

A PROBLEM INVOLVING A NONLOCAL OPERATOR

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Abstract. The aim of this paper is to deal with the elliptic pdes involving a nonlinear integro-differential operator which are possibly degenerate and covers the case of fractional p -Laplacian operator. We prove the existence of a solution in the weak sense to the problem

$$\begin{aligned} -\mathcal{L}_{\Phi}u &= \lambda|u|^{q-2}u \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

if and only if a weak solution to

$$\begin{aligned} -\mathcal{L}_{\Phi}u &= \lambda|u|^{q-2}u + f, \quad f(\neq 0) \in L^{p'}(\Omega), \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

(p' being the conjugate of p), exists for $q \in (p, p_s^*)$ under certain condition on λ , where $-\mathcal{L}_{\Phi}$ is a general nonlocal integro-differential operator of order $s \in (0, 1)$ and p_s^* is the fractional Sobolev conjugate of p . We further prove a necessary condition for the existence of a weak solution to the problem

$$\begin{aligned} -\mathcal{L}_{\Phi}u &= \lambda|u|^{q-2}u + \mu \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where μ is a measure.

1. Introduction

In the recent years, a great amount of attention has been given to the study of fractional and nonlocal operators of elliptic type, both, for research in pure Mathematics and for concrete real world applications. From a physical point of view, the nonlocal operators play a crucial role in describing several phenomena such as, the thin obstacle problem, optimization, phase transitions, material science, water waves, geophysical fluid dynamics and mathematical finance. For further details on these applications, the reader may refer to [6], [8], [9] and the references therein. For a general reference to this topic, one may refer to the recent article of Vázquez [33].

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Of late there has been a rapid growth in the literature on problems involving non-local operators and hence without being exhaustive we now give a literature survey all of which might not be related to our results but is worth mentioning. A testimony to this can be found in [4], [5], [22], [23], [31], [34] in the form of existence and multiplicity results for nonlocal operators like fractional Laplacian, fractional p -Laplacian in combination with a convex or a concave type nonlinearity. An eigenvalue problem for the fractional p -Laplacian and properties like finding the smallest eigenvalue are studied in [13], [18] and [21]. The Brezis-Nirenberg results for the fractional Laplacian and for the fractional p -Laplacian operator has been considered in [30] and [24] respectively. A Dirichlet boundary value problem in the case of fractional Laplacian with polynomial type nonlinearity using variational methods has been studied in [7], [10], [28] and [29]. In [1], the authors proved the existence of weak solution of the fractional p -Laplacian equations with weight for any datum in L^1 .

Recently, Piersanti and Pucci [26] have proved existence results of nontrivial solutions of a perturbed, nonlinear eigenvalue problem involving the integro-differential nonlocal operator that includes the fractional p -Laplacian case. Kussi et al. [20], has established an existence, regularity and potential theory for a nonlocal integro-differential equations involving measure data. The nonlocal elliptic operators considered here are possibly degenerate and also cover the case of the fractional p -Laplacian operator. Based on some generalization of the Wolff potential theory, the authors obtained the existence of a weak solution belonging to a suitable fractional Sobolev space.

Motivated by the interest shared by the mathematical community in this topic, we study here the equivalence of the following two problems involving a nonlocal operator,

$$\begin{aligned}
 P_1 : \quad & -\mathcal{L}_\Phi u = \lambda |u|^{q-2}u \text{ in } \Omega, \\
 & u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
 \end{aligned}
 \tag{1}$$

and

$$\begin{aligned}
 P_2 : \quad & -\mathcal{L}_\Phi u = \lambda |u|^{q-2}u + f, \quad f(\neq 0) \in L^{p'}(\Omega), \\
 & u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
 \end{aligned}
 \tag{2}$$

in the sense that if one problem has a nontrivial weak solution then the other one also has a nontrivial weak solution. We emphasize that the λ appearing in the problem P_1 is different from the λ in the problem P_2 . Here Ω is a bounded open subset of \mathbb{R}^N for $N \geq 2$ and $-\mathcal{L}_\Phi$ is a nonlocal operator (refer, [20]) which is defined as

$$\langle -\mathcal{L}_\Phi u, \varphi \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u(x) - u(y))(\varphi(x) - \varphi(y))K(x, y)dx dy,
 \tag{3}$$

for every smooth function φ with compact support, i.e., $\varphi \in C_c^\infty(\mathbb{R}^N)$. The function Φ is a real valued continuous function over \mathbb{R} , satisfying $\Phi(0) = 0$ together with the following monotonicity property

$$\Lambda^{-1}|t|^p \leq \Phi(t) \leq \Lambda|t|^p, \quad \forall t \in \mathbb{R}.
 \tag{4}$$

