Fractional
Differential
Calculus
Volume 8, Number 1 (2018), 191–203 doi:10.7153/fdc-2018-08-12

COEFFICIENT FUNCTIONAL FOR THE KTH ROOT TRANSFORM OF ANALYTIC FUNCTION AND APPLICATIONS TO FRACTIONAL DERIVATIVES

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(*Communicated by J. A. Tenreiro Machado*)

Abstract. In the present investigation, the authors introduce certain subclass of analytic function and obtain the sharp upper bounds for the coefficient functional $|b_{2k+1} - \nu b_{k+1}^2|$ corresponding to the kth root transformation of certain normalized analytic function defined on the unit disk Δ in the complex plane. As an application of the main results, we obtain the Fekete-Szegö inequalities for the function defined by fractional derivatives. Similar problems are investigated for the inverse function of f and for the function $\frac{z}{f(z)}$. Our results generalize and unify the work of earlier researchers in this direction.

1. Introduction and definition

Denote by $\mathscr{A}$, the class of functions of the form

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathbb{S}$ denote the family of functions $f(z) \in \mathscr{A}$ which are univalent.

For two analytic functions f and g in Δ, the function f is subordinate to g, written as $f(z) \prec g(z) \quad (z \in \Delta)$ if there exists a Schwarz function w, which (by definition) is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)) \quad (z \in \Delta)$. It follows from the Schwarz lemma that $f(z) \prec g(z) \quad (z \in \Delta) \Longrightarrow f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. If the function g is univalent in Δ then (see [14])

$$f(z) \prec g(z) \quad (z \in \Delta) \Longleftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

The Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for normalized univalent function $f(z)$ of the form (1) is well known for its rich history in the theory of geometric function theory. In 1933, Fekete-Szegö disproof the conjecture of Littlewood and Parley that the coefficient

Mathematics subject classification (2010): 30C45, 30C80.
Keywords and phrases: Analytic function, Fekete-Szegö inequality, kth root transformation, subordination, fractional derivatives.
The authors would like to thank to the editor and anonymous referees for their comments and suggestions which improve the contents of the manuscript. Further, the present investigation of the first-named author is supported by CSIR research project scheme no: 25(0278)/17/EMR-II, New Delhi, India.

of odd univalent functions are bounded by unity (see [10]). Since then, it received a great attention among many researchers (for details, see [1, 3, 5, 6, 7, 8, 9, 12, 16]). The technique used by Ma and Minda (see [13]) for the Fekete-Szegö problem for a univalent function $f(z)$ of the form (1) for subclasses of convex and starlike functions were used by many authors to solve the same problem for other classes.

Let k be a positive integer. A domain D is said to be k-fold symmetric if a rotation of D about the origin through an angle $\frac{2\pi}{k}$ carries D to itself. A function f is said to be k-fold symmetric in Δ if $f(e^{\frac{2\pi i}{k}}z) = e^{\frac{2\pi i}{k}}f(z)$ for every $z \in \Delta$. If f is regular and k-fold symmetric in Δ, then

$$f(z) = b_1 z + b_{k+1} z^{k+1} + b_{2k+1} z^{2k+1} + \cdots. \tag{2}$$

Conversely, if f is given by (2), then f is k-fold symmetric inside the circle of convergence of the series. For an univalent function f of the form (1), the kth root transformation is defined by

$$G(z) = \left[f(z^k)\right]^{\frac{1}{k}} = z\left[\frac{f(z^k)}{z^k}\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1}, \tag{3}$$

where the initial coefficients are

$$\begin{aligned} b_{k+1} &= \frac{a_2}{k}, \qquad b_{2k+1} = \frac{a_3}{k} + \frac{1-k}{2k^2}a_2^2 \\ b_{3k+1} &= \frac{a_4}{k} + \frac{1-k}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{3!k^3}a_2^3. \end{aligned}$$

Since f is univalent, so $\frac{f(z^k)}{z^k}$ is non-vanishing in Δ implies that the kth root is an analytic in Δ. The Fekete-Szegö coefficient functional of the associated function $G(z)$ is given by $|b_{2k+1} - \nu b_{k+1}^2|$. This quantity is known as Fekete-Szegö problem of the kth root transform G.

