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EXISTENCE THEORY FOR FRACTIONAL–ORDER NEUTRAL BOUNDARY VALUE PROBLEMS

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Abstract. A new class of Dirichlet boundary value problems of Caputo-Hadamard type fractional neutral differential equations and inclusions is studied. Expressions for Green's functions are derived to obtain the integral equation equivalent to the associated single-valued problem. Existence and uniqueness results are proved for single-valued and multivalued problems at hand. Examples demonstrating the application of the main results are presented. Finally we extend our discussion to the case of three-point nonlocal boundary conditions.

1. Introduction

Neutral functional differential equations appear in many mathematical models for several kinds of biological and physical phenomena [1].

Riemann-Liouville, Caputo and Hadamard type fractional differential equations and inclusions have been studied by many researchers and a variety of results ranging from theoretical development to analytic/numerical methods for solving these equations can be found in the recent related works. This investigation has been motivated by extensive applications of fractional-order equations in the mathematical modelling of numerous social and scientific problems. For applications details, we refer the reader to the works [2]–[3], while the results on functional fractional differential equations and inclusions can be found in [4]–[10]. However, the study on functional fractional neutral differential equations equipped with boundary conditions is yet to be initiated.

The main theme of the present work is to develop the existence criteria for the solutions of Caputo-Hadamard type fractional neutral differential equations and inclusions supplemented with Dirichlet and nonlocal boundary conditions. We begin our investigation with the following problem:

$$\begin{cases} D^{\omega}[D^{\kappa}y(t)-h(t,y(t))]=f(t,y(t)), \ \ t\in J:=[1,\mathfrak{T}], \mathfrak{T}>1, \\ y(1)=0, \ \ y(\mathfrak{T})=0, \end{cases} \tag{1}$$

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where D^{ρ} denotes the Caputo-Hadamard fractional derivatives of order $\rho \in (0,1)$, $\rho = \omega, \kappa$ and $f, h : J \times \mathbb{R} \to \mathbb{R}$ are single-valued appropriate functions. In Section 2, we discuss the existence and uniqueness of solutions for the problem (1).

In the second problem, we study the inclusions analog of (1) given by

$$\begin{cases} D^{\omega}[D^{\kappa}y(t) - h(t,y(t))] \in F(t,y(t)), \ \ t \in J := [1,\mathfrak{T}], \\ y(1) = 0, \ \ y(\mathfrak{T}) = 0, \end{cases} \tag{2}$$

where $F : J \times \mathbb{R} \to \mathscr{X}(\mathbb{R})$ is a multivalued map, $\mathscr{X}(\mathbb{R})$ denotes the family of all nonempty subsets of $\mathbb{R}$. The existence results for the problem (2) are presented in Section 3.

In Section 4, we demonstrate the application of the results obtained for problems (1) and (2). Section 5 is concerned with the study of fractional neutral differential equation considered in (1) with nonlocal three-point boundary conditions. We emphasize that the work presented in this article is a useful contribution towards theoretical development of fractional neutral differential equations and inclusions with Dirichlet and nonlocal three-point boundary conditions.

2. Main results for the problem (1)

We begin this section with some necessary definitions.

DEFINITION 1. [4] The Hadamard derivative of fractional order p for a function $\sigma : [1,\infty) \to \mathbb{R}$ is defined as

$$D^{p}\sigma(x) = \frac{1}{\Gamma(n-p)}\left(x\frac{d}{dx}\right)^{n}\int_{1}^{x}\left(\log\frac{x}{r}\right)^{n-p-1}\frac{\sigma(r)}{r}dr, \ \ n-1 < p < n, n = [p]+1,$$

where $[p]$ denotes the integer part of the real number p and $\log(\cdot) = \log_{e}(\cdot)$.

DEFINITION 2. [4] The Hadamard fractional integral of order p for a function σ is defined as

$$I^{p}\sigma(x) = \frac{1}{\Gamma(p)}\int_{1}^{x}\left(\log\frac{x}{r}\right)^{p-1}\frac{\sigma(r)}{r}dr, \ \ p > 0,$$

provided the integral exists.

In the following lemma, we obtain an integral equation equivalent to the problem (1).

LEMMA 1. *The problem (1) is equivalent to the following integral equation:*

$$y(t) = \int_{1}^{\mathfrak{T}} G_{h}(t,u)h(u,y(u))du + \int_{1}^{\mathfrak{T}} G_{f}(t,u)f(u,y(u))du, \tag{3}$$

where

$$G_h(t,u) = \begin{cases} \dfrac{A}{u}\Big[\Big(\log\dfrac{t}{u}\Big)^{\kappa-1}(\log\mathfrak{T})^{\kappa} - (\log t)^{\kappa}\Big(\log\dfrac{\mathfrak{T}}{u}\Big)^{\kappa-1}\Big], & u \leqslant t, \\ -\dfrac{A(\log t)^{\kappa}\Big(\log\frac{\mathfrak{T}}{u}\Big)^{\kappa-1}}{u}, & t \leqslant u, \end{cases} \tag{4}$$

$$G_f(t,u) = \begin{cases} \dfrac{B}{u}\Big[\Big(\log\dfrac{t}{u}\Big)^{\omega+\kappa-1}(\log\mathfrak{T})^{\kappa} - (\log t)^{\kappa}\Big(\log\dfrac{\mathfrak{T}}{u}\Big)^{\omega+\kappa-1}\Big], & u \leqslant t, \\ -\dfrac{B(\log t)^{\kappa}\Big(\log\frac{\mathfrak{T}}{u}\Big)^{\omega+\kappa-1}}{u}, & t \leqslant u, \end{cases} \tag{5}$$

$$A = \frac{1}{(\log\mathfrak{T})^{\kappa}\Gamma(\kappa)}, \qquad B = \frac{1}{(\log\mathfrak{T})^{\kappa}\Gamma(\omega+\kappa)}. \tag{6}$$

Proof. We know that the solution of Hadamard differential equation in (1) can be written as [4]

$$\begin{aligned} y(t) = {} & \frac{1}{\Gamma(\kappa)}\int_1^t \Big(\log\frac{t}{u}\Big)^{\kappa-1}\frac{h(u,y(u))}{u}du \\ & + \frac{1}{\Gamma(\omega+\kappa)}\int_1^t \Big(\log\frac{t}{u}\Big)^{\omega+\kappa-1}\frac{f(u,y(u))}{u}du + \frac{(\log t)^{\kappa}}{\Gamma(\kappa+1)}b_1 + b_2, \end{aligned} \tag{7}$$

where $b_1, b_2 \in \mathbb{R}$ are arbitrary constants. In view of the conditions $y(1) = 0, y(\mathfrak{T}) = 0$, it follows from (7) that $b_2 = 0$ and

$$\begin{aligned} b_1 = -\frac{\Gamma(\kappa+1)}{(\log\mathfrak{T})^{\kappa}}\Bigg\{ & \frac{1}{\Gamma(\kappa)}\int_1^{\mathfrak{T}} \Big(\log\frac{\mathfrak{T}}{u}\Big)^{\kappa-1}\frac{h(u,y(u))}{u}du \\ & + \frac{1}{\Gamma(\omega+\kappa)}\int_1^{\mathfrak{T}} \Big(\log\frac{\mathfrak{T}}{u}\Big)^{\omega+\kappa-1}\frac{f(u,y(u))}{u}du\Bigg\}. \end{aligned}$$

