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# SOLVABILITY OF A NON–LOCAL PROBLEM WITH INTEGRAL GLUING CONDITION FOR MIXED TYPE EQUATION WITH ERDELYI–KOBER OPERATORS

OBIDJON KH. ABDULLAEV

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*Abstract.* In this paper the existence and the uniqueness of solution of non-local problem with integral gluing condition for mixed type equation are investigated. Considering loaded parabolic-hyperbolic equation involve the Caputo fractional derivative and Erdelyi-Kober integrals. The uniqueness of solution is proved by the method of integral energy and the existence is proved by the method of integral equations.

## 1. Introduction

There has been significant development in fractional differential equations in recent years; see the monographs of A. A. Kilbas, H. M. Srivastava, J. J. Trujillo [1], K. S. Miller and B. Ross [2], I. Podlubny [3], S. G. Samko, A. A. Kilbas, O. I. Marichev [4] and the references therein. Various phenomena in physics, like diffusion in a disordered or fractal medium, or in image analysis, or in risk management have been modeled by means of fractional partial differential equations. In general, there exists no method that yields an exact solution for these equations. Indeed, we can find numerous applications in viscoelasticity, neurons, electrochemistry, control, porous media, electromagnetism, etc., (for details, see [5], [6], [7], [8], [9]).

In research papers [10], [11] the authors considered some classes of initial value problems for functional differential equations involving Riemann-Liouville and Caputo fractional derivatives of order. BVPs for the mixed type equations involving the Caputo and the Riemann-Liouville fractional differential operators were investigated too (see works [12], [13] and references therein).

Note that with intensive research on problem of optimal control of the agroeconomical system, regulating the label of ground waters and soil moisture, it has become necessary to investigate a new class of equations called “loaded equations”. For the first time it was given the most general definition of a “loaded equations” and various loaded equations are classified in detail by A. M. Nakhushev (see [14]). In this direction, in works [15], [12], [16], [17] was investigated, some local and non-local problems for the loaded mixed type equations with integral and integral-differential operators.

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## 2. Preliminaries

### 2.1. Integral and differential operators fractional order

DEFINITION 1. Let $f(x)$ be an absolutely continuous function over $(a,b)$. Then the left and right Riemann-Liouville fractional integrals order $\alpha$ ($\alpha \in R^{+}$) (respectively) are (see [1], p. 69)

$$\left(I_{a+}^{\alpha} f\right) x = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} f(t)(x-t)^{\alpha-1} dt, \quad x>a \tag{1}$$

$$\left(I_{-b}^{\alpha} f\right) x = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} f(t)(t-x)^{\alpha-1} dt, \quad x<b. \tag{2}$$

The Riemann-Liouville fractional derivatives $D_{ax}^{\alpha} f$ and $D_{xb}^{\alpha} f$ of order $\alpha$ ($\alpha \in R^{+}$) are defined by (see [1], p. 26):

$$(D_{ax}^{\alpha} f) x = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad n=[\alpha]+1, \quad x>a; \tag{3}$$

$$(D_{xb}^{\alpha} f) x = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^{n} \int_{x}^{b} \frac{f(t)}{(t-x)^{\alpha-n+1}} dt, \quad n=[\alpha]+1, \quad x<b; \tag{4}$$

respectively, where $[\alpha]$ is the integer part of $\alpha$.

In particular, for $\alpha = N \cup \{0\}$ we have

$$\left(D_{ax}^{0} f\right) x = f(x), \quad \left(D_{xb}^{0} f\right) x = f(x), \quad \left(D_{ax}^{n} f\right) x = f^{(n)}(x);$$

$$\left(D_{xb}^{n} f\right) x = (-1)^{n} f^{(n)}(x), \quad n \in N.$$

where $f^{(n)}(x)$ is the usual derivative of $f(x)$ of order $n$.

DEFINITION 2. Caputo fractional derivatives ${}_{C}D_{ax}^{\alpha} f$ and ${}_{C}D_{xb}^{\alpha} f$ of order $\alpha > 0$ ($\alpha \notin N \cup \{0\}$) are defined by (see [1], p. 92):

$$({}_{C}D_{ax}^{\alpha} f) x = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad n=[\alpha]+1, \quad x>a; \tag{5}$$

$$({}_{C}D_{xb}^{\alpha} f) x = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \quad n=[\alpha]+1, \quad x<b; \tag{6}$$

respectively.

