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FAEDO–GALERKIN APPROXIMATE SOLUTIONS FOR NONLOCAL FRACTIONAL DIFFERENTIAL EQUATION OF SOBOLEV TYPE

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Abstract. This paper studies a fractional differential equation of Sobolev type with nonlocal initial conditions in an arbitrary separable Hilbert space. We study the associated integral equation and then, consider a sequence of approximate integral equations obtained by projection of considered associated integral equation onto finite dimensional space. The sufficient condition for providing the existence and uniqueness of mild solution to every approximate integral equation is obtained via the techniques of Banach fixed point theorem and analytic semigroup theory. By utilizing the Faedo-Galerkin approximations, we establish some convergence results for approximate solutions. Finally, an example is given to explain the applicability of the discussed abstract results.

1. Introduction

Recently, the theory of fractional calculus which provides the integration and differentiation of any order, not necessarily an integer, has proved to be an important tool in the modeling of dynamical systems associated with phenomena such as fractals and chaos. Differential equation of fractional order has been applied in different fields such as physics, chemistry, electronics, mechanics, medicine, nonlinear oscillation of earthquake, models of population growth, electrodynamics of complex medium and many other branches of sciences and technology. For further applications of differential equations with fractional order in other domains and useful backgrounds, we refer to the references [1]–[5], [11], [20]–[22]. The existence of a solution for abstract Cauchy differential equation with nonlocal conditions in a Banach space has been considered first by Byszewski [6]. Many authors have considered and studied the existence of the mild solution to the nonlocal conditions, see [7], [9], [11], [12], [13], [18], [19], [22], [28], [32]. In physical science, the nonlocal condition may be connected with better effect in applications than the classical initial condition since nonlocal conditions are normally more exact for physical estimations than the classical initial condition.

On the other hand, Sobolev type differential equations with fractional order arise in control theory of dynamical systems, when the controller is characterized by a Sobolev type differential equation with fractional order. Moreover, the mathematical modeling and simulations of systems and phenomena are focused around the description of their

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properties in terms of Sobolev type differential equations with fractional order. These new models are more satisfactory than previously utilized integer order models. Differential equations of Sobolev type have not been considered by the authors, extensively. In [10], authors have studied the existence of solutions to Sobolev-type partial neutral differential equations by utilizing fixed point theorem. In [11], authors have considered the Sobolev type fractional differential equation and established the existence of the mild solution for the considered system by virtue of the theory of propagation family via the techniques of the condensing maps and the measure of noncompactness. In [13], Fečkan et al. have discussed the controllability of Sobolev type fractional differential equation via the techniques of a fixed point theorem and characteristic solution operators. Very recently, Debbouche et al. [18] have considered a new kind of Sobolev type nonlinear fractional differential equations in terms of two linear operators. To describe the solution of the problem, authors have introduced two new characteristic solution operators and obtained the existence results by using Leray-Schauder fixed point theorem. For more study of Sobolev type differential equations, we refer to papers [10]–[19] and references cited therein.

To the solvability of evolution problems in the time domain, we have various approaches, namely, the evolution family approach and an approach using finite-dimensional approximations known as Faedo-Galerkin approximations. The Faedo-Galerkin approach may be used for the study of more regular solutions, imposing higher regularity on the data. In [25], author has extended the results of the [24] and considered the Faedo-Galerkin approximations of the solutions for functional Cauchy problem in a separable Hilbert space with the help of analytic semigroup theory and Banach fixed point theorem. In [26], authors have studied the Faedo-Galerkin approximations of the solutions to a class of functional integro-differential equation extended the results of [25]. Muslim et al. [29] have studied the fractional order differential equation and proved some convergence results for Faedo-Galerkin approximations. For more study on Faedo-Galerkin approximations of solutions, we refer to papers [24]–[32] and references cited therein.

Motivated by above mentioned work, our main purpose of this paper is to study the following nonlocal functional differential equation of Sobolev type illustrated by

$$ {}^{c}\mathbf{D}_t^{\beta}[By(t)] = Ly(t) + F(t, y(t), y(h(t))), \quad t \in J = [0, T], \tag{1} $$

$$ g(y) = \phi \in \mathbb{X}. \tag{2} $$

where $\beta \in (0,1)$, $0 < T < \infty$ and ${}^{c}\mathbf{D}_t^{\beta}$ denotes the fractional derivative in Caputo sense. In (1), we assume that the operator $B : D(B) \subset \mathbb{X} \to \mathbb{Y}$ and $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ are closed, positive and self-adjoint operators, where $\mathbb{X}$ and $\mathbb{Y}$ are the Hilbert spaces such that $\mathbb{Y}$ is continuously and densely embedded in $\mathbb{X}$, the state $y(\cdot)$ takes its values in $\mathbb{X}$ and functions $F : [0,T] \times \mathbb{X} \to \mathbb{X}$, $h : [0,T] \to [0,T]$ and $g : C([0,T], \mathbb{X}) \to \mathbb{X}$ are appropriate functions.

The article is organized as follows: Section 2 presents some basic definitions, lemmas and theorems which will be required to prove the result. Section 3 studies the existence and uniqueness of solution for every approximate integral equation by virtue of the theory of analytic semigroup via Banach fixed point theorem. Section 4 proves the convergence of the solution to each of the approximate integral equations with the limiting function which satisfies the associated integral equation and Section 5 focuses on the convergence of the approximate Faedo-Galerkin solutions. Section 6 provides an application for illustrating the discussed abstract results.

2. Preliminaries

In this section, we provide some essential facts about fractional calculus, semigroup theory, theorems and lemmas which will be required to obtain our results.

Throughout the paper, we assume that $(\mathbb{X}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(\mathbb{Y}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ are separable Hilbert spaces. The space $C([0,T],\mathbb{X})$ represents the Banach space of continuous functions from $[0,T]$ into $\mathbb{X}$ with the norm $\|y\|_{[0,T]} = \sup\{\|y(t)\| : \ t \in [0,T]\}$. Let $L(\mathbb{X})$ be the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ endowed with the norm $\|f\|_{L(\mathbb{X})} = \sup\{\|f(y)\| : \|y\| = 1\}$.

Now, we state some basic definitions and properties of fractional calculus.

DEFINITION 2.1. The Riemann-Liouville fractional integral operator $\mathscr{I}_t^{\beta}$ is given by

$$\mathscr{I}_t^{\beta} F(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F(s)ds, \tag{3}$$

where $F \in L^1((0,T),\mathbb{X})$ and $\beta > 0$ is the order of the fractional integration.

DEFINITION 2.2. The Riemann-Liouville fractional derivative is given by

$$^{RL}D_t^{\beta} F(t) = D_t^{\delta} \mathscr{I}_t^{\delta-\beta} F(t), \ \delta - 1 < \beta < \delta, \ \delta \in \mathbb{N}, \tag{4}$$

where $D_t^{\delta} = \frac{d^{\delta}}{dt^{\delta}}$, $\mathscr{F} \in L^1((0,T),\mathbb{X})$, $\mathscr{I}_t^{\delta-\beta} F \in W^{\delta,1}((0,T),\mathbb{X})$. Here, the notation $W^{\delta,1}((0,T),\mathbb{X})$ stands for the Sobolev space defined by

$$\begin{aligned} W^{\delta,1}((0,T),\mathbb{X}) = \Big\{ & y \in \mathbb{X} : \exists\, z \in L^1((0,T),\mathbb{X}) : \\ & y(t) = \sum_{k=0}^{\delta-1} d_k \frac{t^k}{k!} + \frac{t^{\delta-1}}{(\delta-1)!} * z(t), \ t \in (0,T) \Big\}. \end{aligned}$$

Note that $z(t) = y^{\delta}(t)$, $d_k = y^k(0)$.