Inserting the values of b_1 and b_2 in (7), and using (4) and (5), we get the solution (3). By direct computation, one can show that $y(t)$ given by (3) satisfies the problem (1). This completes the proof. □

Relative to the problem (1), we introduce a fixed point operator $\mathscr{N} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by using Lemma 1 as follows

$$\mathscr{N}(y)(t) = \int_1^{\mathfrak{T}} G_h(t,u)h(u,y(u))du + \int_1^{\mathfrak{T}} G_f(t,u)f(u,y(u))du, \tag{8}$$

where $G_g(t,u)$ and $G_f(t,u)$ are respectively given by (4) and (5).

We need the following estimates in the sequel.

$$\begin{aligned} \max_{t\in[1,\mathfrak{T}]}\Big\{\int_1^{\mathfrak{T}} |G_h(t,u)|du\Big\} &= \frac{2(\log\mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} = \vartheta_1, \\ \max_{t\in[1,\mathfrak{T}]}\Big\{\int_1^{\mathfrak{T}} |G_f(t,u)|du\Big\} &= \frac{2(\log\mathfrak{T})^{\omega+\kappa}}{\Gamma(\omega+\kappa+1)} = \vartheta_2. \end{aligned} \tag{9}$$

THEOREM 1. (Uniqueness result) *Let* $f,h : J\times\mathbb{R}\to\mathbb{R}$ *be continuous functions satisfying the following conditions:*

(A_1) $|f(t,y)-f(t,x)|\leqslant \ell\|y-x\|, \ell>0,$ *for* $t\in J$ *and every* $y,x\in\mathbb{R}$*;*

(A_2) $|h(t,y)-h(t,x)|\leqslant k\|y-x\|, k\geqslant 0,$ *for* $t\in J$ *and every* $y,x\in\mathbb{R}$.

Then the problem (1) has a unique solution on the interval J *if*

$$k\vartheta_1+\ell\vartheta_2<1, \tag{10}$$

where $\vartheta_i (i=1,2)$ *are given by* (9)*.*

Proof. In order to apply contraction mapping principle, we need to establish that the operator $\mathcal{N}$ given by (8) is a contraction. To do so, we take $y,x\in C(J,\mathbb{R})$. Then

$$\begin{aligned}
&|\mathcal{N}(y)(t)-\mathcal{N}(x)(t)|\\
\leqslant &\int_1^{\mathfrak{T}}|G_h(t,u)[h(u,y(u))-h(u,x(u))]|du+\int_1^{\mathfrak{T}}|G_f(t,u)[f(u,y(u))-f(u,x(u))]|du\\
\leqslant &\left\{k\int_1^{\mathfrak{T}}|G_h(t,u)|du+\ell\int_1^{\mathfrak{T}}|G_f(t,u)|du\right\}\|y-x\|.
\end{aligned}$$

Taking the norm of the above inequality for $t\in[1,\mathfrak{T}]$ and using (9), we get

$$\|\mathcal{N}(y)-\mathcal{N}(x)\|\leqslant(k\vartheta_1+\ell\vartheta_2)\|y-x\|,$$

which shows that the operator $\mathcal{N}$ is a contraction in view of (10). In consequence, $\mathcal{N}$ has a unique fixed point by the conclusion of Banach's contraction mapping principle, which corresponds to a unique solution of the problem (1). □

Next we show the existence of solutions for the problem (1) by means of Krasnoselskii's fixed point theorem [11].

THEOREM 2. *Let* $f,g : J\times\mathbb{R}\to\mathbb{R}$ *be continuous functions and that* (A_2) *holds. In addition it is assumed that* $|f(t,y)|\leqslant\mu_1(t),\ |h(t,y)|\leqslant\mu_2(t),\forall(t,y)\in J\times\mathbb{R},$ *and* $\mu_1,\mu_2\in C(J,\mathbb{R}^+)$*. Then there exists at least one solution for the problem (1) on* J *provided that*

$$\frac{2k(\log\mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)}<1. \tag{11}$$

Proof. Let us define a set $E_\rho=\{y\in C(J,\mathbb{R}):\|y\|\leqslant\rho\}$, where

$$\rho\geqslant\|\mu_2\|\vartheta_1+\|\mu_1\|\vartheta_2,\ \|\mu_i\|=\sup_{t\in[1,\mathfrak{T}]}|\mu_i(t)|,$$

and $\vartheta_i (i=1,2)$ are given by (9). Introduce operators $\mathcal{N}_1$ and $\mathcal{N}_2$ on E_ρ as

$$\mathcal{N}_1(y)(t)=\int_1^T G_h(t,s)h(s,y(s))ds,\ \mathcal{N}_2(y)(t)=\int_1^T G_f(t,s)f(s,y(s))ds. \tag{12}$$

For any $y,z \in E_\rho$, we have

$$|\mathscr{N}_1 y(t) + \mathscr{N}_2 z(t)| \leqslant \int_1^{\mathfrak{T}} |G_h(t,s)||h(s,y(s))|ds + \int_1^{\mathfrak{T}} |G_f(t,s)||f(s,z(s))|ds$$
$$\leqslant \|\mu_2\|\vartheta_1 + \|\mu_1\|\vartheta_2 \leqslant \rho.$$

Hence $\|\mathscr{N}_1 y + \mathscr{N}_2 z\| \leqslant \rho$, which shows that $\mathscr{N}_1 y + \mathscr{N}_2 z \in E_\rho$. In view of the condition (11), one can easily verify that $\mathscr{N}_1$ is a contraction. Further the operator $\mathscr{N}_2$ is continuous by virtue of continuity of f and is uniformly bounded on E_ρ as

$$\|\mathscr{N}_2 y\| \leqslant \frac{2\|\mu_1\|(\log \mathfrak{T})^{\omega+\kappa}}{\Gamma(\omega+\kappa+1)}.$$