From (3)–(6), as a conclusion we will have:

$$({}_CD_{ax}^{\alpha}f)\,x = \mathrm{si}\,gn^k(x-a)\left(I_{ax}^{\alpha-k}f^{(k)}\right)x,\ \ k-1<\alpha\leqslant k,\ k\in N;$$

consequently, while for $\alpha\in N\cup\{0\}$ we have

$$\left({}_CD_{ax}^{0}f\right)x=f(x),\ \ \left({}_CD_{xb}^{0}f\right)x=f(x),\ \ \left({}_CD_{ax}^{n}f\right)x=f^{(n)}(x);$$

$$({}_CD_{xb}^{n}f)\,x=(-1)^n f^{(n)}(x),\ \ n\in N.$$

The right- and left-hand sided Erdelyi-Kober fractional integrals of the orders $\delta$ and $\alpha$, respectively, are defined by

$$\left(I_{\beta}^{\gamma,\delta}f\right)(x)=\frac{\beta}{\Gamma(\delta)}x^{-\beta(\gamma+\delta)}\int\limits_0^x\left(x^{\beta}-t^{\beta}\right)^{\delta-1}t^{\beta(\gamma+1)-1}f(t)dt,\ \ \delta,\beta>0,\ \gamma\in R, \quad (7)$$

$$\left(J_{\beta}^{\gamma,\alpha}f\right)(x)=\frac{\beta}{\Gamma(\alpha)}x^{\beta\gamma}\int\limits_x^{\infty}\left(t^{\beta}-x^{\alpha}\right)^{\alpha-1}t^{-\beta(\gamma+\alpha-1)-1}f(t)dt,\ \ \alpha,\beta>0,\ \gamma\in R, \quad (8)$$

These operators have been used many authors, in particular, to obtain solutions of the single, dual and triple integral equations possessing special functions of mathematical physics as their kernels. For the theory and applications of Erdelyi-Kober fractional integrals see [21].

## 3. Problem formulation and main functional relations

This paper deals the existence and uniqueness of solution of the non-local problem with integral gluing condition for loaded mixed type equation involving the Caputo fractional derivative.

We consider the equation:

$$0=\begin{cases}u_{xx}-{}_CD_{oy}^{\alpha}u+p(x,y)\left(I_{\beta}^{\gamma,\delta}u\right)x, & \text{at } y>0\\ u_{xx}-u_{yy}-q(\xi,\eta)\left(I_{\beta}^{\gamma,\delta}u\right)\eta, & \text{at } y<0\end{cases} \quad (9)$$

with operators (see (5) and (7)):

$${}_CD_{oy}^{\alpha}u=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^y\frac{u_t(x,t)}{(y-t)^{\alpha}}dt, \quad (10)$$

$$\left(I_{\beta}^{\gamma,\delta}u\right)x=\frac{\beta}{\Gamma(\delta)}x^{-\beta(\gamma+\delta)}\int\limits_0^x\frac{t^{\beta(\gamma+1)-1}}{\left(x^{\beta}-t^{\beta}\right)^{1-\delta}}u(t,0)dt \quad (11)$$

where $\xi=x+y$, $\eta=x-y$, $0<\alpha,\ \beta,\ \gamma,\ \delta<1$, moreover $0<\gamma+\delta<1$.

Let's, $\Omega$ is domain, bounded with segments:$A_1A_2 = \{(x,y):\ x=1,\ 0<y<h\}$, $B_1B_2 = \{(x,y):\ x=0,\ 0<y<h\}$, $B_2A_2 = \{(x,y):\ y=h,\ 0<x<1\}$ at the $y>0$, and characteristics: $A_1C: x-y=1$; $B_1C: x+y=0$ of the equation (1) at $y<0$, where $A_1(1;0), A_2(1;h), B_1(0;0)$, $B_2(0;h)$, $C\left(\frac{1}{2};-\frac{1}{2}\right)$.

Introduce designations: $\theta(x) = \frac{x}{2} - i\cdot\frac{x}{2}$, $i^2=-1$. $\Omega^+ = \Omega\cap(y>0)$, $\Omega^- = \Omega\cap(y<0)$, $I_1 = \{x: 0<x<1\}$, $I_2 = \{y: 0<y<h\}$. In the domain of $\Omega$ the following problem is investigated.