DEFINITION 2.3. The Caputo fractional derivative is given by

$$^{c}\mathbf{D}_t^{\beta} F(t) = \frac{1}{\Gamma(\delta-\beta)} \int_0^t (t-s)^{\delta-\beta-1} F^{\delta}(s)ds, \ \ \delta - 1 < \beta < \delta, \tag{5}$$

where $F \in C^{\delta-1}((0,T),\mathbb{X}) \cap L^1((0,T),\mathbb{X})$ and the following holds

$$\mathscr{I}_t^{q}(^{c}\mathbf{D}_t^{q} F(t)) = F(t) - \sum_{k=0}^{\delta-1} \frac{t^k}{k!} F^k(0). \tag{6}$$

Next, we impose following additional data on operators L and B:

(C1) $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear, closed operator and $B : D(B) \subset \mathbb{X} \to \mathbb{Y}$ is a linear operator.

(C2) $D(B) \subset D(L)$ and B is bijective operator.

(C3) The operators $B^{-1} : \mathbb{Y} \to D(B) \subset \mathbb{X}$ is assumed to be linear, continuous operator with $Im(B^{-1}) \subset D(L)$ and $Im(L) \subset D(B^{-1})$ such that $LB^{-1} = B^{-1}L$.

By the hypothesis (C3), it follows that B^{-1} is closed and injective. Thus, its inverse is also closed i.e., B is closed. By the hypotheses (C1)–(C3) and closed graph theorem, we conclude that the boundedness of the linear operator $LB^{-1} : \mathbb{Y} \to \mathbb{Y}$. Since B is invertible positive operator, therefore, the operator B^{-1} is positive operator. Thus, it follows that LB^{-1} is bounded, positive and self-adjoint operator. Therefore, LB^{-1} is the infinitesimal generator of a semigroup $\{\mathscr{S}(t), t \geqslant 0\}$, $\mathscr{S}(t) := e^{LB^{-1}}$. Thus, without loss of generality, we may assume that $N_0 := \sup_{t \geqslant 0} \|\mathscr{S}(t)\| < \infty$ and $W_1 = \|B^{-1}\|$.

According to previous definitions, we have that if the following integral

$$By(t) = By(0) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}[Ly(s) + F(s, y(s))]ds, \tag{7}$$

exists a.e. for $t \in [0, T]$. Then, the system (1)–(2) is equivalent to the integral equation (7).

In this work, $A = LB^{-1} : D(A) \subset \mathbb{Y} \to \mathbb{Y}$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $\mathscr{S}(\cdot)$. Thus, it follows that there exists a positive constant $N_0 \geqslant 1$ such that $\|\mathscr{S}(t)\| \leqslant N_0$ for each $t \geqslant 0$. We assume that $0 \in \rho(A)$, $\rho(A)$ means resolvent set of A. Therefore, we may define the fractional power A^α for $\alpha \in (0, 1]$ as closed linear operator with domain $D(A^\alpha)$ with inverse $A^{-\alpha}$. Moreover, the subspace $D(A^\alpha)$ is dense in $\mathbb{X}$ with the norm $\|y\|_\alpha = \|A^\alpha y\|$ for $y \in D(A^\alpha)$. Thus, it is not difficult to show that $D(A^\alpha)$ is a Banach space with supremum norm. Hence, we signify the space $D(A^\alpha)$ by $\mathbb{X}_\alpha$ endowed with the α-norm $(\|\cdot\|_\alpha)$. We also have that for $\mathbb{X}_\eta \hookrightarrow \mathbb{X}_\alpha$ for $0 < \alpha < \eta$ and therefore, the embedding is continuous. Then, we may define $\mathbb{X}_{-\alpha} = (\mathbb{X}_\alpha)^*$ for each $\alpha > 0$, dual space of $\mathbb{X}_\alpha$, is a Banach space endowed with the norm $\|y\|_{-\alpha} = \|A^{-\alpha} y\|$ for $y \in \mathbb{X}_{-\alpha}$. For more details on the fractional powers of closed linear operators, we refer to book by Pazy [23].

Now, we present the following lemma follows from the results [20], [21] which will be used to establish the required result.

LEMMA 2.1. *Let us assume that A is the infinitesimal generator of an analytic semigroup $\mathscr{S}(t)$, $t \geqslant 0$ and $0 \in \rho(A)$. Then,*

(i) $\mathscr{S}(t) : X \to D(A^\alpha)$ for every $t > 0$ and $\alpha \geqslant 0$.

(ii) $\mathscr{S}(t)A^\alpha y = A^\alpha \mathscr{S}(t) y$ for each $y \in D(A^\alpha)$.

(iii) For each $t > 0$,

$$\left\| \frac{d^j}{dt^j} \mathscr{S}(t) \right\| \leqslant N_j, \quad j = 1, 2, \tag{8}$$

where $N_j,\ j = 1, 2$ *are some positive constants.*

(iv) The operator $A^{\alpha}\mathscr{S}(t)$ *is bounded and* $\|A^{\alpha}\mathscr{S}(t)\| \leqslant N_{\alpha}t^{-\alpha}e^{-\delta t}$ *for each* $t > 0$.

(v) For each $\alpha \in (0, 1]$ *and* $y \in D(A^{\alpha})$, *then* $\|\mathscr{S}(t)y - y\| \leqslant C_{\alpha}t^{\alpha}\|A^{\alpha}y\|$.

REMARK 2.1. [18] The operator $A^{-\alpha}$ is a bounded linear operator in X such that $D(A^{\alpha}) = Im(A^{-\alpha})$.

We denote by $\mathbb{X}_{\alpha}(T) = C([0, T], \mathbb{X}_{\alpha})$ Banach space of all $\mathbb{X}_{\alpha}$-valued continuous functions on $[0, T]$ endowed with the supremum norm $\|y\|_{\mathbb{X}_{\alpha}(T)} = \sup_{t \in [0,T]} \|y(t)\|_{\alpha}$ for each $y \in \mathbb{X}_{\alpha}(T)$.

3. Existence of approximate solutions

In this section, the sufficient condition for the existence and uniqueness of α-mild solution for system (1)–(2) is derived. To prove the result, we impose following assumptions on the data of the system (1)–(2).