The kernel $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a measurable function satisfying the following ellipticity property

$$\frac{1}{\Lambda|x-y|^{N+sp}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{N+sp}}, \forall x,y \in \mathbb{R}^N, x \neq y, \tag{5}$$

where $\Lambda \geq 1, s \in (0, 1), p > 2 - \frac{s}{N}$ with $q \in (p, \frac{Np}{N-sp} = p_s^*)$ and $p' = \frac{p}{p-1}$, the conjugate of p . Assumptions made in (4) and (5) makes the nonlocal operator $-\mathcal{L}_\Phi$ to be an elliptic operator. Note that, upon taking the special case $\Phi(t) = |t|^{p-2}t$ with $K(x,y) = |x-y|^{-(N+sp)}$ in (3), we recall the fractional p -Laplacian [27] which is defined as

$$(-\Delta_p)^s u = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}[u(x) - u(y)]}{|x-y|^{N+ps}} dy. \tag{6}$$

In other words, the nonlocal operator $-\mathcal{L}_\Phi$, is a generalization of the fractional p -Laplacian for $1 \leq p < \infty$ and $s \in (0, 1)$.

A similar type of equivalence result for problems P_1 and P_2 , but involving the local operator $-\Delta_p$, defined as $\nabla \cdot (|\nabla(\cdot)|^{p-2}\nabla(\cdot))$, in place of nonlocal operator $-\mathcal{L}_\Phi$, are obtained in the work of Giri and Choudhuri [16]. Few other classical and noteworthy existence and multiplicity results has been studied for the case of local operators like Laplacian and p -Laplacian operators by several mathematicians. For instance, Azorero and Alonso [15] have found the conditions on p, q and λ for the existence of infinitely many nontrivial solutions to the problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2} u \text{ in } \Omega \subset \mathbb{R}^N, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Bahri [2] considered nonlinear elliptic equations of the type

$$\begin{aligned} -\Delta u &= |u|^{p-1} u + h(x) \text{ in } \Omega \subset \mathbb{R}^N, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where the function $h \in L^2(\Omega)$ and has proved the existence and multiplicity of the solutions. Bahri and Lions [3] have found a range of $p, 1 < p < \frac{N+2(1-\alpha)}{N-2}, 0 \leq \alpha < 2$, for which the elliptic equation

$$\begin{aligned} -\Delta u &= |u|^{p-1} u - f(x, u) \text{ in } \Omega \subset \mathbb{R}^N, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $f(x, t)$ is Caratheodory function on $\Omega \times \mathbb{R}$ with some growth conditions, has infinitely many solutions. Further, Tarantello [32] studied the Dirichlet problem

$$\begin{aligned} -\Delta u &= |u|^{p-2} u + f \text{ in } \Omega \subset \mathbb{R}^N, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

with $p = \frac{2N}{N-2}$, $N \geq 3$, $f \in H^{-1}(\Omega)$ and has proved the existence of two solutions in $H_0^1(\Omega)$.

In this paper, we use the variational approach to the nonlocal framework. Inspired by the fractional Sobolev spaces, we will work in a functional analytic set up in order to correctly encode the Dirichlet boundary datum in the variational formulation. The paper has been organized as follows. In section 2, we will present some useful tools and preliminaries that we will use throughout this paper like fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ and embedding results of $W^{s,p}(\mathbb{R}^N)$ into the Lebesgue space. We will also define the weak sense in which the solutions to the problems P_1 and P_2 are defined. In section 3, we will discuss a few preliminary and main results. In section 4, we will prove a necessary condition for the existence of a weak solution to the problem

$$\begin{aligned} -\mathcal{L}_\Phi u &= \lambda |u|^{q-2}u + \mu \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where $\mu \in \mathcal{M}(\Omega)$.

2. Functional analytic setup and Main tools

In this section, we discuss the functional analytic setting that will be used below. Due to the nonlocal character of \mathcal{L}_Φ defined in (3), it is natural to work with Sobolev space $W^{s,p}(\mathbb{R}^N)$ and express the Dirichlet condition in $\mathbb{R}^N \setminus \Omega$ rather than $\partial\Omega$. Though fractional Sobolev spaces are well known since the beginning of the last century especially in the field of harmonic analysis, they have become increasingly popular in the last few years under the impulse of the work of Caffarelli & Silvestre [9] and the references therein. We now turn to our problem for which we provide the variational setting on a suitable function space for (1) and (2), jointly with some preliminary results. For all measurable functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$, we set

$$\begin{aligned} \|u\|_{L^p(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N} |u(x)|^p dx \right)^{\frac{1}{p}}, \\ [u]_{s,p} &= \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}, \end{aligned}$$

where $p \in (1, \infty)$ and $s \in (0, 1)$. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as the space of all function $u \in L^p(\mathbb{R}^N)$ such that $[u]_{s,p}$ is finite and endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|_{L^p(\mathbb{R}^N)}^p + [u]_{s,p}^p \right)^{\frac{1}{p}}.$$

More on fractional Sobolev space can be found in Nezza et al. [11] and the references therein. We now define a closed linear subspace of $W^{s,p}(\mathbb{R}^N)$:

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

It is trivial to see that the norms $\|\cdot\|_{L^p(\mathbb{R}^N)}$ and $\|\cdot\|_{L^p(\Omega)}$ agree on $W_0^{s,p}(\Omega)$. We also have a Poincaré type inequality which is as follows:

$$\|u\|_{L^p(\Omega)} \leq C[u]_{s,p}, \text{ for all } u \in W_0^{s,p}(\Omega).$$

Thus, we define the norm on the space $W_0^{s,p}(\Omega)$ by setting $\|u\|_{W_0^{s,p}(\Omega)} = [u]_{s,p}$. Let $p_s^* = \frac{Np}{N-sp}$, with the agreement that $p_s^* = \infty$ if $N \geq sp$. It is well known that $(W_0^{s,p}(\Omega), \|\cdot\|_{W_0^{s,p}(\Omega)})$ is a uniformly convex reflexive Banach space, continuously embedded into $L^r(\Omega)$, for all $r \in [1, p_s^*]$ if $sp < N$, for all $1 \leq r < \infty$ if $N = sp$ and into $L^\infty(\Omega)$ if $N < sp$. It is also compactly embedded in $L^r(\Omega)$ for any $r \in [1, p_s^*]$ if $N \geq sp$ and in $L^\infty(\Omega)$ for $N < sp$. Furthermore, $C_c^\infty(\Omega)$ is a dense subspace of $W_0^{s,p}(\Omega)$ with respect to $\|\cdot\|_{W_0^{s,p}(\Omega)}$. For further detail on the embedding results, we refer the reader to [17], [25] and the references therein.

We define an associated energy functional to the problem P_1 as

$$I_{P_1}(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{P}\Phi(u(x) - u(y))K(x, y)dx dy - \frac{\lambda}{q} \int_{\Omega} |u|^q dx,$$

where $\mathcal{P}\Phi(t) := \int_0^{|t|} \Phi(\tau)d\tau$ being the primitive of Φ . Thus by (4) we have

$$\Lambda^{-1} \frac{|t|^p}{p} \leq \mathcal{P}\Phi(t) \leq \Lambda \frac{|t|^p}{p}, \tag{7}$$

for $t \neq 0$ and $\mathcal{P}\Phi(0) = 0$. The Fréchet derivative of I_{P_1} , which is in $W_0^{-s,p'}(\Omega)$, the dual space of $W_0^{s,p}(\Omega)$ where $p' = \frac{p}{p-1}$ is defined as

$$\langle I'_{P_1}(u), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u(x) - u(y))(v(x) - v(y))K(x, y)dx dy - \lambda \int_{\Omega} |u|^{q-2}uv dx, \tag{8}$$

for every $v \in W_0^{s,p}(\Omega)$.

DEFINITION 1. We say that $u \in W_0^{s,p}(\Omega)$ is a weak (energy) solution to the problem P_1 if

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u(x) - u(y))(\varphi(x) - \varphi(y))K(x, y)dx dy = \lambda \int_{\Omega} |u|^{q-2}u\varphi dx,$$

holds for every $\varphi \in C_c^\infty(\Omega)$.

The weak solutions of the problem P_1 are the critical points of the energy functional I_{P_1} . Similarly, let the corresponding associated energy functional to the problem P_2 be denoted by I_{P_2} which is defined as follows:

$$I_{P_2}(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{P}\Phi(u(x) - u(y))K(x, y)dx dy - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \int_{\Omega} f u dx \tag{9}$$

whose Fréchet derivative is defined as

$$\langle I'_{P_2}(u), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u(x) - u(y))(v(x) - v(y))K(x, y)dx dy - \lambda \int_{\Omega} |u|^{q-2}uv dx - \int_{\Omega} f v dx, \tag{10}$$

for every $v \in W_0^{s,p}(\Omega)$.

DEFINITION 2. We say that $u \in W_0^{s,p}(\Omega)$ is a weak (energy) solution of the problem P_2 if

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u(x) - u(y))(\varphi(x) - \varphi(y))K(x, y)dxdy - \lambda \int_{\Omega} |u|^{q-2}u\varphi dx - \int_{\Omega} f\varphi dx = 0,$$

for all $\varphi \in C_c^\infty(\Omega)$.

The main results in this paper, when stated heuristically, are as follows. The problem P_1 has a nontrivial weak solution if and only if the problem P_2 has a non trivial weak solution. In the implication from P_1 to P_2 , the main tool we will use is the Mountain Pass Theorem [12, 19]. For the converse part, we guarantee the existence of a weak solution to the problem P_1 by considering a sequence of P_2 type problems whose nonhomogeneous part will be denoted by (f_n) . In addition to this we will assume that $f_n \rightarrow 0$ in $L^{p'}(\Omega)$. The corresponding sequence of weak solutions (u_n) to the problem P_2 , will be shown to have a strongly convergent subsequence. The subsequence will still be denoted by (u_n) whose limit is, say u . We complete the proof of the converse part by showing that this limit u is a weak solution to P_1 . One common result, which will be used in proving both the implications is as follows:

THEOREM 1. *If the sequence $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, then*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\Phi(u_n(x) - u_n(y)) - \Phi(u(x) - u(y)))(\varphi(x) - \varphi(y))K(x, y)dxdy \rightarrow 0$$

for all $\varphi \in C_c^\infty(\Omega)$.