PROBLEM I. To find a solution $u(x,y)$ of the equation (9) from the following class of functions:

$$W = \left\{u(x,y):\ u(x,y)\in C(\bar{\Omega})\cap C^2(\Omega^-),\ u_{xx}\in C\left(\Omega^+\right), {}_C D^{\alpha}_{oy}u \in C\left(\Omega^+\right)\right\}$$

satisfies boundary conditions:

$$u(x,y)\Big|_{A_1A_2} = \varphi(y),\ \ 0\leqslant y\leqslant h, \tag{12}$$

$$u(x,y)\Big|_{B_1B_2} = \psi(y),\ \ 0\leqslant y\leqslant h, \tag{13}$$

$$\frac{d}{dx}u(\theta(x)) = a(x)u_y(x,0) + b(x)u_x(x,0) + c(x)u(x,0) + d(x),\ \ x\in I_1. \tag{14}$$

and gluing condition:

$$\lim_{y\to+0} y^{1-\alpha}u_y(x,y) = \lambda_1(x)u_y(x,-0) + \lambda_2(x)\int_0^x r(t)u(t,0)dt,\ \ (x,0)\in A_1B_1, \tag{15}$$

where $\varphi(y)$, $\psi(y)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$ and $\lambda_j(x)$, are given functions, such that $\sum\limits_{j=1}^{2}\lambda_j^2(x)\neq 0$.

In fact the equation (9) at $y\leqslant 0$ on the characteristics coordinate $\xi = x+y$ and $\eta = x-y$ totally looks like:

$$u_{\xi\eta} = \frac{\beta}{4\Gamma(\delta)}q(\xi,\eta)\eta^{-\beta(\gamma+\delta)}\int_0^{\eta}\frac{t^{\beta(\gamma+1)-1}}{\left(\eta^\beta - t^\beta\right)^{1-\delta}}u(t,0)dt. \tag{16}$$

Well known, that a solution of the Cauchy problem for equation (9) in the domain of $\Omega^-$ with initial dates

$$u(x,0) = \tau(x),\ \ 0\leqslant x\leqslant 1;\ \ u_y(x,-0) = \nu^-(x),\ \ 0<x<1 \tag{17}$$

can be represented as follows:

$$u(x,y) = \frac{\tau(x+y)+\tau(x-y)}{2} - \frac{1}{2}\int_{x+y}^{x-y}\nu^-(t)dt$$
$$+\frac{\beta}{4\Gamma(\delta)}\int_{x+y}^{x-y}d\eta\int_{x+y}^{\eta}\eta^{-\beta(\gamma+\delta)}q(\xi,\eta)d\xi\int_0^{\eta}\frac{t^{\beta(\gamma+1)-1}}{\left(\eta^\beta - t^\beta\right)^{1-\delta}}u(t,0)dt. \tag{18}$$

After using condition (14) and taking (11) into account, from (18) we will get:

$$(2a(x)+1)\,\nu^-(x) = \frac{\beta}{2\Gamma(\delta)}\int_0^x x^{-\beta(\gamma+\delta)}q(\xi,x)d\xi\int_0^x \frac{t^{\beta(\gamma+1)-1}}{\left(x^\beta-t^\beta\right)^{1-\delta}}\tau(t)dt$$
$$+(1-2b(x))\,\tau'(x)-2c(x)\tau(x)-2d(x) \tag{19}$$

Due to equality:

$$\int_0^x \frac{t^{\beta\gamma}}{\left(x^\beta-t^\beta\right)^{1-\delta}}\tau(t)dt^\beta = \int_0^{x^\beta}\frac{t^\gamma}{\left(x^\beta-t\right)^{1-\delta}}\tau\left(t^{1/\beta}\right)dt$$

functional relation (19) we can rewrite as:

$$(2\tilde{a}(x)+1)\,\tilde{\nu}^-(x) = Q(x)\int_0^x \frac{t^\gamma}{(x-t)^{1-\delta}}\tilde{\tau}(t)dt+\left(1-2\tilde{b}(x)\right)\tilde{\tau}'(x)-2\tilde{c}(x)\tilde{\tau}(x)-2\tilde{d}(x) \tag{20}$$

where $\tilde{a}(x)=a\left(x^{1/\beta}\right)$ (and for other functions), $Q(x)=\frac{1}{2\Gamma(\beta)}\int\limits_0^{x^{1/\beta}}\frac{q(\xi,x^{1/\beta})}{x^{\gamma+\delta}}d\xi$. Considering designations and gluing condition (15) we have

$$\nu^+(x)=\lambda_1(x)\nu^-(x)+\lambda_2(x)\int_0^x r(t)\tau(t)dt \tag{21}$$

On the other hand, from the Eq. (9) at $y\to+0$ taking (10), (11), (21) into account, and due to $\lim\limits_{y\to 0}D_{0y}^{\alpha-1}f(y)=\Gamma(\alpha)\lim\limits_{y\to 0}y^{1-\alpha}f(y)$ we get:

$$\tau''(x)-\Gamma(\alpha)\lambda_1(x)\nu^-(x)-\Gamma(\alpha)\lambda_2(x)\int_0^x r(t)\tau(t)dt$$
$$+\frac{x^{-\beta(\gamma+\delta)}}{\Gamma(\delta)}p(x,0)\int_0^{x^\beta}\frac{t^\gamma}{\left(x^\beta-t\right)^{1-\delta}}\tau\left(t^{1/\beta}\right)dt=0.$$