(O1) A is a closed, positive definite and self-adjoint linear operator from $D(A) \subset \mathbb{Y}$ into $\mathbb{Y}$. We assume that operator A has the pure point spectrum

$$0 < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m \leqslant \cdots, \tag{9}$$

with $\lambda_m \to \infty$ as $m \to \infty$ and a corresponding complete orthonormal system of eigenfunctions $\{\phi_j\}$, i.e.,

$$A\phi_j = \lambda_j\phi_j, \quad \text{and} \quad \langle \phi_l, \phi_j \rangle = \delta_{lj}, \tag{10}$$

where

$$\delta_{lj} = \begin{cases} 1, & j = l, \\ 0, & \text{otherwise.} \end{cases}$$

(O2) The mapping $F : [0, \infty) \times \mathbb{X}_{\alpha} \times \mathbb{X}_{\alpha} \to \mathbb{Y}$ is continuous and there exists a increasing function $m_R : [0, \infty) \to (0, \infty)$ that depends $R > 0$ such that

$$\|F(t, z, w)\| \leqslant m_R(t), \tag{11}$$

$$\|F(t_1, z_1, w_1) - F(t_2, z_2, w_2)\| \leqslant m_R(t)[|t_1 - t_2|^{\theta_1} + \|z_1 - z_2\|_{\alpha}], \tag{12}$$

for all (t, z, w), (t_1, z_1, w_1), $(t_2, z_2, w_2) \in [0, \infty) \times \mathscr{B}_R(\mathbb{X}_{\alpha}) \times \mathscr{B}_R(\mathbb{X}_{\alpha})$, where $\mathscr{B}_R(\mathbb{X}) = \{z \in \mathbb{X} : \|z\|_{\mathbb{X}} \leqslant R\}$ and $\theta_1 \in (0, 1)$.

(O3) $h:[0,T]\to[0,T]$ is a nonlinear function such that

(i) h satisfies the following condition $h(t)\leqslant t,\ \forall\ t\in[0,T]$.

(ii) There exists a constant $L_h>0$ such that

$$|h(t_1)-h(t_2)|\leqslant L_h|t_1-t_2|,\ \ t_1,t_2\in[0,T]. \tag{13}$$

(O4) (i) There exists a function $\chi\in C([0,T],\mathbb{X}_\alpha)$ such that $g(\chi)=\phi$.

(ii) There function $\chi(t)\in\mathbb{X}_\alpha$ is locally Lipschitz continuous on $[0,T]$.

Motivated by the paper [18], we present the following definition of mild solution of system (1)–(2).

DEFINITION 3.1. A continuous function $y:[0,T]\to\mathbb{X}_\alpha$ is said to be a mild solution of system (1)–(2) if $y(0)=y_0$ and $y(\cdot)$ satisfies the following integral equation

$$y(t)=\mathscr{S}_\beta(t)[B]\chi(0)+\int_0^t(t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F(s,y(s),y(h(s)))ds,\ \ t\in[0,T], \tag{14}$$

where

$$\begin{aligned}
\mathscr{S}_\beta(t) &= \int_0^\infty B^{-1}\xi_\beta(\zeta)\mathscr{S}(t^\beta\zeta)d\zeta,\\
\mathscr{T}_\beta(t) &= \int_0^\infty \beta B^{-1}\zeta\xi_\beta(\zeta)\mathscr{S}(t^\beta\zeta)d\zeta,\\
\xi_\beta(\zeta) &= \frac{1}{\beta}\zeta^{-1-\frac{1}{\beta}}\psi_\beta(\zeta^{\frac{-1}{\beta}})\geqslant 0,\\
\psi_\beta(\zeta) &= \frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\zeta^{-\beta n-1}\frac{\Gamma(n\beta+1)}{n!}\sin(n\pi\beta),\ \zeta\in(0,\infty),
\end{aligned}$$

and $\xi_\beta(\zeta)$ the probability density function defined on $(0,\infty)$, i.e., $\xi_\beta(\zeta)\geqslant 0,\ \zeta\in(0,\infty)$ with $\int_0^\infty\xi_\beta(\zeta)d\zeta=1$.

REMARK 3.1. [21] For each $\nu\in[0,1]$

$$\int_0^\infty\zeta^\nu\xi_\beta(\zeta)d\zeta=\int_0^\infty\zeta^{-\beta\nu}\psi_\beta(\zeta)d\zeta=\frac{\Gamma(1+\nu)}{\Gamma(1+\beta\nu)}. \tag{15}$$

LEMMA 3.1. [18] *Let A be the infinitesimal generator of a semigroup of uniformly bounded linear operators $\mathscr{S}(t)$, $t\geqslant 0$. Then, the operator $\mathscr{S}_\beta(t)$ and $\mathscr{T}_\beta(t)$, $t\geqslant 0$ are bounded linear operator such that*

(1) *We have $\|\mathscr{S}_\beta(t)y\|\leqslant W_1N_0\|y\|$ and $\|\mathscr{T}_\beta(t)y\|\leqslant\frac{W_1N_0}{\Gamma(\beta)}\|y\|$ for each $y\in\mathbb{X}$.*

(2) *The families $\{\mathscr{S}_\beta(t),\ t\geqslant 0\}$ and $\{\mathscr{T}_\beta(t),\ t\geqslant 0\}$ are strongly continuous i.e., for $0\leqslant\tau_1<\tau_2\leqslant T$ and $y\in\mathbb{X}$, we have $\|\mathscr{S}_\beta(\tau_2)y-\mathscr{S}_\beta(\tau_1)y\|\to 0$ and $\|\mathscr{T}_\beta(\tau_2)y-\mathscr{T}_\beta(\tau_1)y\|\to 0$ as $\tau_2\to\tau_1$.*

(3) *If $\mathscr{S}(t)$, $t\geqslant 0$ is compact, then $\mathscr{S}_\beta(t)$ and $\mathscr{T}_\beta(t)$, $t\geqslant 0$ are compact operator.*

(4) *For each* $y \in \mathbb{X}$, $\eta \in (0,1)$ *and* $\alpha \in (0,1)$, *we have* $A\mathscr{T}_\beta(t)y = A^{1-\eta}\mathscr{T}_\beta A^\eta y$ *for* $t \in [0,T]$. *We also have* $\|A^\alpha \mathscr{T}_\beta(t)\| \leqslant \frac{\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} t^{-\alpha\beta}$ *for each* $t \in (0,T]$.

(5) *For any* $y \in \mathbb{X}_\alpha$ *and fixed* $t \geqslant 0$, *we have* $\|\mathscr{S}_\beta(t)y\|_\alpha \leqslant W_1 N_0 \|y\|_\alpha$ *and* $\|\mathscr{T}_\beta(t)y\|_\alpha \leqslant \frac{W_1 N_0}{\Gamma(\beta)}\|y\|_\alpha$.

Let $T_0 > 0$ be arbitrarily fixed such that $0 < T < T_0 < \infty$ and

$$\Psi = \frac{W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} m_R(T_0) \frac{T^{\beta(1-\alpha)}}{(1-\alpha)} < 1. \tag{16}$$

Next, we consider that $\mathbb{H}_n$, spanned by $\{\phi_0, \phi_1, \cdots, \phi_n\}$, is the finite dimensional subspace of Hilbert space $\mathbb{X}$ and for each n, let $P^n : \mathbb{X} \to \mathbb{H}_n$ be the corresponding projection operator, where $n = 0, 1, \cdots,$. Now, consider the function $F_n : [0,T] \times \mathbb{X}_\alpha \to \mathbb{X}$ defined by

$$F_n(t, y(t), y(h(t))) = F(t, P^n y(t), P^n y(h(t))), \tag{17}$$

and the operator $\mathbb{Q}_n : \mathscr{B}_R(\mathbb{X}_\alpha(T)) \to \mathscr{B}_R(\mathbb{X}_\alpha(T))$ given by

$$(\mathbb{Q}_n y)(t) = \mathscr{S}_\beta(t)B\chi(0) + \int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F_n(s, y(s), y(h(s)))ds, \ \ t \in [0,T]. \tag{18}$$