Recently, Ali et al. [2] (also see [20]) have investigated the Fekete-Szegö coefficient functional for the kth root transform of functions belonging to various subclasses of analytic functions by means of subordination. Further, Annamalai et al. [4] have obtained sharp bound of the Fekete-Szegö coefficient functional for the Janowski α-spirallike functions associated with the kth root transformation.

Motivated by the works of Ali et al. [2], Sharma et al. [20] and Annamalai et al. [4], in this paper the authors define generalized subclass of analytic functions of complex order and investigate the Fekete-Szegö coefficient functional associated with kth root transformation of the function f and for the function defined through convolution and fractional derivative in these classes. Similar approach is used to obtain the Fekete-Szegö inequalities for the inverse function of f and for the function $\frac{z}{f(z)}$.

DEFINITION 1. Let $\phi(z)$ be a univalent analytic function with positive real part on Δ with $\phi(0) = 1$ and $\phi'(0) > 0$ where $\phi(z)$ maps Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Let b be a non-zero complex number and γ be a real number such that $0 < \gamma \leqslant 1$. A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$ if

$$1+\frac{1}{b}\left[(1-\beta)\left(\frac{f(z)}{z}\right)^{\alpha}+\beta\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}-1\right] \prec [\phi(z)]^{\gamma} \quad (0 \leqslant \beta \leqslant 1,\ \ \alpha \geqslant 0). \tag{4}$$

The powers are taken with their principal values.

Note that, by specializing the parameters b,α,β and γ, the class $\mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$ reduces to the following classes studied by various earlier researchers.

- $\mathscr{R}^{\gamma}_{b,0,1}(\phi) = S^{\gamma}_{b}(\phi)$ introduced and studied by Sharma et al. [20].
- $\mathscr{R}^{1}_{1,0,1}(\phi) = S^{*}(\phi)$ introduced and studied by Ma and Minda [13] (also see [2]).
- $\mathscr{R}^{1}_{b,0,1}(\phi) = S_{b}(\phi)$ introduced and studied by Ravichandran et al. [18].
- $\mathscr{R}^{1}_{1,\alpha,1}(\phi) = B^{\alpha}(\phi)$ introduced and studied by Ravichandran et al. [19].
- $\mathscr{R}^{1}_{b,\alpha,\beta}(\phi) = R_{b,p,\alpha,\beta}(\phi)$ (with p=1) introduced and studied by Ramachandran et al. [17].

The paper is organized in the following manner. In Section 3, sharp upper bounds for the Fekete-Szegö coefficient functional $|b_{2k+1} - \nu b^{2}_{k+1}|$ associated with kth root transform of the function f belonging to the above mentioned class is investigated. In Section 4, applications for the functions defined by fractional derivatives is obtained. Similar results have been derived for the function $\frac{z}{f(z)}$ in Section 5 and inverse of the function f in Section 6.

2. Preliminaries

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$ satisfying the condition $|w(z)| < 1$.

We need the following lemmas in order to prove our main results:

LEMMA 1. (see [3]) *If $w \in \Omega$ and*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \Delta),$$

then for any real numbers t, we have

$$|w_2 - t w_1^2| \leqslant \begin{cases} -t & t \leqslant -1 \\ 1 & -1 \leqslant t \leqslant 1 \\ t & t \geqslant 1. \end{cases}$$

For $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < t < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality *holds for* $t=-1$ *if and only if* $w(z)=z\left(\frac{\varepsilon+z}{1+\varepsilon z}\right)$ $(0\leqslant\varepsilon\leqslant 1)$ *or one of its rotations, while for* $t=1$, *equality holds if and only if* $w(z)=-z\left(\frac{\varepsilon+z}{1+\varepsilon z}\right)$ $(0\leqslant\varepsilon\leqslant 1)$ *or one of its rotations.*

LEMMA 2. (see [11]) *If* $w\in\Omega$, *then*

$$|w_2-tw_1^2|\leqslant\max\{1,|t|\},$$

for any complex number t. *The result is sharp for the function* $w(z)=z^2$ *or* $w(z)=z$.

