{n_0-1}\left[\sum_{i=1}^m a_i(\xi)I_{\tau+}^{n_0-\alpha_i}\right]^k\right. \\
&\quad \left. \times \sum_{i=1}^m a_i(\xi)\Phi_{n_0-\alpha_i}(\xi-\tau)g(\tau)d\xi\right]d\tau \\
&\quad \text{(Interchanging the order of integration by Dirichlet's formula)} \\
&= I_{0+}^{n_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{\Gamma(n_0)}\int_0^t (t-\xi)^{n_0-1}\left[\int_0^\xi \left[\sum_{i=1}^m a_i(\xi)I_{\tau+}^{n_0-\alpha_i}\right]^k\right. \\
&\quad \left. \times \sum_{i=1}^m a_i(\xi)\Phi_{n_0-\alpha_i}(\xi-\tau)g(\tau)d\tau\right]d\xi \\
&= I_{0+}^{n_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}I_{0+}^{n_0}\left[\int_0^t \left[\sum_{i=1}^m a_i(t)I_{\tau+}^{n_0-\alpha_i}\right]^k \sum_{i=1}^m a_i(t)\Phi_{n_0-\alpha_i}(t-\tau)g(\tau)d\tau\right] \\
&= I_{0+}^{n_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}I_{0+}^{n_0}\left[\sum_{i=1}^m a_i(t)I_{\tau+}^{n_0-\alpha_i}\right]^k \sum_{i=1}^m a_i(t)\int_0^t \Phi_{n_0-\alpha_i}(t-\tau)g(\tau)d\tau \\
&= I_{0+}^{n_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}I_{0+}^{n_0}\left[\sum_{i=1}^m a_i(t)I_{0+}^{n_0-\alpha_i}\right]^k \sum_{i=1}^m a_i(t)I_{0+}^{n_0-\alpha_i}g(t) \\
&= I_{0+}^{n_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}I_{0+}^{n_0}\left[\sum_{i=1}^m a_i(t)I_{0+}^{n_0-\alpha_i}\right]^{k+1} g(t) \\
&= I_{0+}^{n_0}g(t) + \sum_{k=1}^{\infty}(-1)^{k}I_{0+}^{n_0}\left[\sum_{i=1}^m a_i(t)I_{0+}^{n_0-\alpha_i}\right]^{k} g(t) \\
&= \sum_{k=0}^{\infty}(-1)^{k}I_{0+}^{n_0}\left[\sum_{i=1}^m a_i(t)I_{0+}^{n_0-\alpha_i}\right]^{k} g(t).
\end{aligned}$$

It can also be seen that $y(t) \in C^{n_0}[0,T]$. □

REMARK 2. The method of finding out the solution of (1), (2) by using Green function of Caputo fractional differential operator ${}^cL(D_{\tau+})$ like in Theorem 1 is called *Classical Green Function Method.*

3.2. Representation of Solution of the initial value problem of non-homogeneous equation by generalized Green function method

In the previous subsection, it was shown that when $\alpha_0 = n_0$, $n_0 > n_1$, classical Green function method could be used even for Caputo fractional differential equations.

The following theorem shows a generalized Green function method which may be applied to Caputo fractional differential equations for which the classical Green function approach could not be used.

THEOREM 2. *Let $n_0 - 1 < \alpha_0 < n_0$, $n_0 > n_1$. Assume that $g(t)$, $a_i(t) \in C[0,T]$, $i = 1,\cdots,m$. Then there exists the unique solution $y(t) \in C^{\alpha_0,n_0-1}[0,T]$ of the initial value problem (1), (2) and it is represented by*

$$y(t) = \int_0^t G_c(t;\tau)\Xi_{0+}^{n_0-\alpha_0} g(\tau)d\tau = \sum_{k=0}^{\infty}(-1)^k I_{0+}^{\alpha_0}\left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i}\right]^k g(t), \tag{14}$$

where $G_c(t;\tau)$ is the Green function of the Caputo fractional differential operator ${}^cL(D_{\tau+})$ in case of $n_0 > n_1$ and differential symbol $\Xi_{0+}^{n_0-\alpha_0}$ denotes Riemann-Liouville fractional derivative $D_{0+}^{n_0-\alpha_0}$ or Caputo type ${}^cD_{0+}^{n_0-\alpha_0}$.

Proof. From our assumption, Lemma 1 yields the existence and the uniqueness of the solution of the initial value problem (1), (2).

