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GENERALIZED ALMOST AUTOMORPHIC AND GENERALIZED ASYMPTOTICALLY ALMOST AUTOMORPHIC SOLUTIONS OF ABSTRACT VOLTERRA INTEGRO–DIFFERENTIAL INCLUSIONS

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Abstract. The main aim of this paper is to investigate generalized almost automorphy and generalized asymptotical almost automorphy of solutions for certain classes of abstract Volterra integro-differential inclusions and abstract (semilinear) fractional differential inclusions in Banach spaces. We illustrate our abstract results with several examples and possible applications.

1. Introduction and preliminaries

Almost periodic and asymptotically almost periodic solutions of differential equations in Banach spaces have been considered by many authors so far (for the basic information on the subject, we refer the reader to the monographs [3], [5], [8], [13], [17], [29], [31], [38], [42] and [58]).

S. Bochner has introduced the notion of a scalar-valued almost automorphic function in [10], generalizing so the notion of an almost periodic function. The first extensive study of almost automorphic functions on topological groups has been conducted by W. A. Veech [52]–[53]. For the basic information about almost automorphic functions, asymptotically almost automorphic functions, their generalizations and various applications to differential and functional differential equations in Banach spaces, we refer the reader to [1]–[2], [9]–[11], [14], [16]–[26], [28]–[29], [38], [41], [43], [45], [48], [51]–[53], [56]–[57] and [59].

Just a few words about some applications contained in the above-mentioned research papers. Almost automorphic solutions to a class of semilinear fractional differential equations of the form

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t,u(t)), \quad t \in \mathbb{R},$$

where $1 < \alpha < 2$, A is a sectorial operator with domain and range in a Banach space X, of negative sectorial type $\omega < 0$, $f : \mathbb{R} \times X \to X$ is an almost automorphic function in time for any fixed element $x \in X$, satisfying certain Lipschitz conditions, and

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$D_t^\alpha u(t)$ denotes a fractional derivative of the Riemann-Liouville type, have been examined by C. Cuevas and C. Lizama in [16] (for almost automorphic solutions of semilinear Cauchy problems, we also refer to T. Diagana, G. M. N'Guérékata [19] and J. A. Goldstein, G. M. N'Guérékata [28]; the nonautonomous case has been analyzed by H.-S. Ding, J. Liang and T.-J. Xiao [22]).

Concerning Stepanov class of almost automorphic functions, mention should be made of the paper [23] by H.-S. Ding, J. Liang and T.-J. Xiao as well as the paper [2], where S. Abbas, V. Kavitha and R. Murugesu have examined Stepanov-like weighted pseudo almost automorphic solutions to the following fractional order abstract integro-differential equation:

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t), Ku(t)), \quad t \in \mathbb{R},$$

where $Ku(t) = \int_{-\infty}^t k(t-s)h(s,u(s))\,ds,\ t \in \mathbb{R},\ 1 < \alpha < 2,\ A$ is a sectorial operator with domain and range in X, of negative sectorial type $\omega < 0$, the function $k(t)$ is exponentially decaying, the functions $f : \mathbb{R} \times X \times X \to X$ and $h : \mathbb{R} \times X \to X$ are Stepanov-like weighted pseudo almost automorphic in time for each fixed elements of $X \times X$ and X, respectively, satisfying some extra conditions (cf. also T. Diagana [20] for similar results in this direction). It is worth noting that the class of weighted Stepanov-like pseudo almost automorphic functions has been introduced by Z. Xia and M. Fan in [56], where the authors have analyzed the existence and uniqueness of such solutions for the following abstract semilinear integro-differential equation:

$$u(t) = g(t) + \int_{-\infty}^t a(t-s) f(s,u(s))\,ds, \quad t \in \mathbb{R},$$

under certain conditions. Finally, we want to observe that T. Diagana, V. Nelson and G. M. N'Guérékata have introduced the notion of an $S_p^{(n)}$-almost automorphic function in [21], providing also some results about $C^{(m+N)}$-pseudo almost automorphic solutions to the higher-order abstract differential equation