3. Coefficient bounds

In this section, the bounds for the functional $|b_{2k+1}-\nu b_{k+1}^2|$ corresponding to the kth root transformation for the function f in the above class is derived.

THEOREM 1. *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *with* $B_1>0$, $B_2\geqslant 0$ *and* B_n*'s real. If* $f\in\mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$ *and* G *is the* k*th root transformation of* f *given by (3), then for any complex number* ν, *we have*

$$|b_{2k+1}-\nu b_{k+1}^2|\leqslant\frac{|b|\gamma B_1}{(\alpha+2\beta)k}\max\left\{1,\left|\frac{b\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}[2\nu-(1-\alpha k)]-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right|\right\}. \tag{5}$$

The estimate is sharp.

Proof. Let $f\in\mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$. Then by Definition 1, there exists a Schwarz's function $w(z)\in\Omega$ with $w(0)=0$ and $|w(z)|<1$ such that

$$1+\frac{1}{b}\left[(1-\beta)\left(\frac{f(z)}{z}\right)^{\alpha}+\beta\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}-1\right]=[\phi(w(z))]^{\gamma}. \tag{6}$$

Since

$$\left(\frac{f(z)}{z}\right)^{\alpha}=1+\alpha a_2z+\left(\alpha a_3+\frac{\alpha(\alpha-1)}{2}a_2^2\right)z^2+\cdots$$

and

$$\frac{zf'(z)}{f(z)}=1+a_2z+(2a_3-a_2^2)z^2+\cdots,$$

it follows from above that

$$\begin{aligned}&1+\frac{1}{b}\left[(1-\beta)\left(\frac{f(z)}{z}\right)^{\alpha}+\beta\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}-1\right]\\&=1+\frac{\alpha+\beta}{b}a_2z+\left(\frac{\alpha+2\beta}{b}a_3+\frac{(\alpha-1)(\alpha+2\beta)}{2b}a_2^2\right)z^2+\cdots.\end{aligned} \tag{7}$$

Also,

$$[\phi(w(z))]^\gamma = [1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\cdots]^\gamma$$
$$= 1+\gamma B_1w_1z+\{\gamma B_1w_2+\gamma B_2w_1^2+\frac{\gamma(\gamma-1)}{2}B_1^2w_1^2\}z^2+\cdots. \quad (8)$$

Making use of (7) and (8) in (6), we obtain

$$a_2 = \frac{b\gamma B_1w_1}{\alpha+\beta} \quad (9)$$

and

$$a_3 = \frac{b\gamma}{\alpha+2\beta}\left[B_1w_2+B_2w_1^2+\frac{\gamma-1}{2}B_1^2w_1^2-\frac{(\alpha-1)(\alpha+2\beta)}{2(\alpha+\beta)^2}b\gamma B_1^2w_1^2\right]. \quad (10)$$

If $G(z)$ is the kth root transformation of $f(z)$, then

$$G(z) = [f(z^k)]^{\frac{1}{k}} = z+\frac{a_2}{k}z^{k+1}+\left[\frac{a_3}{k}-\frac{k-1}{2k^2}a_2^2\right]z^{2k+1}+\cdots$$
$$= z+\sum_{n=1}^{\infty}b_{nk+1}z^{nk+1}.$$

Equating the coefficients of z^{k+1} and z^{2k+1}, we get

$$b_{k+1} = \frac{a_2}{k} = \frac{b\gamma B_1w_1}{k(\alpha+\beta)}, \quad (11)$$

and

$$b_{2k+1} = \frac{a_3}{k}-\frac{k-1}{2k^2}a_2^2 = \frac{b\gamma B_1}{(\alpha+2\beta)k}\left[w_2+\frac{B_2}{B_1}w_1^2+\frac{\gamma-1}{2}B_1w_1^2\right.$$
$$\left.+\frac{(1-\alpha)(\alpha+2\beta)}{2(\alpha+\beta)^2}b\gamma B_1w_1^2-\left(1-\frac{1}{k}\right)\frac{\alpha+2\beta}{2(\alpha+\beta)^2}b\gamma B_1w_1^2\right]. \quad (12)$$