THEOREM 3.2. *If the assumptions* (O1)–(O2) *are fulfilled, then there exists a unique fixed point* $y_n \in \mathscr{B}_R(\mathbb{X}_\alpha(T))$ *of the operator* $\mathbb{Q}_n$ *i.e.* $\mathbb{Q}_n y_n = y_n$ *for each* $n = 0, 1, 2, \cdots$ *and* y_n *fulfills the approximate integral equation*

$$y_n(t) = \mathscr{S}_\beta(t)B\chi(0) + \int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F_n(s, y_n(s), y(h(s)))ds, \ \ t \in [0,T]. \tag{19}$$

Proof. Firstly, we consider the operator $\mathbb{Q}_n : \mathscr{B}_R(\mathbb{X}_\alpha(T)) \to \mathscr{B}_R(\mathbb{X}_\alpha(T))$ defined by

$$(\mathbb{Q}_n y)(t) = \mathscr{S}_\beta(t)B\chi(0) + \int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F_n(s, y(s), y(h(s)))ds, \ t \in [0,T]. \tag{20}$$

We firstly show that $\mathbb{Q}_n$ is well-defined map. To this end, it is sufficient to show that $t \mapsto (\mathbb{Q}_n y)(t)$ is continuous from $[0,T]$ into $\mathbb{X}_\alpha$ with respect to α-norm $(\|\cdot\|_\alpha)$. For $t_1, t_2 \in [0,T]$ with $t_2 > t_1$, we get

$$\begin{aligned}
&\|(\mathbb{Q}_n y)(t_2) - (\mathbb{Q}_n y)(t_1)\|_\alpha \\
\leqslant\ & \|[\mathscr{S}_\beta(t_2) - \mathscr{S}_\beta(t_1)]B\chi(0)\|_\alpha + \left\| \int_{t_1}^{t_2} (t_2-s)^{\beta-1}\mathscr{T}_\beta(t_2-s)F_n(s, y(s), y(h(s)))ds \right\|_\alpha \\
& + \left\| \int_0^{t_1} (t_2-s)^{\beta-1}\mathscr{T}_\beta(t_2-s)F_n(s, y(s), y(h(s)))ds \right. \\
& \left. - \int_0^{t_1} (t_1-s)^{\beta-1}\mathscr{T}_\beta(t_1-s)F_n(s, y(s), y(h(s)))ds \right\|_\alpha
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|[\mathscr{S}_\beta(t_2)-\mathscr{S}_\beta(t_1)]B\chi(0)\|_\alpha \\
&\quad+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\|A^\alpha\mathscr{T}_\beta(t_2-s)\|\cdot\|F_n(s,y(s),y(h(s)))\|ds \\
&\quad+\int_0^{t_1}(t_1-s)^{\beta-1}\|A^\alpha[\mathscr{T}_\beta(t_1-s)-\mathscr{T}_\beta(t_2-s)]\|\cdot\|F_n(s,y(s),y(h(s)))\|ds \\
&\quad+\int_0^{t_1}[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}]\|A^\alpha\mathscr{T}_\beta(t_2-s)\|\times\|F_n(s,y(s),y(h(s)))\|ds \\
&\leqslant \|[\mathscr{S}_\beta(t_2)-\mathscr{S}_\beta(t_1)]B\chi(0)\|_\alpha+\frac{\beta W_1N_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}m_R(T_0)\frac{(t_2-t_1)^{\beta(1-\alpha)}}{\beta(1-\alpha)} \\
&\quad+\frac{\beta W_1N_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}m_R(T_0)\int_0^{t_1}(t_1-s)^{\beta-1}[(t_1-s)^{-\alpha\beta}-(t_2-s)^{-\alpha\beta}]ds \\
&\quad+\frac{\beta W_1N_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}m_R(T_0)\int_0^{t_1}[(t_1-s)^{\beta-1}-(t_1-s)^{\beta-1}](t_2-s)^{-\alpha\beta}ds. \qquad (21)
\end{aligned}
$$

For $y\in\mathbb{X}$, we have

$$
[\mathscr{S}(t_2^\beta\zeta)-\mathscr{S}(t_1^\beta\zeta)]y=\int_{t_1}^{t_2}\frac{d}{dt}\mathscr{S}(t^\beta\zeta)ydt=\int_{t_1}^{t_2}\zeta\beta t^{\beta-1}A\mathscr{S}(t^\beta\zeta)dt. \qquad (22)
$$

Thus, we get

$$
\begin{aligned}
&\int_0^\infty B^{-1}\xi_\beta(\zeta)\|\mathscr{S}(t_2^\beta\zeta)-\mathscr{S}(t_1^\beta\zeta)\|\|A^\alpha B\chi(0)\|d\zeta \\
&\leqslant \int_0^\infty B^{-1}\xi_\beta(\zeta)[\int_{t_1}^{t_2}\|\frac{d}{dt}\mathscr{S}(t^\beta\zeta)\|dt]B\|A^\alpha\chi(0)\|d\zeta \\
&\leqslant \int_0^\infty B^{-1}\xi_\beta(\zeta)[N_1(t_2-t_1)]\|B\|\|\chi(0)\|_\alpha d\zeta \\
&\leqslant R_1(t_2-t_1), \qquad (23)
\end{aligned}
$$

where $R_1=N_1W_1\|B\|\times\|\chi(0)\|_\alpha$. Also, we have

$$
\begin{aligned}
&\int_0^{t_1}(t_1-s)^{\beta-1}[(t_1-s)^{-\alpha\beta}-(t_2-s)^{-\alpha\beta}]ds \\
&\leqslant \vartheta d_1^{\vartheta-1}(1-h)^{-p_1(1-\vartheta)-1}(t_2-t_1)^{p_1(1-\vartheta)}, \qquad (24)
\end{aligned}
$$

where $h=[1-(\vartheta/p_1)^{1/(\vartheta p_1)}]$, $p_1=1-\beta\alpha$, $\vartheta=(1-\beta)/(1-\beta\alpha)$ and $0<d_1\leqslant 1$. Moreover, it follows that

$$
\begin{aligned}
&\int_0^{t_1}[(t_1-s)^{\beta-1}-(t_1-s)^{\beta-1}](t_2-s)^{-\alpha\beta}ds \\
&\leqslant \frac{N_{1+\alpha}}{\alpha}b_1^{\alpha-1}(1-h_1)^{-\beta(1-\alpha)-1}(t_2-t_1)^{\beta(1-\alpha)}, \qquad (25)
\end{aligned}
$$

where $h_1=(1-(\alpha/\beta)^{1/(\alpha\beta)})$, $0<b_1\leqslant 1$ and $N_{1+\alpha}$ is some positive constant with $\|A^{\alpha+1}\mathscr{S}(t)\|\leqslant N_{1+\alpha}t^{-1-\alpha}$, $\forall\, t>0$. Thus, from the inequalities (21)–(25) and $(O2)$, we deduce that the mapping $t \mapsto F_n(t, y(t))$ is uniformly Hölder continuous on $[0,T]$. Now, we claim that $\mathbb{Q}_n(\mathscr{B}_R(\mathbb{X}_\alpha(T))) \subseteq \mathscr{B}_R(\mathbb{X}_\alpha(T))$. Let $y \in \mathscr{B}_R(\mathbb{X}_\alpha(T))$ and $t \in [0,T]$. Then, we have

$$\begin{aligned} \|(\mathbb{Q}_n y)(t)\|_\alpha &\leqslant \|\mathscr{S}_\beta(t)B\chi(0)\|_\alpha + \|\int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F_n(s,y(s),y(h(s)))ds\|_\alpha \\ &\leqslant W_1\|B\|N_0\|\chi(0)\|_\alpha + \frac{\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_0^t (t-s)^{\beta(1-\alpha)-1} m_R(T_0)ds \\ &\leqslant W_1\|B\|N_0\|\chi(0)\|_\alpha + \frac{W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \times m_r(T_0)\frac{T^{\beta(1-\alpha)}}{(1-\alpha)}. \end{aligned} \tag{26}$$