Similarly to Theorem 1, it is sufficient to show (14) by calculating the first part of (14) because that the second part of (14) is equal to the expression (12), which is the solution of the initial value problem from Lemma 1.

① When $\Xi_{0+}^{n_0-\alpha_0} = D_{0+}^{n_0-\alpha_0}$, we have

$$\begin{aligned} y(t) &= \int_0^t G_c(t;\tau) D_{0+}^{n_0-\alpha_0} g(\tau)d\tau \\ &= \int_0^t \left[\Phi_{n_0}(t-\tau) + \sum_{k=0}^{\infty}(-1)^{k+1} I_{\tau+}^{\alpha_0}\left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i}\right]^k \sum_{i=1}^{m} a_i(t)\Phi_{n_0-\alpha_i}(t-\tau)\right] \\ &\quad \times D_{0+}^{n_0-\alpha_0} g(\tau)d\tau \\ &= \int_0^t \Phi_{n_0}(t-\tau) D_{0+}^{n_0-\alpha_0} g(\tau)d\tau + \int_0^t \sum_{k=0}^{\infty}(-1)^{k+1} I_{\tau+}^{\alpha_0}\left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i}\right]^k \\ &\quad \times \sum_{i=1}^{m} a_i(t)\Phi_{n_0-\alpha_i}(t-\tau) D_{0+}^{n_0-\alpha_0} g(\tau)d\tau \end{aligned}$$

$$\Bigg(\text{Taking account of}$$

$$\int_0^t \Phi_{n_0}(t-\tau)D_{0+}^{n_0-\alpha_0}g(\tau)d\tau = \int_0^t \frac{(t-\tau)^{n_0-1}}{\Gamma(n_0)}D_{0+}^{n_0-\alpha_0}g(\tau)d\tau$$
$$= \frac{1}{\Gamma(n_0)}\int_0^t (t-\tau)^{n_0-1}(D_{0+}^{n_0-\alpha_0}g)(\tau)d\tau = I_{0+}^{n_0}D_{0+}^{n_0-\alpha_0}g(t)\Bigg)$$

$$= I_{0+}^{n_0}D_{0+}^{n_0-\alpha_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}I_{0+}^{\alpha_0}\left[\sum_{i=1}^{m}a_i(t)I_{\tau+}^{\alpha_0-\alpha_i}\right]^k \sum_{i=1}^{m}a_i(t)I_{0+}^{n_0-\alpha_i}D_{0+}^{n_0-\alpha_0}g(t), \quad (15)$$

where

$$I_{0+}^{n_0}D_{0+}^{n_0-\alpha_0}g(t) = I_{0+}^{n_0-\alpha_0+\alpha_0}D_{0+}^{n_0-\alpha_0}g(t) = I_{0+}^{\alpha_0}I_{0+}^{n_0-\alpha_0}D_{0+}^{n_0-\alpha_0}g(t). \quad (16)$$

In (16), it must be true that $g(t)\in I_{0+}^{n_0-\alpha_0}(L_1)$ to hold true that $I_{0+}^{n_0-\alpha_0}D_{0+}^{n_0-\alpha_0}g(t) = g(t)$. On the other hand, $g_{1-(n_0-\alpha_0)}(t) = I_{0+}^{1-(n_0-\alpha_0)}g \in AC[0,T]$ since $0 < n_0-\alpha_0 \leqslant 1$. And $g_{1-(n_0-\alpha_0)}(0) = I_{0+}^{1-(n_0-\alpha_0)}g(0) = 0$ because $1-(n_0-\alpha_0) > 0$, $g\in C[0,T]$.

So the relation $g(t)\in I_{0+}^{n_0-\alpha_0}(L_1)$ can be obtained from Lemma 1 and the expression (16) can be expressed as

$$I_{0+}^{\alpha_0}I_{0+}^{n_0-\alpha_0}D_{0+}^{n_0-\alpha_0}g(t) = I_{0+}^{\alpha_0}g(t).$$

Moreover, since $n_0-\alpha_h = n_0-\alpha_h+\alpha_0-\alpha_0 = \alpha_0-\alpha_h+n_0-\alpha_0 > 0$ and $I_{0+}^{n_0-\alpha_0}D_{0+}^{n_0-\alpha_0}g(t) = g(t)$, the expression $I_{0+}^{n_0-\alpha_i}D_{0+}^{n_0-\alpha_0}g(t)$ from the second term of (15) is reduced to

$$I_{0+}^{n_0-\alpha_i}D_{0+}^{n_0-\alpha_0}g(t) = I_{0+}^{\alpha_0-\alpha_i}I_{0+}^{n_0-\alpha_0}D_{0+}^{n_0-\alpha_0}g(t) = I_{0+}^{\alpha_0-\alpha_i}g(t)$$