Thus, for any complex number ν, we have

$$b_{2k+1}-\nu b_{k+1}^2 = \frac{b\gamma B_1}{(\alpha+2\beta)k}[w_2-tw_1^2], \quad (13)$$

where

$$t = \frac{b\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}[2\nu-(1-\alpha k)]-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1.$$

Therefore,

$$|b_{2k+1}-\nu b_{k+1}^2| = \frac{|b|\gamma B_1}{(\alpha+2\beta)k}|w_2-tw_1^2|. \quad (14)$$

An application of Lemma 2 to the right hand side of (14) gives the desire result as stated in Theorem 1. The estimation is sharp and followed by

$$|b_{2k+1}-\nu b_{k+1}^2| = \begin{cases} \frac{|b|\gamma B_1}{(\alpha+2\beta)k} & (w(z)=z^2) \\ \frac{|b|\gamma B_1}{(\alpha+2\beta)k}\left\{\left|\frac{b\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k))-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right|\right\} & (w(z)=z). \end{cases}$$

This complete the proof of Theorem 1. □

REMARK 1. (i) Putting $\alpha=0$ and $\beta=1$ in Theorem 1 gives the result obtained by Sharma et al. [20].

(ii) Letting $\alpha=0,\ \beta=b=\gamma=1$ we get the result due to Ma and Minda [13] (also, see [2]).

Now, we determine the bounds for the functional $|b_{2k+1}-\nu b_{k+1}^2|$ for real ν for the class $\mathscr{R}^{\gamma}_{1,\alpha,\beta}(\phi)$. We denote such class by $\mathscr{R}^{\gamma}_{\alpha,\beta}(\phi)$.

THEOREM 2. *If $f\in\mathscr{R}^{\gamma}_{\alpha,\beta}(\phi)$ and G is the kth root transformation of the function f defined by (3), then for any real number ν and for*

$$d_1 = \frac{2k(\alpha+\beta)^2[B_2-B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2}$$

$$d_2 = \frac{2k(\alpha+\beta)^2[B_2+B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2}$$

we have

$$|b_{2k+1}-\nu b_{k+1}^2| \leqslant \begin{cases} \frac{\gamma B_1}{(\alpha+2\beta)k}\left[\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-\frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k))\right] & \nu\leqslant d_1, \\ \frac{\gamma B_1}{(\alpha+2\beta)k} & d_1\leqslant\nu\leqslant d_2 \\ \frac{\gamma B_1}{(\alpha+2\beta)k}\left[\frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k))-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right] & \nu\geqslant d_2. \end{cases} \tag{15}$$

Each of the estimates in (15) is sharp.

Proof. Let $f\in\mathscr{R}^{\gamma}_{\alpha,\beta}(\phi)$. From (13) we have

$$b_{2k+1}-\nu b_{k+1}^2 = \frac{\gamma B_1}{(\alpha+2\beta)k}[w_2-tw_1^2], \tag{16}$$

where

$$t = \frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}[2\nu-(1-\alpha k)]-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1.$$

Taking modulus on both sides of (16), we get

$$|b_{2k+1}-\nu b_{k+1}^2| = \frac{\gamma B_1}{(\alpha+2\beta)k}|w_2-tw_1^2|. \tag{17}$$

An application of Lemma 1 on right hand side of (17) gives the following cases.

Case 1: If $\nu \leqslant d_1$, then

$$\nu \leqslant \frac{2k(\alpha+\beta)^2[(B_2-B_1)+\frac{\gamma-1}{2}B_1^2]+(1-\alpha k)(\alpha+2\beta)\gamma B_1^2}{2(\alpha+2\beta)\gamma B_1^2}.$$

$$\Longrightarrow t \leqslant -1 \Longrightarrow |w_2-tw_1^2| \leqslant -t \Longrightarrow$$

$$|b_{2k+1}-\nu b_{k+1}^2| \leqslant \frac{\gamma B_1}{(\alpha+2\beta)k}\left[\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-\frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k))\right]. \tag{18}$$