Now, we can choose the positive integer R such that

$$R = W_1\|B\|N_0\|\chi(0)\|_\alpha + \frac{W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \times m_r(T_0)\frac{T^{\beta(1-\alpha)}}{(1-\alpha)}. \tag{27}$$

Thus, we deduce that $\mathbb{Q}_n(\mathscr{B}_R(\mathbb{X}_\alpha(T))) \subseteq \mathscr{B}_R(\mathbb{X}_\alpha(T))$. Next, we show that $\mathbb{Q}_n$ is a contraction mapping on $\mathscr{B}_R(\mathbb{X}_\alpha(T))$. For $y_1, y_2 \in \mathscr{B}_R(\mathbb{X}_\alpha(T))$ and $t \in [0,T]$, we obtain

$$\begin{aligned} &\|(\mathbb{Q}_n y_1)(t) - (\mathbb{Q}_n y_2)(t)\|_\alpha \\ \leqslant &\|\int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)[F_n(s,y_1(s),y_1(h(s))) - F_n(s,y_2(s),y_2(h(s)))]ds\|_\alpha \\ \leqslant &\frac{W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} m_R(T_0)\frac{T^{\beta(1-\alpha)}}{(1-\alpha)}\|y_1 - y_2\|_{\mathbb{X}_\alpha(T)} \\ \leqslant &\Lambda\|y_1 - y_2\|_{\mathbb{X}_\alpha(T)}. \end{aligned} \tag{28}$$

where $\Lambda = \frac{W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} m_R(T_0)\frac{T^{\beta(1-\alpha)}}{(1-\alpha)} < 1$. Thus, we conclude that $\mathbb{Q}_n$ is a contraction on $\mathscr{B}_R(\mathbb{X}_\alpha(T))$. Thus, there exists a unique $y_n \in \mathscr{B}_R(\mathbb{X}_\alpha(T))$ such that $\mathbb{Q}_n y_n = y_n$ which is solution for the integral equation (19). This finishes the proof of the theorem. $\square$

PROPOSITION 3.3. *Let us assume that* (O1)–(O2) *are fulfilled. If* $\chi(0) \in D(A^\alpha)$ *for some* $0 < \alpha < 1$*, then* $y_n(t) \in D(A^\upsilon)$ *for each* $t \in (0,T]$ *and* $0 \leqslant \upsilon < 1$*. Moreover, if* $\chi(0) \in D(A)$*, then* $y_n(t) \in D(A^\upsilon)$ *for all* $t \in [0,T]$ *and* $0 \leqslant \upsilon < 1$.

Proof. From the above theorem, we have that there exists a unique $y_n \in \mathscr{B}_R(\mathbb{X}_\alpha(T))$ which satisfies approximate integral equation (19). By the theorem 6.13(a) in [23], we have that $\mathscr{S}(t) : \mathbb{X} \to D(A^\upsilon)$ for $t > 0$ and $0 \leqslant \upsilon < 1$ and for $0 \leqslant \upsilon < \eta < 1$, $D(B^\eta) \subseteq D(B^\upsilon)$. We also have that $\mathscr{S}(t)y \in D(A)$ if $y \in D(A)$ using Theorem 2.4 in [23]. The result follows from these facts and the fact that $D(A) \subseteq D(A^\upsilon)$ for $0 \leqslant \upsilon \leqslant 1$. This finishes the proof of proposition. $\square$

LEMMA 3.4. *Suppose that the hypotheses* (O1)–(O2) *are satisfied. If* $\chi(0) \in D(A^\alpha)$ *for* $0 < \alpha < 1$ *and* $t_0 \in (0,T]$*, then there is a constant* S_{t_0}*, independent of* *n such that*

$$\|y_n(t)\|_\upsilon \leqslant S_{t_0}, \quad 0 \leqslant \upsilon < 1, \quad t \in [t_0, T].$$

Furthermore, if $\chi(0) \in D(A)$*, then there is a constant* $S_0 > 0$ *such that*

$$\|y_n(t)\|_\upsilon \leqslant S_0, \quad 0 \leqslant \upsilon < 1, \quad t \in [0, T],$$

and S_0 *is independent of* n*.*

Proof. Let $\chi(0) \in D(A^\alpha)$. Then, for $t \in [t_0, T]$, we apply A^υ on both the sides in (19) and get

$$\|y_n(t)\|_\upsilon \leqslant N_\upsilon W_1 \|B\| t_0^{-\upsilon} \|\chi(0)\| + \frac{\beta N_\upsilon W_1 \Gamma(2-\upsilon)}{\Gamma(1+\beta(1-\upsilon))} m_R(T_0) \frac{T^{\beta(1-\upsilon)}}{\beta(1-\upsilon)} \leqslant S_{t_0}. \tag{29}$$

Moreover, let $\chi(0) \in D(A)$. Then $\chi(0) \in D(A^\upsilon)$ for each $0 \leqslant \upsilon \leqslant 1$ and we get

$$\|y_n(t)\|_\upsilon \leqslant N_0 W_1 \|B\| \times \|\chi(0)\|_\upsilon + \frac{\beta N_\upsilon W_1 \Gamma(2-\upsilon)}{\Gamma(1+\beta(1-\upsilon))} m_R(T_0) \frac{T^{\beta(1-\upsilon)}}{\beta(1-\upsilon)} \leqslant S_0. \tag{30}$$

Thus, the proof of the lemma is completed. □

4. Convergence of solutions

The convergence of the solution $y_n \in \mathbb{X}_\alpha(T)$ of the approximate integral equations (19) to a unique solution $y(\cdot) \in \mathbb{X}_\alpha(T)$ of the equation (14) on $[0, T]$ is discussed in this section.