In order to show the compactness $\mathscr{N}_2$, let us set $\sup_{(t,y)\in[1,\mathfrak{T}]\times B_\rho} |f(t,y)| = \bar{f} < \infty$, and consequently, for $1 < \tau_1 < \tau_2 < \mathfrak{T}$, we have

$$|\mathscr{N}_2 y(\tau_2) - \mathscr{N}_2 y(\tau_1)| \leqslant \frac{\bar{f}}{\Gamma(\omega+\kappa+1)}\Big[|(\log\tau_2)^\kappa - (\log\tau_1)^\kappa|(\log T)^\omega$$
$$+\left|(\log\tau_2)^{\omega+\kappa} - (\log\tau_1)^{\omega+\kappa}\right| + |\log\tau_2/\tau_1|^{\omega+\kappa}\Big] \to 0,$$

independent of y when $\tau_2 - \tau_1 \to 0$. Thus, $\mathscr{N}_2$ is equicontinuous. So $\mathscr{N}_2$ is relatively compact on B_ρ. Hence, it follows by the Arzel$\acute{a}$-Ascoli theorem that $\mathscr{N}_2$ is compact on B_ρ. Thus the hypotheses of Krasnoselskii's fixed point theorem hold true and consequently there exists at least one solution for the problem (1) on J. □

3. Existence results for the problem (2)

Let $C(J,\mathbb{R})$ represent the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|,\ t\in J\}$, while the space of functions $x : J \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_1^{\mathfrak{T}} |x(t)|dt$ is denoted by $L^1(J,\mathbb{R})$.

For each $y \in C(J,\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,y(t)) \text{ on } J\}.$$

Here we recall that a multivalued operator $\mathscr{Q} : \mathscr{M} \to \mathscr{X}_{cl}(\mathscr{M})$ is said to be (a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(\mathscr{Q}(u),\mathscr{Q}(v)) \leqslant \gamma d(u,v)$ for each $u,v \in \mathscr{M}$; (b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$, where $H_d : \mathscr{X}(\mathscr{M}) \times \mathscr{X}(\mathscr{M}) \to \mathbb{R}\cup\{\infty\}$ is given by $H_d(U,V) = \max\{\sup_{u\in U} d(u,V), \sup_{v\in V} d(U,v)\}$ with $d(U,v) = \inf_{u\in U} d(u;v)$ and $d(u,V) = \inf_{v\in V} d(u;v)$. Furthermore, note that $(\mathscr{X}_{cl,b}(\mathscr{M}),H_d)$ is a metric space (see [13]), where $\mathscr{X}_{cl,b}(\mathscr{M}) = \{\mathscr{W} \in \mathscr{X}(\mathscr{M}) : \mathscr{W} \text{ is closed and bounded}\}$.

DEFINITION 3. A function $y \in C^2([1,\mathfrak{T}],\mathbb{R})$ is called a solution of the problem (2) if $y(1) = 0, y(\mathfrak{T}) = 0$ and there exists function $v \in L^1([0,\mathfrak{T}],\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. on $[1,\mathfrak{T}]$ and

$$y(t) = \int_1^{\mathfrak{T}} G_h(t,u)h(u,y(u))du + \int_1^{\mathfrak{T}} G_f(t,u)v(u)du,$$

where $G_h(t,s)$ and $G_f(t,s)$ are respectively given by (4) and (5).

Our first result concerning the existence of solutions for the problem (2) relies on the following theorem.

LEMMA 2. (Covitz and Nadler [12]) *Let* $(\mathscr{M},d)$ *be a complete metric space and* $\mathscr{X}_{cl}(\mathscr{M})=\{\mathscr{W}\in\mathscr{X}(\mathscr{M}):\mathscr{W}$ *is closed*$\}$. *If* $\mathscr{Q}:\mathscr{M}\to\mathscr{X}_{cl}(\mathscr{M})$ *is a contraction, then* $Fix\mathscr{Q}\neq\emptyset$.

THEOREM 3. *Assume that*

(K_0) *there exists a nonnegative constant* $k<\dfrac{\Gamma(\kappa+1)}{2}(\log\mathfrak{T})^{-\kappa}$ *such that*

$$|h(t,u)-h(t,v)|\leqslant k\|u-v\|,\quad \text{for } t\in J \text{ and every } u,v\in\mathbb{R};$$

(K_1) $F:[1,\mathfrak{T}]\times\mathbb{R}\to\mathscr{X}_{cp}(\mathbb{R})$ *is such that* $F(\cdot,y):[1,T]\to\mathscr{X}_{cp}(\mathbb{R})$ *is measurable for each* $y\in\mathbb{R}$, *where* $\mathscr{X}_{cp}(\mathbb{R})=\{\mathscr{W}\in\mathscr{X}(\mathbb{R}):\mathscr{W}$ *is compact*$\}$;

(K_2) $H_d(F(t,y),F(t,\bar{y}))\leqslant m(t)|y-\bar{y}|$ *for almost all* $t\in[1,\mathfrak{T}]$ *and* $y,\bar{y}\in\mathbb{R}$ *with* $m\in L^1([1,\mathfrak{T}],\mathbb{R}^+)$ *and* $d(0,F(t,0))\leqslant m(t)$ *for almost all* $t\in[1,\mathfrak{T}]$.

Then the problem (2) has at least one solution on $[1,\mathfrak{T}]$ *if*

$$\widehat{\delta}:=\frac{2k(\log\mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)}+\frac{2}{\Gamma(\omega+\kappa)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{s}\right)^{\omega+\kappa-1}\frac{m(u)}{u}du<1. \tag{13}$$

Proof. We transform the problem (2) into a fixed point problem by introducing an operator $\widehat{\mathscr{N}}:C(J,\mathbb{R})\longrightarrow\mathscr{X}(C(J,\mathbb{R}))$ as follows

$$\widehat{\mathscr{N}}(y)=\left\{\chi\in C(J,\mathbb{R}):\chi(t)=\int_1^{\mathfrak{T}}G_h(t,u)h(u,y(u))du+\int_1^{\mathfrak{T}}G_f(t,u)w(u)du\right\} \tag{14}$$

for $w\in S_{F,x}$, where $G_h(t,u)$ and $G_f(t,u)$ are respectively given by (4) and (5).

It follows by the assumption (K_1) that the set $S_{F,y}$ is nonempty for each $y\in C([1,\mathfrak{T}],\mathbb{R})$, so F has a measurable selection (see Theorem III.6 [14]). Now we show that the operator $\widehat{\mathscr{N}}$ satisfies the assumptions of Lemma 2. We show that $\widehat{\mathscr{N}}(y)\in\mathscr{X}_{cl}((C[1,\mathfrak{T}],\mathbb{R}))$ for each $y\in C([1,\mathfrak{T}],\mathbb{R})$. Let $\{a_n\}_{n\geqslant 0}\in\widehat{\mathscr{N}}(y)$ be such that $a_n\to a$ $(n\to\infty)$ in $C([1,\mathfrak{T}],\mathbb{R})$. Then $a\in C([1,\mathfrak{T}],\mathbb{R})$ and there exists $w_n\in S_{F,y_n}$ such that, for each $t\in[1,\mathfrak{T}]$,

$$a_n(t)=\int_1^{\mathfrak{T}}G_h(t,u)h(u,y(u))du+\int_1^{\mathfrak{T}}G_f(t,u)w_n(u)du,\ \ t\in J.$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that w_n converges to w in $L^1([1,\mathfrak{T}],\mathbb{R})$. Thus, $w\in S_{F,y}$ and for each $t\in[1,\mathfrak{T}]$, we have

$$a_n(t)\to w(t)=\int_1^{\mathfrak{T}}G_h(t,u)h(u,y(u))du+\int_1^{\mathfrak{T}}G_f(t,u)w(u)du,\ \ t\in J.$$

Hence, $a \in \widehat{\mathcal{N}}(y)$.