In the sequel, replacing $x^\beta\sim x$, finally we get

$$\tilde{\tau}''(x)-\Gamma(\alpha)\tilde{\lambda}_1(x)\tilde{\nu}^-(x)-\Gamma(\alpha)\tilde{\lambda}_2(x)\int_0^x t^{1/\beta-1}\tilde{r}(t)\tilde{\tau}(t)dt$$
$$+\frac{x^{-(\gamma+\delta)}}{\Gamma(\delta)}\tilde{p}(x,0)\int_0^x\frac{t^\gamma}{(x-t)^{1-\delta}}\tilde{\tau}(t)\,dt=0. \tag{22}$$

## 4. Uniqueness of solution of the Problem I

Known that if homogeneous problem has only trivial solution, then we can state that original problem has unique solution.

With this aim, we multiply to $\tilde{\tau}(x)$ the equation (22) and integrate from 0 to 1:

$$\int_0^1 \tilde{\tau}''(x)\tilde{\tau}(x)dx - \Gamma(\alpha)\int_0^1 \tilde{\lambda}_2(x)\tilde{\tau}(x)dx \int_0^x \frac{\tilde{r}(t)}{t^{1-1/\beta}}\tilde{\tau}(t)dt$$

$$-\Gamma(\alpha)\int_0^1 \tilde{\lambda}_1(x)\tilde{\tau}(x)\tilde{\nu}^-(x)dx + \int_0^1 \tilde{\tau}(x)\frac{\tilde{p}(x,0)}{x^{\gamma+\delta}}D_{0x}^{-\delta}x^{\gamma}\tilde{\tau}(x)dx = 0. \tag{23}$$

Obviously, that

$$\int_0^1 \tilde{\tau}''(x)\tilde{\tau}(x)dx \leqslant 0, \quad \text{at}, \quad \tau(1)=0,\ \tau(0)=0. \tag{24}$$

Investigating the integral:

$$\Gamma(\alpha)\int_0^1 \tilde{\lambda}_2(x)\tilde{\tau}(x)dx \int_0^x \frac{\tilde{r}(t)}{t^{1-1/\beta}}\tilde{\tau}(t)dt$$

$$= \frac{\Gamma(\alpha)}{2}\int_0^1 x^{1-1/\beta}\frac{\tilde{\lambda}_2(x)}{\tilde{r}(x)} d\left(\int_0^x \frac{\tilde{r}(t)}{t^{1-1/\beta}}\tilde{\tau}(t)dt\right)^2$$

$$= \frac{\Gamma(\alpha)}{2}\left[\frac{\tilde{\lambda}_2(1)}{\tilde{r}(1)}\left(\int_0^1 \frac{\tilde{r}(t)}{t^{1-1/\beta}}\tilde{\tau}(t)dt\right)^2 - \lim_{x\to 0}\left\{\frac{\tilde{\lambda}_2(x)}{x^{1/1-\beta}\tilde{r}(x)}\left(\int_0^x \frac{\tilde{r}(t)}{t^{1-1/\beta}}\tilde{\tau}(t)dt\right)^2\right\}\right]$$

$$-\frac{\Gamma(\alpha)}{2}\int_0^1 \left(\int_0^x \frac{\tilde{r}(t)}{t^{1-1/\beta}}\tilde{\tau}(t)dt\right)^2 \left(x^{1-1/\beta}\frac{\tilde{\lambda}_2(x)}{\tilde{r}(x)}\right)' dx$$

we infer that

$$\Gamma(\alpha)\int_0^1 \tilde{\lambda}_2(x)\tilde{\tau}(x)dx \int_0^x \frac{\tilde{r}(t)}{t^{1-1/\beta}}\tilde{\tau}(t)dt \geqslant 0$$

at

$$\frac{\tilde{\lambda}_2(1)}{\tilde{r}(1)} \geqslant 0, \quad \text{and} \quad \left(x^{1-1/\beta}\frac{\tilde{\lambda}_2(x)}{\tilde{r}(x)}\right)' \leqslant 0 \tag{25}$$