THEOREM 4.1. *Suppose that the assumptions* $(O1)$ *and* $(O2)$ *are satisfied and* $\chi(0) \in D(A^\alpha)$ *for* $0 < \alpha < 1$*. Then,*

$$\lim_{k \to \infty} \sup_{\{n \geqslant k,\ t \in [t_0, T]\}} \|y_n(t) - y_k(t)\|_\alpha = 0, \tag{31}$$

for each $t_0 \in (0, T]$*.*

Proof. For $n \geqslant k$, we obtain that

$$\begin{aligned}
&\|F_n(t, y_n(t), y_n(h(s))) - F_k(t, y_k(t), y_k(h(t)))\| \\
&\leqslant \|F_n(t, y_n, y_n(h(s))) - F_n(t, y_k, y_k(h(s)))\| + \|F_n(t, y_k, y_k(h(s))) - F_k(t, y_k, y_k(h(s)))\| \\
&\leqslant 2m_R(T_0)\|y_n(t) - y_k(t)\|_\alpha + m_R(T_0)[\|(P^n - P^k)y_k(t)\|_\alpha + \|(P^n - P^k)y_k(h(t))\|_\alpha].
\end{aligned} \tag{32}$$

For $0 < \alpha < \upsilon < 1$, we get

$$\|A^\alpha(P^n - P^k)y_k(t)\| \leqslant \|A^{\alpha-\upsilon}(P^n - P^k)A^\upsilon y_m(t)\| \leqslant \frac{1}{\lambda_k^{\upsilon-\alpha}} \|y_k(t)\|_\upsilon. \tag{33}$$

Thus, from (32) and (33), we obtain

$$\|F_n(t,y_n)-F_k(t,y_k)\| \leqslant 2m_R(T_0)[\|y_n(t)-y_k(t)\|_\alpha + \frac{1}{\lambda_k^{\upsilon-\alpha}}\|y_k(t)\|_\upsilon]. \tag{34}$$

We choose t_0' such that $0<t_0'<t_0<T$, we have

$$\begin{aligned}&\|y_n(t)-y_k(t)\|_\alpha\\ \leqslant &\Big(\int_0^{t_0'}+\int_{t_0'}^t\Big)(t-s)^{\beta-1}\|A^\alpha\mathscr{T}_\beta(t-s)[F_n(s,y_n(s),y_n(h(s)))-F_k(s,y_k(s),y_k(h(s)))]\|ds.\end{aligned} \tag{35}$$

We estimate the first integral of the above inequality as

$$\begin{aligned}&\int_0^{t_0'}(t-s)^{\beta-1}\|A^\alpha\mathscr{T}_\beta(t-s)\|\times\|F_n(s,y_n(s),y_n(h(s)))-F_k(s,y_k(s),y_k(h(s)))\|ds\\ \leqslant &\frac{2\beta W_1N_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}2m_R(T_0)(T-t_0')^{\beta(1-\alpha)-1}t_0'.\end{aligned} \tag{36}$$

The second integral of the inequality (35) can be estimated as

$$\begin{aligned}&\int_{t_0'}^t(t-s)^{\beta-1}\|A^\alpha\mathscr{T}_\beta(t-s)\|\times\|F_n(s,y_n(s),y_n(h(s)))-F_k(s,y_k(s),y_k(h(s)))\|ds\\ \leqslant &\frac{\beta N_\alpha W_1\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}2m_R(T_0)\left[\frac{U_{t_0'}T^{\beta(1-\alpha)}}{\beta(1-\alpha)\lambda_k^{\upsilon-\alpha}}+\int_{t_0'}^t(t-s)^{\beta(1-\alpha)-1}\|y_n-y_k\|_{\mathbb{X}_{\alpha(s)}}ds\right].\end{aligned} \tag{37}$$

Thus, from the inequalities (35) to (37), we conclude

$$\begin{aligned}\|y_n(t)-y_k(t)\|_\alpha \leqslant &\frac{2\beta W_1N_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}2m_R(T_0)(T-t_0')^{\beta(1-\alpha)-1}t_0'\\ &+\frac{\beta N_\alpha W_1\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}2m_R(T_0)\times\frac{U_{t_0'}T^{\beta(1-\alpha)}}{\beta(1-\alpha)\lambda_k^{\upsilon-\alpha}}\\ &+\frac{\beta N_\alpha W_1\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}2m_R(T_0)\times\int_{t_0'}^t(t-s)^{\beta(1-\alpha)-1}\|y_n-y_k\|_{\mathbb{X}_{\alpha(s)}}ds.\end{aligned} \tag{38}$$

From Gronwalls inequality, we deduce that

$$\begin{aligned}\|y_n(t)-y_k(t)\|_\alpha \leqslant &\Big[\frac{2\beta W_1N_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}2m_R(T_0)(T-t_0')^{\beta(1-\alpha)-1}t_0'\\ &+\frac{\beta N_\alpha W_1\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}2m_R(T_0)\times\frac{U_{t_0'}T^{\beta(1-\alpha)}}{\beta(1-\alpha)\lambda_k^{\upsilon-\alpha}}\Big]\times\mathscr{U}.\end{aligned} \tag{39}$$

Letting $k \to \infty$ and taking supremum over $[t_0, T]$, we have the following inequality

$$\|y_n - y_k\|_{\mathbb{X}_\alpha(T)} \leqslant \left[\frac{2\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} 2m_R(T_0)(T - t_0')^{\beta(1-\alpha)-1} t_0'\right] \mathscr{U}. \tag{40}$$

As t_0' is arbitrary, the right hand side may be made as small as desired by taking t_0' sufficiently small. The proof of the theorem is finished. □

COROLLARY 4.1. *Let us assume that* $\chi(0) \in D(A)$. *Then*

$$\lim_{k\to\infty,\ t\in[0,T]} \sup \|y_n(t) - y_k(t)\|_\alpha = 0. \tag{41}$$

Now, we have the following result for the convergence of the solution $y_n(t) \in \mathbb{X}_\alpha(T)$ of the approximate integral equation (19).

THEOREM 4.2. *Let us suppose that hypotheses* (O1)–(O2) *are satisfied and* $\chi(0) \in D(A^\alpha)$. *Then, there exist a unique function* $y_n(t) \in \mathbb{X}_\alpha(T)$ *fulfilling*

$$y_n(t) = \mathscr{S}_\beta(t)B\chi(0) + \int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F_n(s, y_n(s), y_n(h(s)))ds,\ t \in [0,T], \tag{42}$$

and $y(t) \in \mathbb{X}_\alpha(T)$, *satisfying*

$$y(t) = \mathscr{S}_\beta(t)B\chi(0) + \int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F(s, y(s), y(h(s)))ds,\ t \in [0,T], \tag{43}$$

such that $\lim_{n\to} y_n(t) = y(t)$ *in* $\mathbb{X}_\alpha(T)$.

Proof. Let $\chi(0) \in D(A)$. From the Corollary 4.1, it implies that there is a $y \in \mathbb{X}_\alpha(T)$ such that $\lim_{n\to\infty} y_n(t) = y(t)$. Since $y_n \in \mathscr{B}_R(\mathbb{X}_\alpha(T))$ for every n, then we have $y \in \mathscr{B}_R(\mathbb{X}_\alpha(T))$. Moreover, we have

$$\begin{aligned} &\|F_n(t, y_n(t), y_n(h(t))) - F(t, y(t), y(h(s)))\| \\ &= \|F(t, P^n y_n(t), y_n(h(s))) - F(t, y(t), y(h(s)))\| \\ &\leqslant 2m_R(T_0)[\|y_n(t) - y(t)\|_\alpha + \|(P^n - I)y(t)\|_\alpha]. \end{aligned} \tag{44}$$

we take supremum over $[0, T]$ and get

$$\begin{aligned} \sup_{t\in[0,T]} \|F_n(t, y_n(t)) - F(t, y(t))\| &\leqslant 2m_R(T_0)[\|y_n - y\|_{\mathbb{X}_\alpha(T)} + \|(P^n - I)y\|_{\mathbb{X}_\alpha(T)}] \\ &\to 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{45}$$