Case 2: If $d_1 \leqslant \nu \leqslant d_2$, then

$$\frac{2k(\alpha+\beta)^2[B_2-B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2} \leqslant \nu$$
$$\leqslant \frac{2k(\alpha+\beta)^2[B_2+B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2}$$

$$\Longrightarrow -1 \leqslant t \leqslant 1 \Longrightarrow$$

$$|b_{2k+1}-\nu b_{k+1}^2| \leqslant \frac{\gamma B_1}{(\alpha+2\beta)k}. \tag{19}$$

Case-III: If $\nu \geqslant d_2$, then

$$\nu \geqslant \frac{2k(\alpha+\beta)^2[B_2+B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2}$$

$$\Longrightarrow t \geqslant 1 \Longrightarrow |w_2-tw_1^2| \leqslant t \Longrightarrow$$

$$|b_{2k+1}-\nu b_{k+1}^2| \leqslant \frac{\gamma B_1}{(\alpha+2\beta)k}\left[\frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k))-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right]. \tag{20}$$

The results in (15) follows from (18), (19) and (20). We also note the following:

(i) When $\nu \leqslant d_1$, the equality holds if and only if $w(z)=z$ or one of its rotation.

(ii) When $d_1 \leqslant \nu \leqslant d_2$, then the equality holds if and only if $w(z)=z^2$ or one of its rotation.

(iii) When $\nu \geqslant d_2$, then the equality holds when $w(z)=\frac{z(\varepsilon+z)}{1+\varepsilon z}$ $(0 \leqslant \varepsilon \leqslant 1)$ or one of its rotation.

This complete the proof of Theorem 2. □

REMARK 2. (i) Putting $\alpha=0$ and $\beta=1$ in Theorem 2 gives the result due to Sharma et al. [20].

(ii) Taking $\alpha=0$ and $\beta=b=\gamma=1$ in Theorem 2 we obtain the first part result of Ali et al. ([2], Theorem 2.1, p. 122).

4. Applications to functions defined by fractional derivatives

For a fixed $g \in \mathscr{A}$, we define $\mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi)$ to be the class of functions $f \in \mathscr{A}$ for which $(f*g) \in \mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$. In order to introduce the class $\mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi)$, we need the following:

DEFINITION 2. (see [15]) Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order δ is defined by

$$D^{\delta}_{z} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi \quad (0 \leqslant \delta < 1),$$

where the multiplicity of $(z-\xi)^{\delta}$ is removed by requiring that $\log(z-\xi)$ is real for $(z-\xi) > 0$.

Using Definition 2, Owa and Srivastava [15] introduced a fractional derivative operator $\Omega^{\delta} : \mathscr{A} \longrightarrow \mathscr{A}$ defined by

$$\Omega^{\delta} f(z) = \Gamma(2-\delta) z^{\delta} D^{\delta}_{z} f(z) \quad (\delta \neq 2,3,4,\cdots).$$

The class $\mathscr{R}^{\gamma,\delta}_{b,\alpha,\beta}(\phi)$ consists of function $f \in \mathscr{A}$ for which $\Omega^{\delta} f \in \mathscr{T}^{\gamma}_{b,\alpha,\beta}(\phi)$. The class $\mathscr{R}^{\gamma,\delta}_{b,\alpha,\beta}(\phi)$ is the special case of the class $\mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n.$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0).$$

Since $f \in \mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi) \Longrightarrow (f*g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$. Applying Theorem 1 and Theorem 2 for the function

$$(f*g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots$$

we get Theorem 3 and Theorem 4 (mentioned below) respectively.

THEOREM 3. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$ and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$ $(B_1 > 0)$. If $f(z) \in \mathscr{A}$ given by (1) belong to the class $\mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi)$ and F is the kth root transformation of f given by (3). Then for any complex number ν, we have*

$$|b_{2k+1} - \nu b_{k+1}^2| \leqslant \frac{|b|\gamma B_1}{k(\alpha+2\beta)g_3} \max\left\{1, \left|\frac{b\gamma B_1(\alpha+2\beta)g_3}{2k(\alpha+\beta)^2 g_2^2}\left[(2\nu-1)+k\left(1-(1-\alpha)\frac{g_2^2}{g_3}\right)\right] - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right|\right\}.$$

The estimate is sharp.