Note that

$$\lim_{x\to 0}\left\{\frac{\tilde{\lambda}_2(x)}{x^{1/\beta-1}\tilde{r}(x)}\left(\int_0^x \frac{\tilde{r}(t)}{t^{1-1/\beta}}\tilde{\tau}(t)dt\right)^2\right\} = 0,$$

will be get by using an inequality

$$\left(\int_0^x \frac{\tilde{r}(t)}{t^{1-1/\beta}}\tilde{\tau}(t)dt\right)^2 \leqslant const\cdot x^{2/\beta}.$$

Now, we will investigate the integral

$$I=\Gamma(\alpha)\int_0^1 \tilde{\lambda}_1(x)\tilde{\tau}(x)\tilde{\nu}^-(x)dx-\int_0^1 \tilde{\tau}(x)\frac{\tilde{p}(x,0)}{x^{\gamma+\delta}}D_{0x}^{-\delta}x^{\gamma}\tilde{\tau}(x)dx.$$

Taking (20) into account, at $d(x)\equiv 0$ and $1+2\tilde{a}(x)\neq 0$ we get:

$$I=\Gamma(\alpha)\int_0^1 \tilde{\tau}(x)A(x)dx\int_0^x \frac{t^{\gamma}}{(x-t)^{1-\delta}}\tilde{\tau}(t)dt+\Gamma(\alpha)\int_0^1 B(x)\tilde{\tau}(x)\tilde{\tau}'(x)dx$$
$$-\Gamma(\alpha)\int_0^1 C(x)\tilde{\tau}^2(x)dx-\frac{1}{\Gamma(\delta)}\int_0^1 \tilde{\tau}(x)\frac{\tilde{p}(x,0)}{x^{\gamma+\delta}}dx\int_0^x \frac{t^{\gamma}}{(x-t)^{1-\delta}}\tilde{\tau}(t)dt \qquad (26)$$

where

$$A(x)=\frac{Q(x)\tilde{\lambda}_1(x)}{1+2\tilde{a}(x)},\qquad B(x)=\frac{1-2\tilde{b}(x)}{1+2\tilde{a}(x)}\tilde{\lambda}_1(x),\qquad C(x)=\frac{2\tilde{c}(x)\tilde{\lambda}_1(x)}{1+2\tilde{a}(x)}. \qquad (27)$$

By using formulate [19]:

$$|x-t|^{-\gamma}=\frac{1}{\Gamma(\gamma)\cos\frac{\pi\gamma}{2}}\int_0^{\infty} z^{\gamma-1}\cos\left[z(x-t)\right]dz,\ \ 0<\gamma<1$$

after some simplifications from (26) we will get:

$$\Gamma(\alpha)\int_0^1 \tilde{\tau}(x)A(x)dx\int_0^x \frac{t^{\gamma}}{(x-t)^{1-\delta}}\tilde{\tau}(t)dt$$
$$=\frac{\Gamma(\alpha)}{2\Gamma(1-\delta)\sin\frac{\pi\delta}{2}}\int_0^{\infty} z^{-\delta}dz\int_0^1 A(x)x^{-\gamma}\left[dM^2(x,z)+dN^2(x,z)\right]$$

where $M(x,z)=\int_0^x \tilde{\tau}(t)t^{\gamma}\cos zt\,dt$, $N(x,z)=\int_0^x \tilde{\tau}(t)t^{\gamma}\sin zt\,dt$. Further, integrating by parts we have

$$\frac{\Gamma(\alpha)A(1)}{2\Gamma(1-\delta)\sin\frac{\pi\delta}{2}}\int\limits_0^\infty z^{-\delta}\left[M^2(1,z)+N^2(1,z)\right]dz$$
$$-\frac{\Gamma(\alpha)}{2\Gamma(1-\delta)\sin\frac{\pi\delta}{2}}\int\limits_0^\infty z^{-\delta}\lim_{x\to 0}\left\{\frac{A(x)}{x^\gamma}\left[M^2(x,z)+N^2(x,z)\right]\right\}dz$$
$$-\frac{\Gamma(\alpha)}{2\Gamma(1-\delta)\sin\frac{\pi\delta}{2}}\int\limits_0^\infty z^{-\delta}dz\int\limits_0^1\left(\frac{A(x)}{x^\gamma}\right)'\left[M^2(x,z)+N^2(x,z)\right]dx.$$