(15) can be expressed as follow:

$$\begin{aligned}
y(t) &= I_{0+}^{\alpha_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}I_{0+}^{\alpha_0}\left[\sum_{i=1}^{m}a_i(t)I_{0+}^{\alpha_0-\alpha_i}\right]^k \sum_{i=1}^{m}a_i(t)I_{0+}^{\alpha_0-\alpha_i}g(t) \\
&= I_{0+}^{\alpha_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}I_{0+}^{\alpha_0}\left[\sum_{i=1}^{m}a_i(t)I_{0+}^{\alpha_0-\alpha_i}\right]^{k+1} g(t) \\
&= I_{0+}^{\alpha_0}g(t) + \sum_{k=1}^{\infty}(-1)^{k}I_{0+}^{\alpha_0}\left[\sum_{i=1}^{m}a_i(t)I_{0+}^{\alpha_0-\alpha_i}\right]^{k} g(t) \\
&= \sum_{k=0}^{\infty}(-1)^{k}I_{0+}^{\alpha_0}\left[\sum_{i=1}^{m}a_i(t)I_{0+}^{\alpha_0-\alpha_i}\right]^{k} g(t),
\end{aligned}$$

Therefore

$$y(t) = \sum_{k=0}^{\infty}(-1)^{k}I_{0+}^{\alpha_0}\left[\sum_{i=1}^{m}a_i(t)I_{0+}^{\alpha_0-\alpha_i}\right]^{k} g(t).$$

can be obtained.

② If $\Xi_{0+}^{n_0-\alpha_0} = {}^cD_{0+}^{n_0-\alpha_0}$, then

$$\begin{aligned} y(t) &= \int_0^t G_c(t;\tau)^c D_{0+}^{n_0-\alpha_0} g(\tau)d\tau \\ &= \int_0^t \left[\Phi_{n_0}(t-\tau) + \sum_{k=0}^{\infty}(-1)^{k+1} I_{\tau+}^{\alpha_0}\left[\sum_{i=1}^m a_i(t) I_{\tau+}^{\alpha_0-\alpha_i}\right]^k \sum_{i=1}^m a_i(t)\Phi_{n_0-\alpha_i}(t-\tau)\right] \\ &\quad \times {}^c D_{0+}^{n_0-\alpha_0} g(\tau)d\tau \\ &= \int_0^t \frac{(t-\tau)^{n_0-1}}{\Gamma(n_0)} {}^c D_{0+}^{n_0-\alpha_0} g(\tau)d\tau \\ &\quad + \sum_{k=0}^{\infty}(-1)^{k+1} I_{\tau+}^{\alpha_0}\left[\sum_{i=1}^m a_i(t) I_{\tau+}^{\alpha_0-\alpha_i}\right]^k \sum_{i=1}^m a_i(t)\int_0^t \Phi_{n_0-\alpha_i}(t-\tau)\ {}^c D_{0+}^{n_0-\alpha_0} g(\tau)d\tau \end{aligned}$$

can be obtained. Taking account of

$${}^c D_{0+}^{n_0-\alpha_0} g(t) = D_{0+}^{n_0-\alpha_0}\left[g(t) - \sum f(0)\Phi_1(t)\right] = D_{0+}^{n_0-\alpha_0} g(t),$$

$$\int_0^t \frac{(t-\tau)^{n_0-1}}{\Gamma(n_0)}\ {}^c D_{0+}^{n_0-\alpha_0} g(\tau)d\tau = \frac{1}{\Gamma(n_0)}\int_0^t (t-\tau)^{n_0-1}(D_{0+}^{n_0-\alpha_0} g)(\tau)d\tau = I_{0+}^{n_0} D_{0+}^{n_0-\alpha_0} g(t),$$

and

$$\begin{aligned} \int_0^t \Phi_{n_0-\alpha_i}(t-\tau)\ {}^c D_{0+}^{n_0-\alpha_0} g(\tau)d\tau &= \int_0^t \Phi_{n_0-\alpha_i}(t-\tau)(D_{0+}^{n_0-\alpha_0} g)(\tau)d\tau \\ &= I_{0+}^{n_0-\alpha_i} D_{0+}^{n_0-\alpha_0} g(t) \\ &= I_{0+}^{n_0-\alpha_i+\alpha_0-\alpha_0} D_{0+}^{n_0-\alpha_0} g(t) = I_{0+}^{\alpha_0-\alpha_i} I_{0+}^{n_0-\alpha_0} D_{0+}^{n_0-\alpha_0} g(t) \\ &= I_{0+}^{\alpha_0-\alpha_i} g(t). \end{aligned}$$