$$u^{(n)}(t) + \sum_{i=1}^{n-1} a_i(t) u^{(i)}(t) = f(t), \quad t \in \mathbb{R}$$

where $a_i : \mathbb{R} \to \mathbb{R}$ satisfy certain conditions ($i \in \mathbb{N}_{n-1}^0$) and the function $f : \mathbb{R} \to \mathbb{R}$ is Stepanov-like $C^{(m)}$-pseudo almost automorphic. Their method is based on the converting of above equation into an equivalent first order matricial system and therefore is not applicable to abstract multi-term fractional differential equations (cf. [30] for some results in this direction).

In [14], V. Casarino has introduced the notions of a (Stepanov) almost automorphic C_0-group and a (Stepanov) asymptotically almost automorphic C_0-group on Banach space, where some equivalence relations between almost periodicity and almost automorphy for orbits of a C_0-group have been proved. We would like to observe that the extensions of her results to (degenerate) C-regularized groups of operators can be proved almost immediately (see [38, Section 2.4] for almost periodic case). The assertion of [38, Proposition 2.5.1] can be also straightforwardly formulated for various classes of (asymptotically) almost automorphic (a,k)-regularized C-resolvent families in Banach spaces. On the other hand, numerous very non-trivial and unpleasant problems occur if we try to reconsider some known assertions on the (asymptotical) almost periodicity of (a,k)-regularized C-resolvent families in Banach spaces, provided that the results from the Bohr-Fourier analysis of almost periodic functions are needed for their proofs (see e.g. [38, Section 2.5, Section 2.6] for more details). Furthermore, asymptotical almost periodicity is the property stable under the action of subordination principle discovered by E. Bazhlekova [6, Theorem 3.1], and it seems very complicated to say anything relevant about the inheritance of asymptotical almost automorphy under the action of this subordination principle; Stepanov and Weyl generalizations are much more delicate to deal with here, even in the case of consideration of asymptotical almost periodicity.

We use the standard notation throughout the paper. By $(X,\|\cdot\|)$ we denote a complex Banach space. If $(Y,\|\cdot\|_Y)$ is also such a space, then by $L(X,Y)$ we denote the space of all continuous linear mappings from X into Y; $L(X)\equiv L(X,X)$. If A is a linear operator acting on X, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively. The symbol I denotes the identity operator on X. If $I=[0,\infty)$ or $I=\mathbb{R}$, then by $C_b(I:X)$ we denote the space consisted of all bounded continuous functions from I into X; the symbol $C_0([0,\infty):X)$ denotes the closed subspace of $C_b([0,\infty):X)$ consisting of functions vanishing at infinity. The space $C_b(I:X)$ becomes one of Banach's when equipped with the sup-norm. If $\zeta>0$, then we put $g_\zeta(t):=t^{\zeta-1}/\Gamma(\zeta)$, $t>0$, where $\Gamma(\cdot)$ denotes the Gamma function.

Fractional calculus started more than three centuries ago, probably with some works of Leibnitz, and developed later by several mathematicians as Euler, Fourier, Liouville, Grunwald, Letnikov and Riemann, among many others. The first conference on fractional calculus and fractional differential equations was held in New Haven (1974) and, from then on, fractional calculus has gained more and more attention due to its wide and invaluable applications in various fields of science, such as mathematical physics, engineering, biology, chemistry, economics etc. For basic information on fractional calculus and fractional differential equations, the reader may consult [6], [32]–[33], [47], [49] and references cited therein.