Thus, from (19), (45), we get

$$y(t) = \mathscr{S}_\beta(t)By_0 + \int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F(s, y(s), y(h(s)))ds,\ t \in [0,T]. \tag{46}$$

Now, let $\chi(0) \in D(A^{\alpha})$. Since, $A^{\alpha}y_n(t)$ converges to $A^{\alpha}y(t)$ for each $t \in (0,T]$ and $y_n(0) = y(0) = \chi(0)$. Then, $A^{\alpha}y_n(t)$ converges to $A^{\alpha}y(t)$ in $\mathbb{X}$. Furthermore, we have that $y_n \in \mathscr{B}_R(\mathbb{X}_{\alpha}(T))$ for each n and also $y \in \mathscr{B}_R(\mathbb{X}_{\alpha}(T))$. For $t \in [t_0,T]$, from the Theorem 4.1, we obtain

$$\lim_{n\to\infty} \sup_{t\in[t_0,T]} \|y_n(t) - y(t)\|_{\alpha} = 0. \tag{47}$$

Also, we have

$$\begin{aligned}&\sup_{t\in[0,T]} \|F_n(t,y_n(t),y_n(h(s))) - F(t,y(t),y(h(s)))\| \\ &\leqslant 2m_R(T_0)[\|y_n - y\|_{\mathbb{X}_{\alpha}(T)} + \|(P^n - I)y\|_{\mathbb{X}_{\alpha}(T)}] \to 0, \quad \text{as} \quad n \to \infty.\end{aligned} \tag{48}$$

Thus, for $0 < t_0 < t$, the integral equation (19) can be rewritten as

$$y_n(t) = \mathscr{S}_{\beta}(t)B\chi(0) + \Big(\int_0^{t_0} + \int_{t_0}^{t}\Big)(t-s)^{\beta-1}\mathscr{T}_{\beta}(t-s)F_n(s,y_n(s),y_n(h(s)))ds \tag{49}$$

we estimate first integral as

$$\Big\|\int_0^{t_0}(t-s)^{\beta-1}\mathscr{T}_{\beta}(t-s)F_n(s,y_n(s),y_n(h(s)))ds\Big\| \leqslant \frac{N_0W_1}{\Gamma(\beta)}2m_R(T_0)\frac{{t_0}^{\beta}}{\beta}. \tag{50}$$

Therefore, we get

$$\Big\|y_n(t)-\mathscr{S}_{\beta}(t)B\chi(0)-\int_{t_0}^{t}(t-s)^{\beta-1}\mathscr{T}_{\beta}(t-s)F_n(s,y_n(s),y_n(h(s)))ds\Big\| \leqslant \frac{N_0W_1}{\Gamma(\beta)}2m_R(T_0)\frac{{t_0}^{\beta}}{\beta}. \tag{51}$$

Taking $n \to \infty$ and getting

$$\Big\|y(t)-\mathscr{S}_{\beta}(t)B\chi(0)-\int_{t_0}^{t}(t-s)^{\beta-1}\mathscr{T}_{\beta}(t-s)F(s,y(s),y(h(s)))ds\Big\| \leqslant \frac{N_0W_1}{\Gamma(\beta)}2m_R(T_0)\frac{{t_0}^{\beta}}{\beta}. \tag{52}$$

As t_0 is arbitrary, we deduce that $y(t)$ fulfills the integral equation (14).

Now, we show the uniqueness. Let y_1 and y_2 be the solution of the integral equation (14). Thus, we have

$$\begin{aligned}&\|y_1(\tau) - y_2(\tau)\|_{\alpha} \\ &\leqslant \int_0^{\tau}(\tau-s)^{\beta-1}\|A^{\alpha}\mathscr{T}_{\beta}(\tau-s)\| \times \|F(s,y_1(s),y_1(h(s))) - F(s,y_2(s),y_2(h(s)))\|ds \\ &\leqslant \frac{\beta N_{\alpha}W_1\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_0^{\tau}(\tau-s)^{\beta(1-\alpha)-1}2m_R(T_0)\|y_1-y_2\|_{\mathbb{X}_{\alpha}(s)}ds.\end{aligned}$$

Taking supremum on $[0,t]$ and obtaining

$$\|y_1-y_2\|_{\mathbb{X}_{\alpha}(T)} \leqslant \frac{\beta N_{\alpha}W_1\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_0^{\tau}(\tau-s)^{\beta(1-\alpha)-1}2m_R(T_0)\|y_1-y_2\|_{\mathbb{X}_{\alpha}(s)}ds. \tag{53}$$

From Gronwalls inequality and the fact that

$$\|y_1(t)-y_2(t)\| \leqslant \frac{1}{\lambda_0^{\alpha}}\|y_1-y_2\|_{\mathbb{X}_\alpha(T)}, \tag{54}$$

we deduce that $y_1 = y_2$ on $[0,T]$. This finishes the proof of the theorem. □

5. Faedo-Galerkin approximations

In this section, we study the Faedo-Galerkin Approximation of a solution and show the convergence results for such an approximation.

We know that for any $0 < T < T_0$, we have a unique $y \in \mathbb{X}_\alpha(T)$ satisfying the following integral equation

$$y(t) = \mathscr{S}_\beta(t)[B]\chi(0) + \int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F(s,y(s),y(h(s)))ds, \quad t \in [0,T]. \tag{55}$$

We also have a unique solution $y_n \in \mathbb{X}_\alpha(T)$ for the approximate integral equation

$$y_n(t) = \mathscr{S}_\beta(t)[B]\chi(0)+\int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)F_n(s,y_n(s),y_n(h(s)))ds, \quad t \in [0,T]. \tag{56}$$

Applying the projection on above equation, then Faedo-Galerkin approximation is given by $v_n(t) = P^n y_n(t)$ satisfying

$$\begin{aligned} P^n y_n(t) = v_n(t) &= \mathscr{S}_\beta(t)BP^n\chi(0)+\int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)P^nF(s,P^ny_n(s),P^ny_n(s))ds \\ &= \mathscr{S}_\beta(t)BP^n\chi(0) + \int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)P^nF(s,v_n(s),v_n(h(s)))ds. \end{aligned} \tag{57}$$

Let solution $y(\cdot)$ of (55) and $v_n(\cdot)$ of (57), have the following representation

$$y(t) = \sum_{i=0}^{\infty}\alpha_i(t)\phi_i, \quad \alpha_i(t) = (y(t),\phi_i), \quad i = 0,1,2\cdots, \tag{58}$$

$$v_n(t) = \sum_{i=0}^{n}\alpha_i^n(t)\phi_i, \quad \alpha_i^n(t) = (v_n(t),\phi_i), \quad i = 0,1,2\cdots, \tag{59}$$

respectively.

From (57) and (59), we obtain the following system of fractional differential equations

$$\frac{d^\beta}{dt^\beta}\alpha_i^n(t) + \lambda_i\alpha_i^n(t) = F_i^n(t,\alpha_0^n(t),\alpha_1^n(t),\cdots,\alpha_n^n(t)), \tag{60}$$

$$\alpha_i^n(0) = z_i, \tag{61}$$

where

$$F_i^n(t,\alpha_0^n(t),\alpha_1^n(t),\cdots,\alpha_n^n(t)) = (B^{-1}F(t,\sum_{i=0}^{n}\alpha_i^n(t)\phi_i,\sum_{i=0}^{n}\alpha_i^n(t)\phi_i),\phi_i), \tag{62}$$

and $z_i = (\chi(0),\phi_i)$ for each $n = 1,2,\cdots,n$.