Therefore the following expression can be lead.

$$\begin{aligned} y(t) &= I_{0+}^{n_0} D_{0+}^{n_0-\alpha_0} g(t) + \sum_{k=0}^{\infty}(-1)^{k+1} I_{0+}^{\alpha_0}\left[\sum_{i=1}^m a_i(t) I_{0+}^{\alpha_0-\alpha_i}\right]^k \sum_{i=1}^m a_i(t) I_{0+}^{\alpha_0-\alpha_i} g(t) \\ &= I_{0+}^{n_0-\alpha_0+\alpha_0} D_{0+}^{n_0-\alpha_0} g(t) + \sum_{k=0}^{\infty}(-1)^{k+1} I_{0+}^{\alpha_0}\left[\sum_{i=1}^m a_i(t) I_{0+}^{\alpha_0-\alpha_i}\right]^{k+1} g(t) \\ &= I_{0+}^{\alpha_0} g(t) + \sum_{k=1}^{\infty}(-1)^k I_{0+}^{\alpha_0}\left[\sum_{i=1}^m a_i(t) I_{0+}^{\alpha_0-\alpha_i}\right]^k g(t). \end{aligned}$$

i.e.

$$y(t) = \int_0^t G_c(t;\tau)\Xi_{0+}^{n_0-\alpha_0} g(\tau)d\tau = \sum_{k=0}^{\infty}(-1)^k I_{0+}^{\alpha_0}\left[\sum_{i=1}^m a_i(t) I_{0+}^{\alpha_0-\alpha_i}\right]^k g(t). \quad \square$$

REMARK 3. The method of finding out the solution of (1) and (2) by using Green function of Caputo fractional differential operator ${}^{c}L(D_{\tau+})$ like Theorem 2 is called *Generalized Green Function Method.*

3.3. Representation of Solution of the initial value problem of non-homogeneous equation by modified Green function method

As seen in section 3.2, the generalized Green function method could be applied to Caputo fractional differential equations only in special cases.

The following theorem shows a modified Green function method which can be applied to Caputo fractional differential equations in every case.

THEOREM 3. *If $g(t),\ a_i(t) \in C[0,T],\ i = 1,\cdots,m$, then there exists the unique solution $y(t) \in C^{\alpha_0,n_0-1}[0,T]$ to the initial value problem of the non-homogeneous equation (1) with homogeneous initial condition (2) and it is written by*

$$y(t) = \int_0^t G_R(t;\tau)g(\tau)d\tau = \sum_{k=0}^{\infty}(-1)^k I_{0+}^{\alpha_0}\left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i}\right]^k g(t), \tag{17}$$

where $G_R(t;\tau)$ is the Green function of the Riemann-Liouville fractional differential operator ${}^{R}L(D_{\tau+})$ represented by (7).

Proof. Similarly to the above theorems, the existence and uniqueness of solution $y(t) \in C^{\alpha_0,n_0-1}[0,T]$ of (1) and (2) can be claimed. Therefore, it just needs to be verified that (17) holds true by calculating the first part of (17).

So,

$$\begin{aligned}
y(t) &= \int_0^t G_R(t;\tau)g(\tau)d\tau \\
&= \int_0^t \Phi_{\alpha_0}(t-\tau)d\tau + \sum_{k=0}^{\infty}(-1)^{k+1}\int_0^t I_{\tau+}^{\alpha_0}\left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i}\right]^k \\
&\quad \times \sum_{i=1}^{m} a_i(t)\Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau)d\tau \\
&= I_{0+}^{\alpha_0}g(t) + \sum_{k=0}^{\infty}(-1)^{k+1}\int_0^t \left[\frac{1}{\Gamma(\alpha_0)}\int_\tau^t (t-\xi)^{\alpha_0-1}\left[\sum_{i=1}^{m} a_i(\xi) I_{\tau+}^{\alpha_0-\alpha_i}\right]^k \right. \\
&\quad \left. \times \sum_{i=1}^{m} a_i(\xi)\Phi_{\alpha_0-\alpha_i}(\xi-\tau)g(\tau)d\xi\right]d\tau
\end{aligned}$$