Due to estimation

$$\left(\int\limits_0^x \tilde{\tau}(t)t^\gamma\cos zt\,dt\right)^2+\left(\int\limits_0^x \tilde{\tau}(t)t^\gamma\sin zt\,dt\right)^2\leqslant const\cdot x^{2\gamma+2}, \tag{28}$$

we deduce, that

$$\lim_{x\to 0}\left\{\frac{A(x)}{x^\gamma}\left[\left(\int\limits_0^x \tilde{\tau}(t)t^\gamma\cos zt\,dt\right)^2+\left(\int\limits_0^x \tilde{\tau}(t)t^\gamma\sin zt\,dt\right)^2\right]\right\}=0.$$

Hence, finally we have:

$$\Gamma(\alpha)\int\limits_0^1\tilde{\tau}(x)A(x)dx\int\limits_0^x\frac{t^\gamma}{(x-t)^{1-\delta}}\tilde{\tau}(t)dt\geqslant 0$$

at

$$A(1)\geqslant 0,\qquad \left(\frac{A(x)}{x^\gamma}\right)'\leqslant 0. \tag{29}$$

Similarly, due to estimation (28) we can get:

$$\Gamma(\alpha)\int\limits_0^1\tilde{\tau}(x)\frac{\tilde{p}(x,0)}{x^{\gamma+\delta}}dx\int\limits_0^x\frac{t^\gamma}{(x-t)^{1-\delta}}\tilde{\tau}(t)dt\leqslant 0$$

at

$$\tilde{p}(1,0)\leqslant 0,\qquad \left(\frac{\tilde{p}(x,0)}{x^{\gamma+\delta}}\right)'\geqslant 0. \tag{30}$$

Due to (29), (30) and taking into account:

$$\Gamma(\alpha)\int\limits_0^1 B(x)\tilde{\tau}(x)\tilde{\tau}'(x)dx=\frac{\Gamma(\alpha)}{2}\int\limits_0^1 B(x)d\left(\tilde{\tau}^2(x)\right)dx$$

from (26) we will deduce that

$$I=\Gamma(\alpha)\int_0^1 \tilde{\lambda}_1(x)\tilde{\tau}(x)\tilde{\nu}^-(x)dx-\int_0^1 \tilde{\tau}(x)\frac{\tilde{p}(x,0)}{x^{\gamma+\delta}}D_{0x}^{-\delta}x^{\gamma}\tilde{\tau}(x)dx\geqslant 0 \tag{31}$$

if $A(1)\geqslant 0$, $\left(\frac{A(x)}{x^{\gamma}}\right)'\leqslant 0$, $\tilde{p}(1,0)\leqslant 0$, $\left(\frac{\tilde{p}(x,0)}{x^{\gamma+\delta}}\right)'\geqslant 0$, $B(x)\geqslant 0$ and $C(x)\leqslant 0$.

Thus, considering (24), (25) and (31) from (23) it is concluded, that $\tau(x)\equiv 0$. Hence, based on the solution of the first boundary problem for the Eq. (9) owing to account (13) and (14) we will get $u(x,y)\equiv 0$ in $\overline{\Omega}^+$. Further, from functional relations (20), taking into account $\tau(x)\equiv 0$ we deduce that $\nu^-(x)\equiv 0$. Consequently, based on the solution (18) we obtain $u(x,y)\equiv 0$ in closed domain $\overline{\Omega}^-$. As a conclusion we can formulate this theorem:

THEOREM 1. *If satisfy conditions*

$$\frac{\tilde{\lambda}_2(1)}{\tilde{r}(1)}\geqslant 0;\ A(1)\geqslant 0;\ \tilde{p}(1,0)\leqslant 0;\ B(x)\geqslant 0;\ C(x)\leqslant 0, \tag{32}$$

$$\left(\frac{A(x)}{x^{\gamma}}\right)'\leqslant 0;\ \left(\frac{\tilde{p}(x,0)}{x^{\gamma+\delta}}\right)'\geqslant 0;\ \left(x^{1-1/\beta}\frac{\tilde{\lambda}_2(x)}{\tilde{r}(x)}\right)'\leqslant 0, \tag{33}$$