Next we show that there exists $\widehat{\delta} < 1$ (defined by (13)), such that

$$H_d(\widehat{\mathcal{N}}(y), \widehat{\mathcal{N}}(\bar{y})) \leqslant \widehat{\delta}\|y - \bar{y}\| \text{ for each } y, \bar{y} \in C^2([1,\mathfrak{T}], \mathbb{R}).$$

Let $y, \bar{y} \in C^2([1,\mathfrak{T}], \mathbb{R})$ and $\chi_1 \in \widehat{\mathcal{N}}(y)$. Then there exists $w_1(t) \in F(t, y(t))$ such that, for each $t \in [1,\mathfrak{T}]$,

$$\chi_1(t) = \int_1^{\mathfrak{T}} G_h(t,u)h(u,y(u))du + \int_1^{\mathfrak{T}} G_f(t,u)w_1(u)du, \quad t \in J.$$

By (K_2), we have

$$H_d(F(t,y), F(t,\bar{y})) \leqslant m(t)|y(t) - \bar{y}(t)|.$$

So, there exists $v \in F(t, \bar{y}(t))$ such that

$$|w_1(t) - v| \leqslant m(t)|y(t) - \bar{y}(t)|, \quad t \in [1,\mathfrak{T}].$$

Define $\mathscr{D} : [1,\mathfrak{T}] \to \mathscr{X}(\mathbb{R})$ by

$$\mathscr{D}(t) = \{v \in \mathbb{R} : |w_1(t) - v| \leqslant m(t)|y(t) - \bar{y}(t)|\}.$$

As the multivalued operator $\mathscr{D}(t) \cap F(t, \bar{y}(t))$ is measurable (Proposition III.4 [14]), there exists a function $w_2(t)$ which is a measurable selection for $\mathscr{D}$. So $w_2(t) \in F(t, \bar{y}(t))$ and for each $t \in [1,\mathfrak{T}]$, we have $|w_1(t) - w_2(t)| \leqslant m(t)|y(t) - \bar{y}(t)|$.

For each $t \in [1,\mathfrak{T}]$, let us define

$$\chi_2(t) = \int_1^{\mathfrak{T}} G_h(t,u)h(u,y(u))du + \int_1^{\mathfrak{T}} G_f(t,u)w_2(u)du, \quad t \in J.$$

Thus,

$$\begin{aligned} |\chi_1(t) - \chi_2(t)| &\leqslant \left| \int_1^{\mathfrak{T}} G_h(t,u)[h(u,y(u)) - h(u,\bar{y}(u))]du + \int_1^{\mathfrak{T}} G_f(t,u)[w_1(u) - w_2(u)]du \right| \\ &\leqslant \left(\frac{2k(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} + \frac{2}{\Gamma(\omega+\kappa)} \int_1^{\mathfrak{T}} \left(\log \frac{\mathfrak{T}}{u}\right)^{\omega+\kappa-1} \frac{m(u)}{u} du \right) \|y - \bar{y}\|. \end{aligned}$$

Hence

$$\|\chi_1 - \chi_2\| \leqslant \left(\frac{2k(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} + \frac{2}{\Gamma(\omega+\kappa)} \int_1^{\mathfrak{T}} \left(\log \frac{\mathfrak{T}}{u}\right)^{\omega+\kappa-1} \frac{m(u)}{u} du \right) \|y - \bar{y}\|.$$

By interchanging the roles of y and $\bar{y}$ in an analogous manner, we can obtain $H_d(\widehat{\mathcal{N}}(y), \widehat{\mathcal{N}}(\bar{y})) \leqslant \widehat{\delta}\|y - \bar{y}\|$, where $\widehat{\delta}$ is defined by (13). So $\widehat{\mathcal{N}}$ is a contraction. Therefore, the conclusion of Lemma 2 applies and $\widehat{\mathcal{N}}$ has a fixed point y, which is a solution of (2). This completes the proof. $\square$

In relation to the forthcoming results, we quickly recall some basic concepts [15].

DEFINITION 4. A multi-valued map $\Theta : \mathcal{M} \to \mathcal{X}(\mathcal{M})$ is said to be measurable if the function $t \longmapsto d(y,\Theta(t)) = \inf\{|y-z| : z \in \Theta(t)\}$ is measurable for every $y \in \mathcal{M}$ and completely continuous if $\Theta(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{X}_b(\mathcal{M})$.

REMARK 1. If the multi-valued map Θ is completely continuous with nonempty compact values, then Θ is upper semicontinuous (u.s.c.) if and only if Θ has a closed graph, that is, $x_n \to x_*,\ y_n \to y_*,\ y_n \in \Theta(x_n)$ imply $y_* \in \Theta(x_*)$.

DEFINITION 5. A multivalued map $\Theta : J \times \mathbb{R} \to \mathcal{X}(\mathbb{R})$ is said to be Carathéodory if (i) $t \longmapsto \Theta(t,x)$ is measurable for each $x \in \mathbb{R}$ and (ii) $x \longmapsto \Theta(t,x)$ is upper semicontinuous for almost all $t \in J$. Further a Carathéodory function Θ is called L^1-Carathéodory if there exists $\varphi_\rho \in L^1(J,\mathbb{R}^+)$ for each $\rho > 0$ such that $\|\Theta(t,x)\| = \sup\{|v| : v \in \Theta(t,x)\} \leqslant \varphi_\rho(t)$ for all $\|x\| \leqslant \rho$ and for a.e. $t \in J$.

Next we state some known results on multivalued maps that we need in the sequel.