can be obtained. (Interchanging the order of integration by Dirichlet's formula)

$$= I_{0+}^{\alpha_0} g(t) + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{\Gamma(\alpha_0)} \int_0^t \left[\int_0^{\xi} (t-\xi)^{\alpha_0-1} \left[\sum_{i=1}^{m} a_i(\xi) I_{\tau+}^{\alpha_0-\alpha_i} \right]^k \right.$$

$$\left. \times \sum_{i=1}^{m} a_i(\xi) \Phi_{\alpha_0-\alpha_i}(\xi-\tau) g(\tau) d\tau \right] d\xi$$

$$= I_{0+}^{\alpha_0} g(t) + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-\xi)^{\alpha_0-1} \left[\int_0^{\xi} \left[\sum_{i=1}^{m} a_i(\xi) I_{\tau+}^{\alpha_0-\alpha_i} \right]^k \right.$$

$$\left. \times \sum_{i=1}^{m} a_i(\xi) \Phi_{\alpha_0-\alpha_i}(\xi-\tau) g(\tau) d\tau \right] d\xi$$

$$= I_{0+}^{\alpha_0} g(t) + \sum_{k=0}^{\infty} (-1)^{k+1} I_{0+}^{\alpha_0} \left[\int_0^t \left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i} \right]^k \sum_{i=1}^{m} a_i(t) \Phi_{\alpha_0-\alpha_i}(t-\tau) g(\tau) d\tau \right]. \quad (18)$$

In the second term of the above,

$$\int_0^t \left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i} \right]^k \sum_{i=1}^{m} a_i(t) \Phi_{\alpha_0-\alpha_i}(t-\tau) g(\tau) d\tau.$$

is calculated. Now

$$A := \int_0^t \left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i} \right]^k \sum_{i=1}^{m} a_i(t) \Phi_{\alpha_0-\alpha_i}(t-\tau) g(\tau) d\tau.$$

It can be inductively calculated for $k = 0, 1, 2, \cdots$.

When $k = 0$,

$$\begin{aligned} A &= \int_0^t \sum_{i=1}^{m} a_i(t) \Phi_{\alpha_0-\alpha_i}(t-\tau) g(\tau) d\tau \\ &= \sum_{i=1}^{m} a_i(t) \int_0^t \Phi_{\alpha_0-\alpha_i}(t-\tau) g(\tau) d\tau \\ &= \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^0 \sum_{i=1}^{m} a_i(t) \int_0^t \Phi_{\alpha_0-\alpha_i}(t-\tau) g(\tau) d\tau. \end{aligned}$$

When $k = 1$,

$$\begin{aligned} A &= \int_0^t \left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i} \right]^1 \sum_{i=1}^{m} a_i(t) \Phi_{\alpha_0-\alpha_i}(t-\tau) g(\tau) d\tau \\ &= \sum_{i=1}^{m} a_i(t) \int_0^t \left[I_{\tau+}^{\alpha_0-\alpha_i} \sum_{i=1}^{m} a_i(t) \Phi_{\alpha_0-\alpha_i}(t-\tau) g(\tau) \right] d\tau \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} a_i(t) \int_0^t \left[\frac{1}{\Gamma(\alpha_0-\alpha_i)} \int_\tau^\xi (\xi-\eta)^{\alpha_0-\alpha_i-1} \sum_{i=1}^{m} a_i(\eta)\Phi_{\alpha_0-\alpha_i}(\eta-\tau)g(\tau)d\eta \right] d\tau \\
&= \sum_{i=1}^{m} a_i(t) \frac{1}{\Gamma(\alpha_0-\alpha_i)} \int_0^\xi \left[\int_0^\eta (\xi-\eta)^{\alpha_0-\alpha_i-1} \sum_{i=1}^{m} a_i(\eta)\Phi_{\alpha_0-\alpha_i}(\eta-\tau)g(\tau)d\tau \right] d\eta \\
&= \sum_{i=1}^{m} a_i(t) \frac{1}{\Gamma(\alpha_0-\alpha_i)} \int_0^\xi (\xi-\eta)^{\alpha_0-\alpha_i-1} \sum_{i=1}^{m} a_i(\eta) \left[\int_0^\eta \Phi_{\alpha_0-\alpha_i}(\eta-\tau)g(\tau)d\tau \right] d\eta \\
&= \sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \sum_{i=1}^{m} a_i(\xi) \left[\int_0^\eta \Phi_{\alpha_0-\alpha_i}(\eta-\tau)g(\tau)d\tau \right] \\
&= \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^1 \sum_{i=1}^{m} a_i(t) \int_0^t \Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau)d\tau.
\end{aligned}
$$