THEOREM 4. *If* $f \in \mathscr{R}_{\alpha,\beta}^{\gamma,g}(\phi)$ *and* F *is the* k*th root transform of the function* f *given by (3), then for any real number* ν *and for*

$$e_1 = \frac{1}{2}\left[1 + \frac{2k(\alpha+\beta)^2 g_2^2}{\gamma B_1(\alpha+2\beta)g_3}\left(\frac{B_2}{B_1} + \frac{\gamma-1}{2}B_1 - 1\right) - k\left(1-(1-\alpha)\frac{g_2^2}{g_3}\right)\right],$$

and

$$e_2 = \frac{1}{2}\left[1 + \frac{2k(\alpha+\beta)^2 g_2^2}{\gamma B_1(\alpha+2\beta)g_3}\left(1 + \frac{B_2}{B_1} + \frac{\gamma-1}{2}B_1\right) - k\left(1-(1-\alpha)\frac{g_2^2}{g_3}\right)\right],$$

we have

$$|b_{2k+1} - \nu b_{k+1}^2| \leqslant \begin{cases} \frac{\gamma B_1}{k(\alpha+2\beta)g_3}\left[\frac{B_2}{B_1} + \frac{\gamma-1}{2}B_1 - \frac{\gamma B_1(\alpha+2\beta)g_3}{2k(\alpha+\beta)^2 g_2^2}\left\{(2\nu-1)+k(1-(1-\alpha)\frac{g_2^2}{g_3})\right\}\right] \\ \hfill (\nu \leqslant e_1) \\ \frac{\gamma B_1}{k(\alpha+2\beta)g_3} \hfill (e_1 \leqslant \nu \leqslant e_2) \\ \frac{\gamma B_1}{k(\alpha+2\beta)g_3}\left[\frac{\gamma B_1(\alpha+2\beta)g_3}{2k(\alpha+\beta)^2 g_2^2}\left\{(2\nu-1)+k\left(1-(1-\alpha)\frac{g_2^2}{g_3}\right) - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right\}\right] \\ \hfill (\nu \geqslant e_2). \end{cases}$$

The result is sharp.

As

$$\Omega^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} a_n z^n \tag{21}$$

we have

$$g_2 = \frac{2}{2-\delta} \tag{22}$$

$$g_3 = \frac{6}{(2-\delta)(3-\delta)}. \tag{23}$$

For g_2 and g_3 given by (22) and (23) respectively, Theorem 3 and Theorem 4 reduces to the following results.

THEOREM 5. *If* $f \in \mathscr{R}_{b,\alpha,\beta}^{\gamma,\delta}(\phi)$ *and* F *is the* k*th root transform of* f *given by (3), then for any complex number* ν*, we have*

$$|b_{2k+1} - \nu b_{k+1}^2| \leqslant \frac{|b|\gamma B_1(2-\delta)(3-\delta)}{6k(\alpha+2\beta)} \max\left\{1, \left|\frac{3b\gamma B_1(\alpha+2\beta)(2-\delta)}{4k(3-\delta)(\alpha+\beta)^2} \times\left\{(2\nu-1)+k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)\right\} - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right|\right\}.$$

THEOREM 6. *If* $f \in \mathscr{R}_{\alpha,\beta}^{\gamma,\delta}(\phi)$ *and* $g_n > 0$ *and* F *is the* k*th root transform of* f *given by (3), then for any real number* ν *and for*

$$e_1 = \frac{1}{2}\left[1 + \frac{4k(\alpha+\beta)^2(3-\delta)}{3(2-\delta)\gamma B_1(\alpha+2\beta)}\left(\frac{B_2}{B_1} + \frac{\gamma-1}{2} - 1\right) - k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)\right],$$

and

$$e_2=\frac{1}{2}\left[1+\frac{4k(\alpha+\beta)^2(3-\delta)}{3(2-\delta)\gamma B_1(\alpha+2\beta)}\left(1+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)\right]$$