We define the graph of Θ to be the set $Gr(\Theta) = \{(x,y) \in X \times Y, y \in \Theta(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

LEMMA 3. ([15, Proposition 1.2]) *If* $\Theta : \mathcal{M} \to \mathcal{X}_{cl}(Y) = \{Y \in \mathcal{P}(\mathcal{M}) : Y \text{ is closed}\}$ *is u.s.c., then* $Gr(\Theta)$ *is a closed subset of* $X \times Y$*, that is, for every sequence* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *and* $\{y_n\}_{n\in\mathbb{N}} \subset Y$*, if when* $n \to \infty$*,* $x_n \to x_*$*,* $y_n \to y_*$ *and* $y_n \in \Theta(x_n)$*, then* $y_* \in \Theta(x_*)$*. Conversely, if* Θ *is completely continuous and has a closed graph, then it is upper semi-continuous.*

LEMMA 4. ([16]) *Let* $\mathcal{M}$ *be a Banach space. Let* $\Theta_1 : J \times \mathbb{R} \to \mathcal{X}_{cp,c}(\mathcal{M}) = \{Y \in \mathcal{X}(\mathcal{M}) : Y \text{ is compact and convex }\}$ *be an* L^1*-Carathéodory multivalued map and let* Φ_1 *be a linear continuous mapping from* $L^1(J,\mathcal{M})$ *to* $C(J,\mathcal{M})$*. Then the operator* $\Phi_1 \circ S_{\Theta_1} : C(J,\mathcal{M}) \to \mathcal{X}_{cp,c}(C(J,\mathcal{M}))$*,* $x \mapsto (\Phi_1 \circ S_{\Theta_1})(x) = \Theta(S_{\Theta_1,x})$ *is a closed graph operator in* $C(J,\mathcal{M}) \times C(J,\mathcal{M})$*.*

LEMMA 5. (Nonlinear alternative for contractive maps (Corollary 3.8 [17])) *Let* $\mathcal{M}$ *be a Banach space, and* D *be a bounded neighborhood of* $0 \in \mathcal{M}$*. Let* $\Theta_a : X \to \mathcal{X}_{cp,c}(\mathcal{M})$ *and* $\Theta_b : \bar{D} \to \mathcal{X}_{cp,c}(\mathcal{M})$ *be two multi-valued operators such that* (a) Θ_a *is contraction, and* (b) Θ_b *is upper semicontinuous and compact. Then, if* $\Theta = \Theta_a + \Theta_b$*, either* (i) Θ *has a fixed point in* $\bar{D}$ *or* (ii) *there is a point* $u \in \partial D$ *and* $\lambda \in (0,1)$ *with* $u \in \lambda\Theta(u)$*.*

The following theorem deals with the case when F has convex values.

THEOREM 4. *Assume that* (K_0) *and the following conditions hold:*

(K_4) $F : J \times \mathbb{R} \to \mathcal{X}_{cp,c}(\mathbb{R})$ *is* L^1*-Carathéodory;*

(K_5) *there exists a continuous nondecreasing function* $\Phi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C(J,\mathbb{R}^+)$ *such that*

$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,u)\} \leqslant p(t)\Phi(\|u\|) \text{ for each } (t,u) \in J \times \mathbb{R};$$

(K_6) *there exists a constant* $C_0 > 0$ *such that*

$$\frac{\left(1-\frac{2k(\log\mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)}\right)C_0}{\frac{2h_0(\log\mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)}+\frac{\Phi(C_0)\|p\|}{\Gamma(\omega+\kappa+1)}(\log\mathfrak{T})^{\omega+\kappa}} > 1, \tag{15}$$

where $h_0 = \sup_{t\in J}|h(t,0)|$.

Then the problem (2) has at least one solution on J.

Proof. Define an operator $\mathscr{U} : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ and a multi-valued operator $\mathscr{V} : C(J,\mathbb{R}) \longrightarrow \mathscr{X}(C(J,\mathbb{R}))$ by

$$\mathscr{U}y(t) = \int_1^{\mathfrak{T}} G_h(t,u)h(u,y(u))du \tag{16}$$

$$\mathscr{V}y(t) = \left\{\varphi \in C(J,\mathbb{R}) : \varphi(t) = \int_1^{\mathfrak{T}} G_f(t,u)w(u)du,\, w \in S_{F,y}\right\}. \tag{17}$$

Notice that $\widehat{\mathscr{N}} = \mathscr{U} + \mathscr{V}$, where the operator $\widehat{\mathscr{N}}$ is given by (14). The proof will be complete once it is established that the operators $\mathscr{U}$ and $\mathscr{V}$ satisfy the hypothesis of Theorem 5 on J. Firstly, we establish that the operators $\mathscr{U},\mathscr{V} : B_r \to \mathscr{X}_{cp,c}(C(J,\mathbb{R}))$ are well defined, where $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \leqslant r\}$ is a bounded set in $C(J,\mathbb{R})$. Observe that the operator $\mathscr{V}$ is equivalent to the composition $\widehat{\mathscr{Q}} \circ S_F$, where $\widehat{\mathscr{Q}}$ is the continuous linear operator on $L^1(J,\mathbb{R})$ into $C(J,\mathbb{R})$, defined by

$$\widehat{\mathscr{Q}}(v)(t) = \int_1^{\mathfrak{T}} G_f(t,u)w(u)du.$$

For an arbitrary $y \in B_r$, let $\{w_n\}$ be a sequence in $S_{F,y}$. Then, by definition of $S_{F,y}$, we get $w_n(t) \in F(t,y(t))$ for almost all $t \in J$. Since $F(t,y(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{w_n(t)\}$ (we denote it by $\{w_n(t)\}$ again) that converges in measure to some $w(t) \in S_{F,y}$ for almost all $t \in J$. On the other hand, $\widehat{\mathscr{Q}}$ is continuous, so $\widehat{\mathscr{Q}}(w_n)(t) \to \widehat{\mathscr{Q}}(w)(t)$ pointwise on J.

To accomplish that the convergence is uniform, we need to show that $\{\widehat{\mathscr{Q}}(w_n)\}$ is an equicontinuous sequence. Let $\tau_1,\tau_2 \in J$ with $\tau_1 < \tau_2$. Then, we have

$$\begin{aligned}
&|\widehat{\mathscr{Q}}(v_n)(\tau_2) - \widehat{\mathscr{Q}}(v_n)(\tau_1)| \\
&= \left|\int_1^{\mathfrak{T}} [G_f(\tau_2,u) - G_f(\tau_1,u)]w_n(u)du\right| \\
&\leqslant \frac{\Phi(r)\|p\|}{\Gamma(\omega+\kappa+1)}\Big[\left|(\log\tau_2)^{\omega+\kappa} - (\log\tau_1)^{\omega+\kappa}\right| + |\log\tau_2/\tau_1|^{\omega+\kappa}\Big] \\
&\quad + \frac{(\log\tau_2)^{\kappa} - (\log\tau_1)^{\kappa}}{\Gamma(\omega+\kappa+1)}\Phi(r)\|p\|(\log T)^{\omega}.
\end{aligned}$$

Clearly the right hand of the above inequality tends to zero as $\tau_2 \to \tau_1$. Thus, the sequence $\{\widehat{\mathscr{Q}}(w_n)\}$ is equicontinuous. Therefore, by the Arzelá-Ascoli theorem, there exists a uniformly convergent subsequence. So, there is a subsequence of $\{w_n\}$ (we denote it again by $\{w_n\}$) such that $\widehat{\mathscr{Q}}(w_n) \to \widehat{\mathscr{Q}}(w)$. Note that $\widehat{\mathscr{Q}}(v) \in \widehat{\mathscr{Q}}(S_{F,y})$. Hence, $\mathscr{V}(y) = \widehat{\mathscr{Q}}(S_{F,y})$ is compact for all $x \in B_r$. So $\mathscr{V}(y)$ is compact.