The proof of this theorem, can be found in the work of Kussi et al. [20], Theorem 1.1.

3. Existence results

We begin the section by assuming that the problem P_1 has a nontrivial weak solution in $W_0^{s,p}(\Omega)$. In order to show the existence of a non trivial weak solution to the problem P_2 , we will use the Mountain pass theorem. To apply the Mountain pass theorem, we need the following technical lemmas.

LEMMA 1. *The function I defined in (9) is a C^1 functional over $W_0^{s,p}(\Omega)$.*

Proof. It is trivial to see that the functional I is differentiable over $W_0^{s,p}(\Omega)$. Thus it is enough to show that $I'(u)$ is continuous. Thus from (10), for each $u \in W_0^{s,p}(\Omega)$ we have

$$\begin{aligned}
 |\langle I'(u), v \rangle| &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\Phi(u(x) - u(y))| |(v(x) - v(y))| K(x, y) dx dy + \lambda \int_{\Omega} |u|^{q-1} |v| dx \\
 &\quad + \int_{\Omega} |f| |v| dx \\
 &\leq \Lambda^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-1} |v(x) - v(y)|}{|x - y|^{N+sp}} dx dy + \lambda \|u\|_{\frac{q}{q-1}} \|v\|_q \\
 &\quad + \|f\|_{p'} \|v\|_p \\
 &\leq \left[\left\| \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{\frac{N+sp}{p'}}} \right\|_{p'} + C_1 \lambda \|u\|_{\frac{q}{q-1}} + C_2 \|f\|_{p'} \right] \|v\|_{W_0^{s,p}(\Omega)},
 \end{aligned}$$

where C_1, C_2 are constants obtained from the embedding of $W_0^{s,p}(\Omega)$ into $L^q(\Omega)$ for $q \in (p, p_s^*)$ and into $L^p(\Omega)$ respectively. Thus I is a C^1 functional over $W_0^{s,p}(\Omega)$. \square

LEMMA 2. *For the functional I given in the lemma 1, there exist $u_0, u_1 \in W_0^{s,p}(\Omega)$ and a positive real number C_3 such that $I(u_0), I(u_1) < C_3$ and $I(v) \geq C_3$, for every v satisfying $\|v - u_0\|_{W_0^{s,p}(\Omega)} = r$ for some $r > 0$.*

Proof. Let us consider $u_0 = 0$ and let $w \in B(0, 1) = \{u \in W_0^{s,p}(\Omega) : \|u\|_{W_0^{s,p}(\Omega)} = 1\}$. Consider $v = u_0 + rw$ for $r > 0$ so that $\|v - u_0\|_{W_0^{s,p}(\Omega)} = r$. We first show that there exists a $r > 0$ such that for each v satisfying $\|v - u_0\|_{W_0^{s,p}(\Omega)} = r$ we have $I(v) \geq C_3$, where $C_3 > 0$. From the monotonicity and ellipticity conditions in (4) and (5) respectively,

$$\begin{aligned}
 I(u) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{P}\Phi(u(x) - u(y))K(x, y) dx dy - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \int_{\Omega} f u dx \\
 &\geq \frac{\Lambda^{-2}}{p} \|u\|_{W_0^{s,p}(\Omega)}^p - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \int_{\Omega} f u dx
 \end{aligned}$$

Thus we have,

$$I(u_0 + rw) - I(u_0) \geq \frac{\lambda^{-2} r^p}{p} \|w\|_{W_0^{s,p}(\Omega)}^p - \frac{\lambda r^q}{q} \|w\|_q^q - r \int_{\Omega} f w dx.$$

Note that $\|w\|_{W_0^{s,p}(\Omega)} = 1$ and $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$, hence $\|w\|_p^p \leq C_4 \|w\|_{W_0^{s,p}(\Omega)} = C_4$.