We also have the following convergence theorem.

THEOREM 5.1. *Suppose that the hypotheses* (O1)–(O2) *are fulfilled. Then, we have*

(1). *If* $y_0 \in D(A^\alpha)$, *then for any* $t_0 \in (0,T]$,

$$\lim_{k\to\infty} \sup_{\{n\geqslant k,\ t\in[t_0,T]\}} \|A^\alpha[v_n(t) - v_k(t)]\| = 0. \tag{63}$$

(2). *If* $y_0 \in D(A)$, *then*

$$\lim_{k\to\infty} \sup_{\{n\geqslant k,\ t\in[0,T]\}} \|A^\alpha[v_n(t) - v_k(t)]\| = 0 \tag{64}$$

Proof. Let $n \geqslant k$ and $0 \leqslant \alpha < \upsilon$. Then, we get

$$\begin{aligned}\|v_n(t) - v_k(t)\|_\alpha &= \|P^n y_n(t) - P^k y_k(t)\|_\alpha \\ &\leqslant \|P^n[y_n(t) - y_k(t)]\|_\alpha + \|(P^n - P^k)y_k\|_\alpha \\ &\leqslant \|y_n(t) - y_k(t)\|_\alpha + \frac{1}{\lambda_k^{\upsilon-\alpha}}\|y_k(t)\|_\upsilon. \end{aligned}\tag{65}$$

By the Theorem 4.1 and Corollary 4.1, we have that $y_n \to y_k$ and $\lambda_k \to \infty$ as $k \to \infty$. Thus, this completes the proof of theorem. □

THEOREM 5.2. *Suppose that the hypotheses* (O1)–(O2) *are fulfilled and* $y_0 \in D(A^\alpha)$. *Then, there exists a unique function* $v_n \in \mathbb{X}_\alpha(T)$ *fulfilling following equation*

$$v_n(t) = \mathscr{S}_\beta(t)BP^n y_0 + \int_0^t (t-s)^{\beta-1}\mathscr{T}_\beta(t-s)P^n F(s, v_n(s))ds,\ \ t \in [0,T], \tag{66}$$

and $y \in \mathbb{X}_\alpha(T)$ *that satisfies equation (55) such that* $v_n \to y$ *as* $n \to \infty$.

Proof. For $y_0 \in D(A^\alpha)$ and $t \in [0,T]$, we have

$$\begin{aligned}\|v_n(t) - y(t)\|_\alpha &= \|P^n y_n(t) - P^n y(t) + P^n y(t) - y(t)\|_\alpha \\ &\leqslant \|P^n(y_n(t) - y(t))\|_\alpha + \|(P^n - I)y(t)\|_\alpha. \end{aligned}\tag{67}$$

By the Theorem 4.2, we have $y_n \to y$ as $n \to \infty$. Thus, the results follow from the Theorem 4.2. □

To prove the convergence of α_i^n to α_i, we have the following theorem.

THEOREM 5.3. *Assume that the conditions* (O1)–(O2) *are satisfied. Then,*

(1). *If* $y_0 \in D(A^\alpha)$, *then for any* $0 < t_0 \leqslant T$,

$$\lim_{n\to\infty} \sup_{t\in[t_0,T]} [\sum_{i=0}^n \lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2] = 0. \tag{68}$$

(2). *If* $y_0 \in D(A)$, *then*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} [\sum_{i=0}^n \lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2] = 0. \tag{69}$$

Proof. We have that

$$A^{\alpha}[y(t)-v_n(t)] = A^{\alpha}[\sum_{i=0}^{\infty}(\alpha_i(t)-\alpha_i^n(t))\phi_i] = \sum_{i=0}^{\infty}\lambda_i^{\alpha}(\alpha_i(t)-\alpha_i^n(t))\phi_i. \tag{70}$$

Thus,

$$\|A^{\alpha}[y(t)-v_n(t)]\|^2 \geqslant \sum_{i=0}^{n}\lambda_i^{2\alpha}[\alpha_i(t)-\alpha_i^n(t)]^2. \tag{71}$$

Thus, the results follows from the Theorem 5.1 and 5.2. □

6. Application

Let us consider the following fractional differential system of Sobolev type illustrated as

$$^{c}D_t^{\beta}[w(t,x)-w_{xx}(t,x)] + \frac{\partial^2 w(t,x)}{\partial x^2} = f(t,w(t,x)),\ \ x\in\mathbb{S},\ t\in[0,1], \tag{72}$$

$$w(0,x) = w_0(x), \quad x\in[0,\pi], \tag{73}$$

$$w(t,0) = w(t,\pi) = 0, \quad 0<t\leqslant 1, \tag{74}$$

where $^{c}D_t^{\beta}$ is the fractional derivative in Caputo sense, $0<\beta<1$. Let $w(t)(x) = w(t,x)$ and $f(t,\cdot) = F(t,\cdot)$.

Now, we take $X = Y = L^2[0,\pi]$ and consider the operators L,B on domains and ranges contained in $L^2[0,\pi]$ defined by

$$By = y - y'',\ \ Ly = -y'' \tag{75}$$

with domain

$$D(B) = D(L) = \{y\in X:\ y,y' \text{ are absolutely continuous } y''\in X,\ y(0) = y(\pi) = 0\}. \tag{76}$$

Thus, B and L can be written, respectively, as

$$By = \sum_{n=1}^{\infty}(1+n^2)(y,u_n)u_n, \text{ and } Ly = \sum_{n=1}^{\infty} -n^2(y,u_n)u_n, \tag{77}$$

where $u_n(t) = \sqrt{\frac{2}{\pi}}\sin(nt)$, $n = 1,2,\cdots$, are eigenfunctions of B corresponding to eigenvalue $\lambda_n = -n^2$. Moreover, we have that for any $y\in X$,

$$B^{-1}y = \sum_{n=1}^{\infty}\frac{1}{1+n^2}(y,u_n)u_n, \text{ and } LB^{-1}y = \sum_{n=1}^{\infty}\frac{-n^2}{1+n^2}(y,u_n)u_n, \tag{78}$$

with

$$\mathscr{S}(t)y = \sum_{n=1}^{\infty}\exp\left(\frac{-n^2 t}{1+n^2}\right)(y,u_n)u_n. \tag{79}$$

Clearly, B^{-1} is continuous, bounded with $\|B^{-1}\| \leqslant 1$ and LB^{-1} generates the above strongly continuous semigroup $\mathscr{S}(t)$ on $L^2[0,\pi]$ with $\|\mathscr{S}(t)\| \leqslant e^{-t} \leqslant 1$.

Therefore, the system (72)–(74) can be reformulated as

$$D_t^q[Bu(t)] = Lu(t) + F(t,u(t)), \quad t > 0, \tag{80}$$

$$u(0) = w_0. \tag{81}$$

Thus, the results of the earlier sections to guarantee the existence of Faedo Galerkin approximations and their convergence to the unique solution of (72)–(74) may be applied with appropriate function F satisfying suitable conditions.

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