In order to show that $\mathscr{V}(y)$ is convex for all $y \in C(J,\mathbb{R})$, let $m_1, m_2 \in \mathscr{V}(y)$. We select $w_1, w_2 \in S_{F,y}$ such that

$$m_i(t) = \int_1^{\mathfrak{T}} G_f(t,u) w_i(u) du, \;\; i = 1,2,$$

for almost all $t \in J$. Let $0 \leqslant \lambda \leqslant 1$. Then, we have

$$[\lambda m_1 + (1-\lambda)m_2](t) = \int_1^{\mathfrak{T}} G_f(t,u)[\lambda u_1(u) + (1-\lambda)u_2(u)]du.$$

Since $S_{F,y}$ is convex (F has convex values), $\lambda w_1(u) + (1-\lambda)w_2(u) \in S_{F,y}$. Thus $\lambda m_1 + (1-\lambda)m_2 \in \mathscr{V}(y)$, which proves that $\mathscr{V}$ is convex-valued.

Obviously, $\mathscr{U}$ is compact and convex-valued. The rest of the proof will be completed in several steps and claims.

Step 1: We show that $\mathscr{U}$ *is a contraction on* $C(J,\mathbb{R})$.

Let $y, x \in C(J,\mathbb{R})$. Then we have

$$|\mathscr{U}(y)(t) - \mathscr{U}(x)(t)| = \left| \int_1^{\mathfrak{T}} G_h(t,u)[h(u,y(u)) - h(u,x(u))]du \right| \leqslant \frac{k(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} \|y - x\|,$$

which implies that

$$\|\mathscr{U}(y) - \mathscr{U}(x)\| \leqslant \frac{k(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} \|y - x\|.$$

In view of the condition (K_0), it follows that $\mathscr{U}$ is a contraction.

Step 2: $\mathscr{V}$ *is compact and upper semi-continuous*. This will be established in several claims.

Claim I: $\mathscr{V}$ *maps bounded sets into bounded sets in* $C(J,\mathbb{R})$. Let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \leqslant r\}$ be a bounded set in $C(J,\mathbb{R})$. Then, for each $h \in \mathscr{V}(y), y \in B_r$, there exists $\sigma \in S_{F,y}$ such that

$$\sigma(t) = \int_1^{\mathfrak{T}} G_f(t,s) w(u) du.$$

Then, for $t \in J$, we have

$$|\sigma(t)| \leqslant \frac{2\Phi(\|y\|)\|p\|}{\Gamma(\omega+\kappa)} \int_1^{\mathfrak{T}} \left(\log \frac{t}{u}\right)^{\omega+\kappa-1} \frac{du}{u} \leqslant \frac{2\Phi(\|y\|)\|p\|}{\Gamma(\omega+\kappa+1)} (\log \mathfrak{T})^{\omega+\kappa}.$$

Thus

$$\|\sigma\| \leqslant \frac{2\Phi(r)\|p\|}{\Gamma(\omega+\kappa+1)} (\log \mathfrak{T})^{\omega+\kappa}.$$

Claim II: $\mathscr{V}$ *maps bounded sets into equi-continuous sets.* Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $y \in B_r$. Then, for each $\sigma \in \mathscr{V}(y)$, one can easily obtain

$$|\sigma(t_2) - \sigma(t_1)| \leqslant \frac{\Phi(r)\|p\|}{\Gamma(\omega+\kappa+1)}\Big[\big|(\log t_2)^{\omega+\kappa} - (\log t_1)^{\omega+\kappa}\big| + |\log t_2/t_1|^{\omega+\kappa}\Big] + \frac{|(\log t_2)^{\kappa} - (\log t_1)^{\kappa}|}{\Gamma(\omega+\kappa+1)}\Phi(r)\|p\|(\log \mathfrak{T})^{\omega}.$$

Obviously the right hand side of the above inequality tends to zero independently of $y \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathscr{V} : C(J,\mathbb{R}) \to \mathscr{X}(C(J,\mathbb{R}))$ is completely continuous.

Next we show that $\mathscr{V}$ is an upper semi-continuous multivalued mapping. It follows by Lemma 3 that $\mathscr{V}$ will be upper semicontinuous if we establish that it has a closed graph as it is already shown to be completely continuous. We establish it in the following claim.

Claim III: $\mathscr{V}$ *has a closed graph.* Let $y_n \to y_*, \sigma_n \in \mathscr{V}(y_n)$ and $\sigma_n \to \sigma_*$. Then we need to show that $\sigma_* \in \mathscr{V}(y_*)$. Associated with $\sigma_n \in \mathscr{V}(y_n)$, there exists $w_n \in S_{F,y_n}$ such that for each $t \in J$,

$$\sigma_n(t) = \int_1^{\mathfrak{T}} G_f(t,u) w_n(u) du.$$

Thus it suffices to show that there exists $w_* \in S_{F,y_*}$ such that for each $t \in J$,

$$\sigma_*(t) = \int_1^{\mathfrak{T}} G_f(t,u) w_*(s) u du.$$

Let us consider the linear operator $\Lambda : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ given by

$$v \mapsto \Lambda(v)(t) = \int_1^{\mathfrak{T}} G_f(t,u) w(u) du.$$

Observe that

$$\|\sigma_n(t) - \sigma_*(t)\| = \left\| \int_1^{\mathfrak{T}} G_f(t,u)(w_n(u) - w_*(u)) du \right\| \to 0 \text{ as } n \to \infty.$$

Thus, it follows by Lemma 4 that $\Lambda \circ S_F$ is a closed graph operator. Further, we have $\sigma_n(t) \in \Lambda(S_{F,y_n})$. Since $y_n \to y_*$, we have that

$$\sigma_*(t) = \int_1^{\mathfrak{T}} G_f(t,u) v_*(u) u du,$$

for some $w_* \in S_{F,y_*}$. Hence $\mathscr{V}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathscr{V}$ is compact valued and upper semi-continuous. Thus the operators $\mathscr{U}$ and $\mathscr{V}$ satisfy all the conditions of Lemma 5 and hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $y \in \lambda\mathscr{U}(y) + \lambda\mathscr{V}(y)$ for $\lambda \in (0,1)$, then there exist $v \in S_{F,y}$ such that

$$y(t) = \lambda\Big(\int_1^{\mathfrak{T}} G_h(t,u) h(u,y(u)) du + \int_1^{\mathfrak{T}} G_f(t,u) w(u) du\Big), \ t \in J.$$