*then, the solution $u(x,y)$ of the Problem I is unique.*

## 5. Existence of solution of the Problem I

THEOREM 2. *If satisfy conditions (32), (33) and*

$$p(x,y)\in C\left(\overline{\Omega^+}\right)\cap C^2\left(\Omega^+\right),\ q(x,y)\in C\left(\overline{\Omega^-}\right)\cap C^2\left(\Omega^-\right), \tag{34}$$

$$\varphi(y),\psi(y)\in C\left(\overline{I_2}\right)\cap C^1\left(I_2\right),\ a(x),b(x),c(x),d(x)\in C^1\left(\overline{I_1}\right)\cap C^2\left(I_1\right) \tag{35}$$

*then the solution of the investigating problem is exist.*

*Proof.* Taking (20) into account from Eq. (22) we will obtain

$$\tau''(x)=f(x) \tag{36}$$

where

$$\begin{aligned} f(x)=&\Gamma(\alpha)\tilde{\lambda}_2(x)\int_0^x t^{1/\beta-1}\tilde{r}(t)\tilde{\tau}(t)dt+\Gamma(\alpha)A(x)\int_0^x (x-t)^{\delta-1}t^{\gamma}\tilde{\tau}(t)dt\\ &-\frac{1}{\Gamma(\delta)}x^{-\delta-\gamma}\tilde{p}(x,0)\int_0^x (x-t)^{\delta-1}t^{\gamma}\tilde{\tau}(t)dt\\ &-\Gamma(\alpha)B(x)\tilde{\tau}'(x)-\Gamma(\alpha)C(x)\tilde{\tau}(x)-D(x) \end{aligned} \tag{37}$$

and $D(x)=\frac{2\Gamma(\alpha)\tilde{\lambda}_1(x)\tilde{d}(x)}{1+2\tilde{a}(x)}$.

Solution of equation (36) together with conditions

$$\tau(0)=\psi(0),\ \ \tau(1)=\varphi(0) \tag{38}$$

has a form

$$\tau(x)=\int_0^x (x-t)f(t)dt-x\int_0^1 (1-t)f(t)dt+(1-x)\psi(0)+x\varphi(0) \tag{39}$$

Further, substituting (37) into (39) we obtain:

$$\begin{aligned}\tilde{\tau}(x)=&\Gamma(\alpha)\int_0^x s^{\gamma}\tilde{\tau}(s)ds\int_s^x \frac{(x-t)}{(t-s)^{1-\delta}}A(t)dt-\Gamma(\alpha)\int_0^x (x-t)C(t)\tilde{\tau}(t)dt\\ &-\Gamma(\alpha)x\int_0^1 s^{\gamma}\tilde{\tau}(s)ds\int_s^1 (1-t)(t-s)^{\delta-1}A(t)dt+\Gamma(\alpha)x\int_0^1 (1-t)C(t)\tilde{\tau}(t)dt\\ &+\Gamma(\alpha)\int_0^x s^{1/\beta-1}\tilde{\tau}(s)\tilde{r}(s)ds\int_s^x (x-t)\tilde{\lambda}_2(t)dt-\Gamma(\alpha)\int_0^x (x-t)B(t)\tilde{\tau}'(t)dt\\ &-\Gamma(\alpha)x\int_0^1 s^{1/\beta-1}\tilde{\tau}(s)\tilde{r}(s)ds\int_s^1 (1-t)\tilde{\lambda}_2(t)dt+\Gamma(\alpha)x\int_0^1 (1-t)B(t)\tilde{\tau}'(t)dt\\ &-\frac{\Gamma(\alpha)}{\Gamma(\delta)}\int_0^x s^{\gamma}\tilde{\tau}(s)ds\int_s^x \frac{(x-t)t^{-\delta-\gamma}}{(t-s)^{\delta-1}}\tilde{p}(t,0)dt\\ &+\frac{\Gamma(\alpha)}{\Gamma(\delta)}x\int_0^1 s^{\gamma}\tilde{\tau}(s)ds\int_s^1 \frac{(1-t)t^{-\delta-\gamma}}{(t-s)^{\delta-1}}\tilde{p}(t,0)dt\\ &-\int_0^x (x-t)D(t)dt+x\int_0^1 (1-t)D(t)dt+(1-x)\psi(0)+x\varphi(0)\end{aligned} \tag{40}$$