we have

$$|b_{2k+1}-\nu b_{k+1}^2| \leqslant \begin{cases} \frac{\gamma B_1(2-\delta)(3-\delta)}{6k(\alpha+2\beta)}\left[\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-\frac{3\gamma B_1(\alpha+2\beta)(2-\delta)}{4k(\alpha+\beta)^2(3-\delta)}\left\{(2\nu-1)+k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)\right\}\right] & (\nu\leqslant e_1) \\ \frac{\gamma B_1(2-\delta)(3-\delta)}{6k(\alpha+2\beta)} & (e_1\leqslant\nu\leqslant e_2) \\ \frac{\gamma B_1(2-\delta)(3-\delta)}{6k(\alpha+2\beta)}\left[\frac{3\gamma B_1(\alpha+2\beta)(2-\delta)}{4k(\alpha+\beta)^2(3-\delta)}\left\{(2\nu-1)+k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right\}\right] & (\nu\geqslant e_2). \end{cases}$$

The result is sharp.

5. Coefficient functional associated with $\frac{z}{f(z)}$

In this section, bounds for Fekete-Szegö coefficient functional associated with the function H defined by

$$H(z)=\frac{z}{f(z)}=1+\sum_{n=1}^{\infty}s_nz^n \tag{24}$$

where f belongs to the class $\mathscr{R}_{b,\alpha,\beta}^{\gamma}(\phi)$ are obtained. The proof of the results obtained in this section are similar to those given in Section 3 and hence we chose to omit the details.

THEOREM 7. *Let $f\in\mathscr{R}_{b,\alpha,\beta}^{\gamma}(\phi)$ and $H(z)=\frac{z}{f(z)}=1+\sum_{n=1}^{\infty}s_nz^n$. Then for any complex number ν, we have*

$$|s_2-\nu s_1^2|\leqslant\frac{|b|\gamma B_1}{\alpha+2\beta}\max\left\{1,\left|(2\nu-1-\alpha)\frac{(\alpha+2\beta)b\gamma B_1}{2(\alpha+\beta)^2}+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right|\right\}. \tag{25}$$

The result is sharp.

Proof. A simple computation gives

$$H(z)=\frac{z}{f(z)}=1-a_2z+(a_2^2-a_3)z^2+\cdots \tag{26}$$

From (24) and (26) we have

$$s_1=-a_2, \tag{27}$$

and

$$s_2=a_2^2-a_3. \tag{28}$$

Using (9) and (10) in (27) and (28), we obtain

$$s_1 = -\frac{b\gamma B_1 w_1}{\alpha+\beta},$$
$$s_2 = \frac{b^2\gamma^2 B_1^2 w_1^2}{(\alpha+\beta)^2} - \frac{b\gamma}{\alpha+2\beta}\left[B_1 w_2 + \left\{B_2 + \frac{\gamma-1}{2}B_1^2 + \frac{(1-\alpha)(\alpha+2\beta)}{2(\alpha+\beta)^2}b\gamma B_1^2\right\} w_1^2\right].$$

For any complex number ν,

$$|s_2 - \nu s_1^2| = \frac{|b|\gamma B_1}{\alpha+2\beta}\left|w_2 - \left\{(1+\alpha-2\nu)\frac{(\alpha+2\beta)b\gamma B_1}{2(\alpha+\beta)^2} - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right\} w_1^2\right|.$$

The desire result can be obtained by application of Lemma 2. This complete the proof of Theorem 7. $\square$

Restricting ν to be real and taking $b=1$, we now obtain the coefficient inequality for the function f in the class $\mathscr{R}^{\gamma}_{\alpha,\beta}(\phi)$. Proceeding similar manner as in Theorem 2 for the function $\frac{z}{f(z)}$ we can obtain the following result.