Similarly, since $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in (p, p_s^*)$, we have $\|w\|_q^q \leq C_5$. This leads to

$$I(u_0 + rw) - I(u_0) \geq r \left[\frac{\Lambda^{-2} r^{p-1}}{p} - \frac{r^{q-1} \lambda}{q} C_5 - C_4^{\frac{1}{p}} \|f\|_{p'} \right].$$

Let $F(r) = \frac{\Lambda^{-2} r^{p-1}}{p} - \frac{r^{q-1} \lambda}{q} C_5 - C_4^{\frac{1}{p}} \|f\|_{p'}$. Then $F(0) < 0$ and for $r_0 = \frac{q(p-1)}{p(q-1)} \cdot \frac{\Lambda^{-2}}{\lambda C_5}$, we see that $F'(r_0) = 0$. Further, a bit of calculus guarantees that $F''(r_0) < 0$ and hence

r_0 is a maximizer of F . Note that, if $0 < \lambda < \lambda_1 = \frac{\Lambda^{-2q(p-1)}}{C_5^p(p-1)} \left(\frac{q-p}{p(q-1)} \cdot \frac{1}{C_4^{1/p} \|f\|_{p'}} \right)^{\frac{q-p}{p-1}}$, then $F(r_0) > 0$. Hence there exist $r_1, r_2 > 0$ and $r_1 < r_0 < r_2$ such that $F(r) > 0$ for each $r \in (r_1, r_2)$. Thus, we choose $r = r_0$ such that $\|v - u_0\|_{W_0^{s,p}(\Omega)} = r_0$ and for which $I(v) \geq C_3 > 0$ for each v such that $\|v - u_0\|_{W_0^{s,p}(\Omega)} = r_0$.

Choice of u_1 : Let w_q be a nontrivial weak solution of the problem P_1 . Then consider $g = kw_q$, $k \in \mathbb{R}$, where we have normalized w_q with respect to the norm of $W_0^{s,p}(\Omega)$ without changing its notation. Now we have,

$$I(u) \leq \frac{\Lambda^2}{p} \|u\|_{W_0^{s,p}(\Omega)}^p - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \int_{\Omega} f u dx.$$

From this it can be seen that

$$I(g) \leq \frac{\Lambda^2 k^p}{p} - \frac{\lambda k^q}{q} \int_{\Omega} |w_q|^q dx - kC,$$

where $C = \int_{\Omega} f w_q dx$. Since $p < q < p_s^*$, we observe that k can be chosen so large, say k_0 such that $\frac{\Lambda^2 k_0^p}{p} - \frac{\lambda k_0^q}{q} \int_{\Omega} |w_q|^q dx - k_0 C < 0$. Then $I(k_0 w_q) < 0$. Thus we can choose $u_1 = k_0 w_q$, where $k_0 > r_0$, due to which $\|u_1 - u_0\|_{W_0^{s,p}(\Omega)} > r_0$. Hence the lemma follows. \square

LEMMA 3. *The functional I given in the lemma 1 satisfies the Palais-Smale condition when $\Lambda \in [1, (\frac{q}{p})^{1/4}]$.*

Proof. Let (u_m) be a sequence in $W_0^{s,p}(\Omega)$ such that $|I(u_m)| \leq M$ and $I'(u_m) \rightarrow 0$ in $W^{-s,p'}(\Omega)$. Now,

$$\begin{aligned} \langle I'(u_m), v \rangle &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u_m(x) - u_m(y))(v(x) - v(y))K(x, y) dx dy \\ &\quad - \lambda \int_{\Omega} |u_m|^{q-2} u_m v dx - \int_{\Omega} f v dx, \end{aligned}$$

for every $v \in W_0^{s,p}(\Omega)$. From the definition of the functional and its derivative we have

$$\begin{aligned} \langle I'(u_m), u_m \rangle &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u_m(x) - u_m(y))(u_m(x) - u_m(y))K(x, y) dx dy \\ &\quad - \lambda \int_{\Omega} |u_m|^q dx - \int_{\Omega} f u_m dx \end{aligned}$$

$$I(u_m) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{P} \Phi(u_m(x) - u_m(y))K(x, y) dx dy - \frac{\lambda}{q} \int_{\Omega} |u_m|^q dx - \int_{\Omega} f u_m dx.$$

From the above two equations, it follows that

$$\begin{aligned}
 & q \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{P}\Phi(u_m(x) - u_m(y))K(x,y)dx dy \\
 & \quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u_m(x) - u_m(y))(u_m(x) - u_m(y))K(x,y)dx dy \\
 & = qI(u_m) - \langle I'(u_m), u_m \rangle + (q - 1) \int_{\Omega} f u_m dx
 \end{aligned} \tag{11}$$

From (4) and (5), we have

$$\begin{aligned}
 \frac{1}{\Lambda^2} \|u_m\|_{W_0^{s,p}(\Omega)}^p & \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u_m(x) - u_m(y))(u_m(x) - u_m(y))K(x,y)dx dy \\
 & \leq \Lambda^2 \|u_m\|_{W_0^{s,p}(\Omega)}^p
 \end{aligned}$$

and hence

$$\frac{1}{\Lambda^2 p} \|u_m\|_{W_0^{s,p}(\Omega)}^p \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{P}\Phi(u_m(x) - u_m(y))K(x,y)dx dy \leq \frac{\Lambda^2}{p} \|u_m\|_{W_0^{s,p}(\Omega)}^p.$$