After some simplifications (40) we will rewrite as integral equation:

$$\tilde{\tau}(x)=\int_0^1 K(x,s)\tilde{\tau}(s)ds+f_1(x). \tag{41}$$

Here

$$K(x,s)=\begin{cases}K_1(x,s); & 0\leqslant s\leqslant x,\\ K_2(x,s); & x\leqslant s\leqslant 1.\end{cases} \tag{42}$$

$$K_1(x,s) = \Gamma(\alpha)\left[s^{\gamma}\int\limits_s^x (x-t)(t-s)^{\delta-1}A(t)dt-(x-s)C(s)\right]$$

$$-\Gamma(\alpha)x\left[s^{\gamma}\int\limits_s^1 (1-t)(t-s)^{\delta-1}A(t)dt-(1-s)C(s)\right]$$

$$+\Gamma(\alpha)\left[s^{1/\beta-1}\tilde{r}(s)\int\limits_s^x (x-t)\tilde{\lambda}_2(t)dt+B(t)-(x-t)B'(t)\right]$$

$$-\Gamma(\alpha)x\left[s^{1/\beta-1}\tilde{r}(s)\int\limits_s^1 (1-t)\tilde{\lambda}_2(t)dt+B(t)-(1-t)B'(t)\right]$$

$$-\frac{\Gamma(\alpha)}{\Gamma(\delta)}s^{\gamma}\left[\int\limits_s^x \frac{(x-t)t^{-\delta-\gamma}}{(t-s)^{\delta-1}}\tilde{p}(t,0)dt-x\int\limits_s^1 \frac{(1-t)t^{-\delta-\gamma}}{(t-s)^{\delta-1}}\tilde{p}(t,0)dt\right], \quad (43)$$

$$K_2(x,s) = \Gamma(\alpha)x\left[(1-s)C(s)-s^{\gamma}\int\limits_s^1 (1-t)(t-s)^{\delta-1}A(t)dt\right]$$

$$-\Gamma(\alpha)x\left[s^{1/\beta-1}\tilde{r}(s)\int\limits_s^1 (1-t)\tilde{\lambda}_2(t)dt+B(t)-(1-t)B'(t)\right]$$

$$+\frac{\Gamma(\alpha)}{\Gamma(\delta)}s^{\gamma}x\int\limits_s^1 \frac{(1-t)t^{-\delta-\gamma}}{(t-s)^{\delta-1}}\tilde{p}(t,0)dt. \quad (44)$$

$$f_1(x) = (1-x)\psi(0)+x\varphi(0)-\int\limits_0^x (x-t)D(t)dt+x\int\limits_0^1 (1-t)D(t)dt. \quad (45)$$

Due to class (34), (35) of the given functions and after some evaluations (43) and (44) from (42) and (45) we will conclude that $|K(x,t)| \leqslant const$, $|f_1(x)| \leqslant const$. Since kernel $K(x,t)$ is continuous and function in right-side $F(x)$ is continuously differentiable, solution of integral equation (41) we can write via resolvent-kernel:

$$\tilde{\tau}(x) = f_1(x)-\int\limits_0^1 \Re(x,s)f_1(s)ds,$$

where $\Re(x,s)$ is the resolvent-kernel of $K(x,s)$. Unknown function $\nu^-(x)$ we will found from (20). Solution of the Problem I in the domain $\Omega^+$ we write as follows [20]:

$$u(x,y)=\int_0^y G_\xi(x,y,0,\eta)\psi(\eta)d\eta-\int_0^y G_\xi(x,y,1,\eta)\varphi(\eta)d\eta+\int_0^1 G_0(x-\xi,y)\tau(\xi)d\xi$$
$$+\int_0^y\int_0^1 G(x,y,\xi,\eta)p(\xi,\eta)\left(I_\beta^{\gamma,\delta}\tau\right)\xi d\xi d\eta$$

Here $G_0(x-\xi,y)=\frac{1}{\Gamma(1-\alpha)}\int_0^y(y-\eta)^{-\alpha}G(x,\eta,\xi,0)d\eta$,

$$G(x,y,\xi,\eta)=\frac{(y-\eta)^{\alpha/2-1}}{2}\sum_{n=-\infty}^{\infty}\left[e_{1,\alpha/2}^{1,\alpha/2}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\alpha/2}}\right)-e_{1,\alpha/2}^{1,\alpha/2}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\alpha/2}}\right)\right]$$

is the Green's function of the first boundary problem Eq. (9) in the domain $\Omega^+$ with the Riemanne-Liouville fractional differential operator instead of the Caputo ones [18],

$$e_{1,\delta}^{1,\delta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\delta-\delta n)}$$

is the Wright type function. Solution of the Problem I in the domain $\Omega^-$ will be found by the formulate (18). Hence, the Theorem 2 is proved. □

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*Obidjon Kh. Abdullaev*
*National University of Uzbekistan*
*Universitetskaya – 4*
*Olmazar, Tashkent, 100125 Uzbekistan*
*e-mail:* obidjon.mth@gmail.com