By our assumptions, and using the estimate: $|h(t,y)| = |h(t,y) - h(t,0) + h(t,0)| \leqslant k\|y\| + h_0$, we obtain

$$|y(t)| \leqslant 2\frac{k\|y\| + h_0}{\Gamma(\kappa+1)}(\log \mathfrak{T})^{\kappa} + 2\frac{\Phi(\|y\|)\|p\|}{\Gamma(\omega+\kappa+1)}(\log \mathfrak{T})^{\omega+\kappa},$$

which leads to

$$\frac{\left(1 - \dfrac{2k(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)}\right)\|y\|}{\dfrac{2h_0(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} + \dfrac{\Phi(\|y\|)\|p\|}{\Gamma(\omega+\kappa+1)}(\log \mathfrak{T})^{\omega+\kappa}} \leqslant 1. \tag{18}$$

If condition (ii) of Theorem 5 holds, then there exists $\lambda \in (0,1)$ and $y \in \partial B_{C_0}$ with $y = \lambda \widehat{\mathscr{N}}(y)$. Then, y is a solution of (2) with $\|y\| = C_0$. Now, by the inequality (18), we get

$$\frac{\left(1 - \dfrac{2k(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)}\right)C_0}{\dfrac{2h_0(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} + \dfrac{\Phi(C_0)\|p\|}{\Gamma(\omega+\kappa+1)}(\log \mathfrak{T})^{\omega+\kappa}} \leqslant 1,$$

which contradicts (15). Hence, $\widehat{\mathscr{N}}$ has a fixed point in J by Lemma 5, and consequently the problem (2) has a solution. This completes the proof. □

LEMMA 6. (Krasnoselskii's fixed point theorem [18]) *Let $\mathscr{M}$ be a Banach space, $Y \in \mathscr{X}_{b,cl,c}(\mathscr{M}) = \{Y \in \mathscr{X}(\mathscr{M}) : Y$ is bounded, closed and convex$\}$ and $\Psi_1, \Psi_2 : Y \to \mathscr{X}_{cp,c}(\mathscr{M})$ be two multivalued operators. If (i) $\Psi_1 y + \Psi_2 y \subset Y$ for all $y \in Y$; (ii) Ψ_1 is contraction; and (iii) Ψ_2 is u.s.c and compact, then there exists $y \in Y$ such that $y \in \Psi_1 y + \Psi_2 y$.*

THEOREM 5. *Let (K_0), (K_4) and the following assumption hold:*

(K_7) *there exists a function $\rho \in C([1,\mathfrak{T}],\mathbb{R}^+)$ such that*

$$\|F(t,w)\|_{\mathscr{X}} := \sup\{|y| : y \in F(t,w)\} \leqslant \rho(t), \ \text{for each } (t,w) \in [1,\mathfrak{T}] \times \mathbb{R}.$$

Then the problem (2) has at least one solution on J.

Proof. As in the proof of the last result, we transform the problem (2) into a fixed point problem by using the operator $\mathscr{N} : C(J,\mathbb{R}) \longrightarrow \mathscr{X}(C(J,\mathbb{R}))$ defined by (14). As before, one can show that the operators $\mathscr{U}$ and $\mathscr{V}$ defined respectively by (16) and (17) are indeed multivalued operators $\mathscr{U}, \mathscr{V} : B_r \to \mathscr{X}_{cp,c}(C(J,\mathbb{R}))$, where $B_r = \{y \in C(J,\mathbb{R}) : \|y\| \leqslant r\}$ is a bounded set in $C(J,\mathbb{R})$, $\mathscr{U}$ is a contraction on $C(J,\mathbb{R})$ and $\mathscr{V}$ is u.s.c. and compact.

Here, we show that $\mathscr{U}(y) + \mathscr{V}(y) \subset B_r$ for all $y \in B_r$. Suppose $y \in B_r$ with

$$r > \left[\frac{2h_0(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} + \frac{2\|q\|(\log \mathfrak{T})^{\omega+\kappa}}{\Gamma(\omega+\kappa+1)}\right]\left(1 - \frac{2k(\log \mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)}\right)^{-1}$$

and an arbitrary element $\sigma \in \mathscr{V}$. Choose $v \in S_{F,y}$ such that

$$\sigma(t) = \int_1^{\mathfrak{T}} G_h(t,s)h(s,y(s))ds + \int_1^{\mathfrak{T}} G_f(t,s)v(s)ds, \quad t \in J.$$

By our assumptions, we obtain

$$\|\sigma\| \leqslant 2\frac{kr+h_0}{\Gamma(\kappa+1)}(\log\mathfrak{T})^{\kappa} + \frac{2\|q\|}{\Gamma(\omega+\kappa+1)}(\log\mathfrak{T})^{\omega+\kappa} < r.$$

Hence $\|\sigma\| \leqslant r$, which means that $\mathscr{U}(y)+\mathscr{V}(y) \subset B_r$ for all $y \in B_r$.

Thus, the operators $\mathscr{U}$ and $\mathscr{V}$ satisfy all the conditions of Lemma 6 and hence its conclusion implies that $y \in \mathscr{U}(y)+\mathscr{V}(y)$ in B_r. Therefore the problem (2) has a solution in B_r and the proof is completed. □

4. Examples

In this section we give examples to illustrate the usefulness of our main results. Let us consider the fractional functional differential equation

$$\begin{cases} D^{2/5}\Big(D^{3/4}y(t) - h(t,y)\Big) = f(t,y(t)), \quad t \in [1,2], \\ y(1) = 0, \quad y(2) = 0, \end{cases} \tag{19}$$

Here $\omega = 2/5$, $\kappa = 3/4$, and $h(t,y)$, $f(t,y)$ will be chosen suitably for the illustration of the obtained results.

(a) For illustrating Theorem 1, we take

$$f(t,y) = \frac{1}{24}(y+\tan^{-1}y)e^t + \frac{1}{\sqrt{t^2+1}}, \; h(t,y) = \frac{1}{15+t}\left(\frac{|y|}{1+|y|}\right) + \sin(t). \tag{20}$$

It is easy to check that $f(t,y)$ and $h(t,y)$ satisfy the conditions (A_1) and (A_2) respectively with $\ell = e^2/12$ and $k = 1/16$. Also

$$\frac{2k(\log\mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} + \frac{2\ell(\log\mathfrak{T})^{\omega+\kappa}}{\Gamma(\omega+\kappa+1)} \approx 0.856309 < 1.$$

Thus all the conditions of Theorem 1 are satisfied. So, by the conclusion of Theorem 1, the problem (19) with $f(t,y)$ and $h(t,y)$ given by (20) has a unique solution on $[1,2]$.