From (11), we get

$$\begin{aligned}
 \frac{(q - p\Lambda^4)}{\Lambda^2 p} \|u_m\|_{W_0^{s,p}(\Omega)}^p & \leq qI(u_m) - \langle I'(u_m), u_m \rangle + (q - 1) \int_{\Omega} f u_m dx \\
 & \leq qM + \|I'(u_m)\|_{-s,p'} \|u_m\|_{W_0^{s,p}(\Omega)} + (q - 1)c \|f\|_{p'} \|u_m\|_{W_0^{s,p}(\Omega)}
 \end{aligned}$$

From this, it follows that the sequence $(\|u_m\|_{W_0^{s,p}(\Omega)})$ is bounded. If not, then divide by $\|u_m\|_{W_0^{s,p}(\Omega)}^p$ in the above and let $m \rightarrow \infty$. On using $\|I'(u_m)\|_{-s,p'} \rightarrow 0$, we get a contradiction viz. $\frac{(q - p\Lambda^4)}{\Lambda^2 p} \leq 0$, by the assumption $q - p\Lambda^4 > 0$. Thus there exists a subsequence of (u_m) , which will still be denoted by (u_m) , converge weakly to u in $W_0^{s,p}(\Omega)$. We will now show that this subsequence (u_m) is strongly convergent in $W_0^{s,p}(\Omega)$.

We know that $W_0^{s,p}(\Omega)$ is compactly embedded in $L^r(\Omega)$, $r \in [1, p_s^*)$ and hence $u_m \rightarrow u$ in $L^r(\Omega)$, for $1 \leq r < p_s^*$. Consider $\tilde{u}_m = u_m - u$. Then $\tilde{u}_m \rightarrow 0$ in $W_0^{s,p}(\Omega)$. Consider,

$$\begin{aligned}
 \langle I'(\tilde{u}_m), \tilde{u}_m \rangle & = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(\tilde{u}_m(x) - \tilde{u}_m(y))(\tilde{u}_m(x) - \tilde{u}_m(y))K(x,y)dx dy \\
 & \quad - \lambda \int_{\Omega} |\tilde{u}_m|^q dx - \int_{\Omega} f \tilde{u}_m dx
 \end{aligned} \tag{12}$$

The second and the third term of the functional approach to 0 as $m \rightarrow \infty$. This is because $\tilde{u}_m \rightarrow 0$ in $W_0^{s,p}(\Omega)$ implies $\int_{\Omega} f \tilde{u}_m dx \rightarrow 0$ and $u_m \rightarrow u$ in $L^r(\Omega)$ for $r \in [1, p_s^*)$ implies $\int_{\Omega} |\tilde{u}_m|^q dx \rightarrow 0$. By the definition of weak convergence, for all $u \in W_0^{s,p}(\Omega)$ we have $\langle I'(u), \tilde{u}_m \rangle \rightarrow 0$, as $m \rightarrow \infty$, i.e. $\lim_{m \rightarrow \infty} \langle I'(u), \tilde{u}_m \rangle = 0$, for all $u \in W_0^{s,p}(\Omega)$. Upon taking $u = \tilde{u}_n$,

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \langle I'(\tilde{u}_n), \tilde{u}_m \rangle = 0.$$

We now consider,

$$\begin{aligned} \langle I'(\tilde{u}_n), \tilde{u}_m \rangle &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(\tilde{u}_n(x) - \tilde{u}_n(y))(\tilde{u}_m(x) - \tilde{u}_m(y))K(x, y) dx dy \\ &\quad - \lambda \int_{\Omega} |\tilde{u}_n|^{q-2} \tilde{u}_n \tilde{u}_m dx - \int_{\Omega} f \tilde{u}_m dx \end{aligned} \tag{13}$$

Since $\tilde{u}_n \rightharpoonup 0$, hence by the work of Kussi et al. [20], we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(\tilde{u}_n(x) - \tilde{u}_n(y))(\tilde{u}_m(x) - \tilde{u}_m(y))K(x, y) dx dy = 0.$$

In addition to this we have, $\lim_{n \rightarrow \infty} \int_{\Omega} |\tilde{u}_n|^{q-2} \tilde{u}_n \tilde{u}_m dx = 0$. Passing to the limit $n \rightarrow \infty$ in (13), we have,

$$\lim_{n \rightarrow \infty} \langle I'(\tilde{u}_n), \tilde{u}_m \rangle = \int_{\Omega} f \tilde{u}_m dx. \tag{14}$$

On further passing to the limit $m \rightarrow \infty$ in (14) we get,

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle I'(\tilde{u}_n), \tilde{u}_m \rangle = 0.$$

Therefore, we have

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle I'(\tilde{u}_n), \tilde{u}_m \rangle = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \langle I'(\tilde{u}_n), \tilde{u}_m \rangle = 0.$$