When $k=2$,

$$
\begin{aligned}
A &= \int_0^t \left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i} \right]^2 \sum_{i=1}^{m} a_i(t)\Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau)d\tau \\
&= \int_0^t \left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i} \right]^1 \left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i} \right]^1 \sum_{i=1}^{m} a_i(t)\Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau)d\tau \\
&\quad \text{(Using the result in case of } k=1) \\
&= \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^1 \sum_{i=1}^{m} a_i(t) \int_0^t \left[I_{\tau+}^{\alpha_0-\alpha_i} \sum_{i=1}^{m} a_i(t)\Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau) \right] d\tau \\
&= \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^1 \int_0^t \left[\sum_{i=1}^{m} a_i(t) I_{\tau+}^{\alpha_0-\alpha_i} \right] \sum_{i=1}^{m} a_i(t)\Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau)d\tau \\
&= \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^1 \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^1 \sum_{i=1}^{m} a_i(t) \int_0^t \Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau)d\tau \\
&= \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^2 \sum_{i=1}^{m} a_i(t) \int_0^t \Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau)d\tau.
\end{aligned}
$$

Therefore the express can be inductively written as

$$
\begin{aligned}
A &= \int_0^t \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^k \sum_{i=1}^{m} a_i(t)\Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau)d\tau \\
&= \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^k \sum_{i=1}^{m} a_i(t) \int_0^t \Phi_{\alpha_0-\alpha_i}(t-\tau)g(\tau)d\tau.
\end{aligned}
$$

and it can be also rewritten by using the definition of fractional integral as follow;

$$A = \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^k \sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} g(t).$$

So, (18) can be expressed as

$$\begin{aligned} y(t) &= I_{0+}^{\alpha_0} g(t) + \sum_{k=0}^{\infty} (-1)^{k+1} I_{0+}^{\alpha_0} \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^k \sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} g(t) \\ &= I_{0+}^{\alpha_0} g(t) + \sum_{k=0}^{\infty} (-1)^{k+1} I_{0+}^{\alpha_0} \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^{k+1} g(t) \\ &= \sum_{k=0}^{\infty} (-1)^{k} I_{0+}^{\alpha_0} \left[\sum_{i=1}^{m} a_i(t) I_{0+}^{\alpha_0-\alpha_i} \right]^{k} g(t). \end{aligned}$$

Therefore (17) has been verified. □

DEFINITION 9. The method of finding out the solution of the initial value problem of linear non-homogeneous Caputo fractional differential equation by using Green function of Riemann-Liouville fractional differential operator ${}^cL(D_{\tau+})$ like in Theorem 3 is called *Modified Green Function Method*

REMARK 4. Theorem 3 yield a new Green function method which find out the solutions of initial value problems of non-homogeneous Caputo fractional differential equations by using the Green function of Riemann-Liouville differential operator not Caputo type and the method can be applied to every type of equations.

4. Conclusion

In this paper, We have discussed that the solution of linear inhomogeneous Caputo fractional differential equation with continuous variable coefficient under homogeneous initial condition can be represented by classical Green function, generalized Green function and modified Green function. For future research, we will study the representation of solution to semi-linear inhomogeneous Caputo fractional differential equation with continuous variable coefficient by using Green function.

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(Received November 28, 2016)

Sunae Pak
Faculty of Mathematics
Kim Il Sung University
Pyongyang, Democratic People's Republic of Korea
e-mail: 15204692651@163.com

Huichol Choi
Faculty of Mathematics
Kim Il Sung University
Pyongyang, Democratic People's Republic of Korea
e-mail: 18348668513@163.com

Kinam Sin
Faculty of Mathematics
Kim Il Sung University
Pyongyang, Democratic People's Republic of Korea
and
Department of Mathematics
Harbin Institute of Technology
Harbin 150001, PR China
e-mail: 15bf12025@hit.edu.cn