Next we consider the following multivalued fractional functional boundary value problem:

$$\begin{cases} D^{2/5}\Big(D^{3/4}y(t) - g(t,y)\Big) \in F(t,y(t)), \quad t \in [1,2], \\ y(1) = 0, \; y(2) = 0, \end{cases} \tag{21}$$

where $\omega = 2/5$, $\kappa = 3/4$, $g(t,y) = |y|/[8(5+t)(4+|y|)] + e^{1-t}/\sqrt{15+t^2}$ and $F(t,y(t))$ will be chosen appropriately.

(b) In order to demonstrate the application of Theorem 3, let us choose

$$F(t,y(t)) = \left[0\ ,\ \frac{\tan^{-1} y}{(2+(t-1)^2)} + \frac{1}{3+t^2}\right]. \tag{22}$$

Clearly

$$H_d(F(t,y),F(t,\bar{y})) \leqslant \frac{1}{(2+(t-1)^2)}\|y-\bar{y}\|.$$

Letting $m(t) = 1/[2+(t-1)^2)]$, it is easy to check that $d(0,F(t,0)) \leqslant m(t)$ holds for almost all $t \in [1,2]$ and that $\delta \leqslant 0.749196$. As the hypotheses of Theorem 3 are satisfied, we conclude that the problem (21) with $F(t,y)$ given by (22) has at least one solution on $[1,2]$.

(c) For the illustration of Theorem 4, we take

$$F(t,y(t)) = \left[\frac{|y|}{10(|y|+5)} + \frac{1}{9}\sin^2(\pi t/2)t\ ,\ \frac{e^{1-t}}{2}\left(\frac{1}{\pi}\tan^{-1} y + \sin y(t) + \frac{1}{2}\right)\right]. \tag{23}$$

Using the given data, we find that $k = 1/12 < [\Gamma(1.75)/2(\log 2)^{3/4}] \approx 0.604917$, $h_0 = 1/4, \|p\| = 1/2,\ \Phi(\|y\|) = 1+\|y\|$ and Condition (15) is satisfied with $C_0 > \widehat{C_0} \approx 1.291949$. Thus, all the conditions of Theorem 4 are satisfied and consequently, there exists at least one solution for the problem (21) with $F(t,y)$ given by (23) on $[1,2]$.

5. Three-point boundary conditions

In this section we replace the Dirichlet boundary conditions in problem (1) by nonlocal three-point boundary conditions and consider the following problem:

$$\begin{cases} D^{\omega}[D^{\kappa}y(t) - g(t,y(t))] = f(t,y(t)), \ \ t \in J := [1,\mathfrak{T}], \\ y(1) = 0, \ \ y(\mathfrak{T}) = \sigma y(\zeta), \ \ 1 < \zeta < \mathfrak{T}, \end{cases} \tag{24}$$

where $\sigma \in \mathbb{R}$ such that $\sigma \neq (\log \mathfrak{T})^{\kappa}/(\log \zeta)^{\kappa}$.

As in Section 2, we introduce a fixed point operator $\widehat{N}_1 : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ associated with the problem (24) as follows

$$\begin{aligned} &\widehat{N}_1(y)(t) \\ &= \frac{1}{\Gamma(\kappa)}\int_1^t \left(\log\frac{t}{s}\right)^{\kappa-1}\frac{g(s,y(s))}{s}ds + \frac{1}{\Gamma(\omega+\kappa)}\int_1^t \left(\log\frac{t}{s}\right)^{\omega+\kappa-1}\frac{f(s,y(s))}{s}ds \\ &\quad + \frac{(\log t)^{\kappa}}{(\log T)^{\kappa} - \sigma(\log\zeta)^{\kappa}}\left\{\frac{\sigma}{\Gamma(\kappa)}\int_1^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\kappa-1}\frac{g(s,y(s))}{s}ds\right. \\ &\quad + \frac{\sigma}{\Gamma(\omega+\kappa)}\int_1^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\omega+\kappa-1}\frac{f(s,y(s))}{s}ds - \frac{1}{\Gamma(\kappa)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{s}\right)^{\kappa-1}\frac{g(s,y(s))}{s}ds \\ &\quad \left. - \frac{1}{\Gamma(\omega+\kappa)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{s}\right)^{\omega+\kappa-1}\frac{f(s,y(s))}{s}ds\right\}. \end{aligned} \tag{25}$$

Furthermore we set

$$\varpi = \frac{k(\log\mathfrak{T})^{\kappa}}{\Gamma(\kappa+1)} + \frac{\ell(\log\mathfrak{T})^{\omega+\kappa}}{\Gamma(\omega+\kappa+1)} + \frac{(\log\mathfrak{T})^{\kappa}}{|(\log\mathfrak{T})^{\kappa}-\sigma(\log\zeta)^{\kappa}|} \times\Big(\frac{k(|\sigma|(\log\zeta)^{\kappa}+(\log\mathfrak{T})^{\kappa})}{\Gamma(\kappa+1)} + \frac{\ell(|\sigma|(\log\zeta)^{\omega+\kappa}+(\log\mathfrak{T})^{\omega+\kappa})}{\Gamma(\omega+\kappa+1)}\Big). \tag{26}$$

The uniqueness result for the problem (24) can be formulated as follows.

THEOREM 6. *Assume that the conditions* (A_1), (A_2) *hold and that* $\varpi < 1$ *hold, where* ϖ *is given by* (26). *Then the problem* (24) *has a unique solution* $[1,\mathfrak{T}]$.

EXAMPLE. Consider the following three-point problem

$$\begin{cases} D^{1/3}\Big(D^{2/3}y(t) - g(t,y)\Big) = f(t,y(t)), \ \ t\in[1,e], \\ y(1)=0, \ \ y(e)=\sigma y(\zeta), \end{cases} \tag{27}$$

where $\omega = 1/3, \kappa = 2/3,\ \sigma = 1/[2(\log 2)^{2/3}]$ and

$$f(t,y) = \frac{a_1}{2}\Big(\sin y + \frac{|y|}{1+|y|}\Big) + \frac{1}{1+\log t},\ g(t,y) = \frac{a_2}{2}\Big(y+\tan^{-1}y\Big) + (2+\log t)^2, \tag{28}$$

a_1 and a_2 are constants to be chosen appropriately. Clearly $f(t,y)$ and $g(t,y)$ satisfy the conditions (A_1) and (A_2) respectively with $\ell = a_1$ and $k = a_2$. Moreover, it is found that $\varpi = \frac{4a_2}{\Gamma(5/3)} + a_1(3+(\log 2)^{1/3})$. For suitable values of a_1 and a_2, one can find that $\varpi < 1$. For instance, if $a_1 = 1/12, a_2 = 1/11$, then $\varpi \approx 0.726561 < 1$. Thus the conclusion of Theorem 6 applies and hence the problem (27) has a unique solution on $[1,e]$.

REMARK 2. The existence results for the problem (24) analog to the ones obtained in Section 3 for the problem (1) can be established with the aid of (25) and (26). We can also discuss the inclusion case of the problem (24) like the problem (2).

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