Hence it follows that $\lim_{m \rightarrow \infty} \langle I'(\tilde{u}_m), \tilde{u}_m \rangle = 0$. Passing to the limit $m \rightarrow \infty$ in (12), we get

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(\tilde{u}_m(x) - \tilde{u}_m(y))(\tilde{u}_m(x) - \tilde{u}_m(y))K(x, y) dx dy = 0.$$

But,

$$0 \leq \frac{1}{\Lambda^2} \|\tilde{u}_m\|_{W_0^{s,p}(\Omega)}^p \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(\tilde{u}_m(x) - \tilde{u}_m(y))(\tilde{u}_m(x) - \tilde{u}_m(y))K(x, y) dx dy.$$

From this it is clear that $\lim_{m \rightarrow \infty} \|\tilde{u}_m\|_{W_0^{s,p}(\Omega)} = 0$, that is $\tilde{u}_m \rightarrow 0$ in $W_0^{s,p}(\Omega)$. Hence $u_m \rightarrow u$ in $W_0^{s,p}(\Omega)$. \square

THEOREM 2. *Suppose that the problem*

$$\begin{aligned} P_1 : \quad & -\mathcal{L}_{\Phi}u = \lambda |u|^{q-2}u \text{ in } \Omega, \\ & u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

has a nontrivial weak solution for some $\lambda > 0$, where $q \in (p, p_s^*)$. Then there exists $\lambda_1 > 0$ such that for all $\lambda \in (0, \lambda_1)$, the problem

$$\begin{aligned} P_2 : \quad & -\mathcal{L}_{\Phi}u = \lambda |u|^{q-2}u + f, \quad f(\neq 0) \in L^{p'}(\Omega), \\ & u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

has a nontrivial weak solution whenever $\Lambda \in [1, (\frac{q}{p})^{1/4})$, where Λ is as in equation (4).

Proof. By the results proved in Lemmas 1, 2 and 3, it follows that the functional I associated with the problem P_2 satisfies all the condition of Mountain Pass theorem. So by the Mountain Pass theorem, an extreme point for I exists in $W_0^{s,p}(\Omega)$, which is a weak solution to the problem P_2 . \square

The Theorem 2 shows that if the problem P_1 has a nontrivial weak solution then the problem P_2 also has a nontrivial weak solution under certain restrictions on λ and Λ respectively. We now show the existence of nontrivial weak solution to the problem P_1 with the assumption that the problem P_2 has a nontrivial weak solution for each $f(\neq 0) \in L^{p'}(\Omega)$ in the set $\mathfrak{M} = \{u \in W_0^{s,p}(\Omega) : \|u\|_q = 1\}$ for some $\lambda > 0$, where $q \in (p, p_s^*)$. Existence of such a solution in the subset \mathfrak{M} of $W_0^{s,p}(\Omega)$ can be guaranteed from the weak lower semicontinuity and coercivity of the associated functional I_{P_2} .

In order to show the existence of nontrivial weak solution of the problem P_1 for $q \in (p, p_s^*)$, let us consider a sequence $(f_n) \subset L^{p'}(\Omega)$ such that $f_n \rightarrow 0$ in $L^{p'}(\Omega)$. Then for each such f_n , there exists a weak solution to the problem P_2 , say u_n . Thus each u_n is a critical point of the functional I_{P_2} , i.e., $\langle I'_{P_2}(u_n), \varphi \rangle = 0$ for every $\varphi \in W_0^{s,p}(\Omega)$. In particular, $\langle I'_{P_2}(u_n), u_n \rangle = 0$. This implies that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u_n(x) - u_n(y))(u_n(x) - u_n(y))K(x, y)dx dy - \lambda \int_{\Omega} |u_n|^q dx = \int_{\Omega} f_n u_n dx. \tag{15}$$

Further, since $\|u_n\|_q = 1$ and using (4) and (5) in (15), we have

$$\Lambda^{-2} \|u_n\|_{W_0^{s,p}(\Omega)}^p - \lambda \leq \int_{\Omega} f_n u_n dx \leq C \|f_n\|_{p'} \|u_n\|_{W_0^{s,p}(\Omega)} \leq CM \|u_n\|_{W_0^{s,p}(\Omega)},$$

where M is such that $\|f_n\|_{p'} \leq M$. It follows that $(\|u_n\|_{W_0^{s,p}(\Omega)})$ is bounded, for if not, then divide by $\|u_n\|_{W_0^{s,p}(\Omega)}^p$ and let $n \rightarrow \infty$ to get a contradiction viz. $\Lambda^{-2} \leq 0$. Hence there exists a subsequence (u_n) which converge weakly to u in $W_0^{s,p}(\Omega)$. Since $W_0^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for $q \in (p, p_s^*)$, hence $u_n \rightarrow u$ in $L^q(\Omega)$. Therefore,

$$\int_{\Omega} |u_n|^{q-2} u_n v dx \rightarrow \int_{\Omega} |u|^{q-2} u v dx$$

and $\int_{\Omega} f_n v dx \rightarrow 0$ for each $v \in W_0^{s,p}(\Omega)$. By Kussi et al. [20] we have,