THEOREM 8. *IF $f \in \mathscr{R}^{\gamma}_{\alpha,\beta}(\phi)$ and $H(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} s_n z^n$, then for any real number ν, and for*

$$p_1 = \frac{(1+\alpha)(\alpha+2\beta)\gamma B_1^2 - 2(\alpha+\beta)^2(B_1 + B_2 + \frac{\gamma-1}{2}B_1^2)}{2(\alpha+2\beta)\gamma B_1^2}$$

and

$$p_2 = \frac{(1+\alpha)(\alpha+2\beta)\gamma B_1^2 - 2(\alpha+\beta)^2(B_2 - B_1 + \frac{\gamma-1}{2}B_1^2)}{2(\alpha+2\beta)\gamma B_1^2}$$

we have

$$|s_2 - \nu s_1^2| \leqslant \begin{cases} \frac{\gamma B_1}{\alpha+2\beta}\left[\frac{(\alpha+2\beta)\gamma B_1}{2(\alpha+\beta)^2}(1+\alpha-2\nu) - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right] & \nu \leqslant p_1 \\ \frac{\gamma B_1}{\alpha+2\beta} & p_1 \leqslant \nu \leqslant p_2 \\ \frac{\gamma B_1}{\alpha+2\beta}\left[\frac{(\alpha+2\beta)\gamma B_1}{2(\alpha+\beta)^2}(2\nu-\alpha-1) + \frac{B_2}{B_1} + \frac{\gamma-1}{2}B_1\right] & \nu \geqslant p_2. \end{cases}$$

6. Coefficient inequality for the inverse of the function $f(z)$

THEOREM 9. *If $f \in \mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} l_n w^n$ is the inverse function of f with $|w| < r_0$, where r_0 is the greater than the radius of the Koebe domain of the class $f \in \mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$, then for any complex number ν, we have*

$$|l_3 - \nu l_2^2| \leqslant \frac{|b|\gamma B_1}{\alpha+2\beta}\max\left\{1, \left|(\alpha+3-2\nu)\frac{b\gamma B_1(\alpha+2\beta)}{2(\alpha+\beta)^2} - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right|\right\}. \quad (29)$$

The result is sharp.

Proof. Since

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} l_n w^n \tag{30}$$

is the inverse function of f, we have

$$f^{-1}(f(z)) = f(f^{-1}(z)) = z \tag{31}$$

From (30) and (31) we have

$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z \tag{32}$$

Equating the coefficients of z and z^2 from equations (30) and (32) we obtain

$$l_2 = -a_2 = -\frac{b\gamma B_1 w_1}{\alpha+\beta},$$

$$l_3 = 2a_2^2 - a_3 = 2\frac{b^2\gamma^2 B_1^2 w_1^2}{(\alpha+\beta)^2} - \frac{b\gamma}{\alpha+2\beta}\left[B_1 w_2 + \left\{B_2 + \frac{\gamma-1}{2}B_1^2 + \frac{(1-\alpha)(\alpha+2\beta)}{2(\alpha+\beta)^2} b\gamma B_1^2\right\} w_1^2\right].$$

For any complex number ν, we have

$$|l_3 - \nu l_2^2| = \frac{|b|\gamma B_1}{(\alpha+2\beta)}\left| w_2 - \left\{\frac{b\gamma B_1(\alpha+2\beta)}{2(\alpha+\beta)^2}(\alpha+3-2\nu) - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right\} w_1^2\right|. \tag{33}$$

Applying Lemma 2 on right hand side of (33) we obtain the require result as stated in (29). The result is sharp and followed by

$$|l_3 - \nu l_2^2| = \begin{cases} \frac{|b|\gamma B_1}{\alpha+2\beta} & w(z) = z^2 \\ \frac{|b|\gamma B_1}{\alpha+2\beta}\left|\frac{b\gamma B_1(\alpha+2\beta)}{2(\alpha+\beta)^2}(\alpha+3-2\nu) - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right| & w(z) = z. \end{cases} \quad \square$$

CONCLUDING REMARK. For $k = 1$, the kth root transformation of f reduces to the given function f itself. Therefore, the estimate given in equations (5) and (15) (with $b = \gamma = 1$) is an extension of the corresponding results for the Fekete-Szegö functional with $p = 1$ studied by Ramachandran et al. [17]. For the class defined in (4), an attempt has made for finding second Hankel determinant for the kth root transform of f by making use of Chebyshev polynomial.

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(Received October 3, 2017)

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