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u_n(x) - u_n(y))(v(x) - v(y))K(x, y)dx dy \\ & \rightarrow \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u(x) - u(y))(v(x) - v(y))K(x, y)dx dy. \end{aligned}$$

Passing to the limit $n \rightarrow \infty$ in the following equation

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u_n(x) - u_n(y))(v(x) - v(y))K(x, y)dx dy - \lambda \int_{\Omega} |u_n|^{q-2} u_n v dx = \int_{\Omega} f_n v dx,$$

we get,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(u(x) - u(y))(v(x) - v(y))K(x, y)dx dy - \lambda \int_{\Omega} |u|^{q-2} u v dx = 0,$$

for each $v \in W_0^{s,p}(\Omega)$. This shows that u is weak solution of the problem P_1 .

Assume that $\lambda \in \left(0, \inf_{u \neq 0 \in W_0^{s,p}(\Omega)} \frac{\|u\|_{W_0^{s,p}(\Omega)}}{\|u\|_q}\right]$. As in the proof of Lemma 3, it is easy to see that

$$\|u_n\|_{W_0^{s,p}(\Omega)} \rightarrow \|u\|_{W_0^{s,p}(\Omega)}.$$

Since $\|u_n\|_q = 1$ and $u_n \rightarrow u$ in $W_0^{s,p}(\Omega)$, we have

$$0 < \lambda \leq \liminf \frac{\|u_n\|_{W_0^{s,p}(\Omega)}}{\|u_n\|_q} = \liminf \|u_n\|_{W_0^{s,p}(\Omega)} = \|u\|_{W_0^{s,p}(\Omega)}$$

This implies that u is a nontrivial weak solution to the problem P_1 . We thus have proved the following theorem.

THEOREM 3. *Suppose that to each $f \in L^{p'}(\Omega)$, the problem*

$$P_2 : \quad -\mathcal{L}_\Phi u = \lambda |u|^{q-2} u + f, \quad f(\neq 0) \in L^{p'}(\Omega), \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

has a nontrivial weak solution in $\mathfrak{M} \subset W_0^{s,p}(\Omega)$ for some $\lambda > 0$, where $q \in (p, p_s^*)$. Then the problem

$$P_1 : \quad -\mathcal{L}_\Phi u = \lambda |u|^{q-2} u, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega$$

has a nontrivial solution in $W_0^{s,p}(\Omega)$, whenever $\lambda \in \left(0, \inf_{u \neq 0 \in W_0^{s,p}(\Omega)} \frac{\|u\|_{W_0^{s,p}(\Omega)}}{\|u\|_q}\right]$.

4. A necessary condition for the existence of a weak solution

We now prove a necessary condition for the existence of a weak solution to the problem

$$-\mathcal{L}_\Phi u = \lambda |u|^{q-2} u + \mu, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \tag{16}$$

where μ is a measure.

DEFINITION 3. Suppose $K \subset \Omega$ is a compact set, then we define the capacity as

$$Cap_{s,q}(K) = \inf\{\|\varphi\|_{W^{s,q}(\Omega)}^q : \varphi \in C_c^\infty(\Omega), 0 \leq \varphi \leq 1, \varphi \equiv 1 \text{ in } K\}.$$

THEOREM 4. *Suppose $2 < p < q$ and u is a weak solution to the problem (16) then $\mu \in L^1(\Omega) + W^{-s,p'}(\Omega)$.*

Proof. Suppose u is a weak solution to (16). Choose a test function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) \equiv 1$ over a compact subset of Ω say K and $0 \leq \varphi \leq 1$ in Ω from which one has $\varphi \geq \chi_K$. Thus

$$\begin{aligned} \mu(K) &\leq \left| \int_{\Omega} -\mathcal{L}_{\Phi} u \varphi dx - \lambda \int_{\Omega} |u|^{q-2} u \varphi dx \right| \\ &\leq (C_6 \| -\mathcal{L}_{\Phi} u \|_{W^{-s,p'}(\Omega)} + C_7 \| u \|_q^{q-1}) \| \varphi \|_{s,q}. \end{aligned} \quad (17)$$

We refer to a result from Gallouët et al [14] that $\mu \in L^1(\Omega) + W^{-s,p}(\Omega)$ iff $\mu(K) = 0$ whenever $Cap_{s,q}(K) = 0$ for K compact subset of Ω . Coming back to our theorem, if we suppose $Cap_{s,q}(K) = 0$ then there exists (φ_n) such that $\| \varphi_n \|_{W^{s,q}(\Omega)}^q \rightarrow 0$. Hence if this sequence (φ_n) is used in (17), one has $\mu(K) = 0$. Thus we have $\mu \in L^1(\Omega) + W^{-s,p'}(\Omega)$. \square

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