The Mittag-Leffler functions and Wright functions naturally occur as solutions of fractional integro-differential equations. Assume that $\alpha>0$ and $\beta\in\mathbb{R}$. Then the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z):=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad z\in\mathbb{C}.$$

Set, for short,

$$E_\alpha(z):=E_{\alpha,1}(z),\quad z\in\mathbb{C}.$$

Let $\gamma\in(0,1)$. The Wright function $\Phi_\gamma(\cdot)$ is defined by

$$\Phi_\gamma(t):=\mathscr{L}^{-1}\big(E_\gamma(-\lambda)\big)(t),\quad t\geqslant 0,$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform. It is well known that the Wright function $\Phi_\gamma(\cdot)$ can be entirely extended to the whole complex plane by the formula

$$\Phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\gamma-\gamma n)}, \quad z \in \mathbb{C}.$$

In this paper, we use the Caputo fractional derivatives [6] and the Weyl-Liouville fractional derivatives [45]. Let $\gamma \in (0,1)$. The Caputo fractional derivative $\mathbf{D}_t^\gamma u(t)$ of order γ is defined for those functions $u \in C([0,\infty) : X)$ for which $g_{1-\gamma} * (u - u(0)) \in C^1([0,\infty) : X)$, by

$$\mathbf{D}_t^\gamma u(t) := \frac{d}{dt}\Big[g_{1-\gamma} * \big(u - u(0)\big)\Big].$$

The Weyl-Liouville fractional derivative $D_{t,+}^\gamma u(t)$ of order γ is defined for those continuous functions $u : \mathbb{R} \to X$ such that $t \mapsto \int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\,ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by

$$D_{t,+}^\gamma u(t) := \frac{d}{dt}\int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\,ds, \quad t \in \mathbb{R}.$$

Set $\mathbf{D}_t^1 u(t) := (d/dt)u(t)$ and $D_{t,+}^1 u(t) := -(d/dt)u(t)$.

Before explaining the organization and main ideas of this paper, the author wishes to express his heartfelt sense of gratitude and sincere thanks to Prof. G. M. N'Guérékata, who initiated the genesis of this paper, and Prof. T. Diagana, for many useful suggestions which have improved the quality of the paper.

In Section 2, we present a short retrospective of definitions and results about multivalued linear operators in Banach spaces. Section 3, containing two separate subsections, is devoted to the recapitulation of some known results on almost automorphic functions, asymptotically almost automorphic functions and their generalizations (in this section, essentially, the only new results are Proposition 1-Proposition 3). Our main results are stated in Section 4, where we investigate the generalized (asymptotically) almost automorphic properties of various types of convolution products. Let $\mathscr{A}$ be an MLO in X; cf. Section 2 for the notion. Of concern is the following abstract Cauchy inclusion of first order

$$u'(t) \in \mathscr{A}u(t) + f(t),\ t \in \mathbb{R}, \tag{1}$$

and its fractional relaxation analogue

$$D_{t,+}^\gamma u(t) \in -\mathscr{A}u(t) + f(t),\ t \in \mathbb{R}, \tag{2}$$

where $D_{t,+}^\gamma$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1)$, and $f : \mathbb{R} \to X$ is a generalized almost automorphic function, as well as their semilinear analogues

$$u'(t) \in \mathscr{A}u(t) + f(t,u(t)),\ t \in \mathbb{R}, \tag{3}$$

and

$$D_{t,+}^\gamma u(t) \in -\mathscr{A}u(t) + f(t,u(t)),\ t \in \mathbb{R}, \tag{4}$$

where $f : \mathbb{R} \times X \to X$ is a generalized almost automorphic function. Moreover, of concern is the following fractional relaxation inclusion

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathscr{A} u(t) + f(t),\ t \geqslant 0, \\ u(0) = x_0, \end{cases}$$

and its semilinear analogue

$$(\mathrm{DFP})_{f,\gamma,s} : \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathscr{A} u(t) + f(t,u(t)),\ t \geqslant 0, \\ u(0) = x_0, \end{cases}$$

where $\mathbf{D}_t^{\gamma}$ denotes the Caputo fractional derivative of order $\gamma \in (0,1]$, $x_0 \in X$ and $f : [0,\infty) \to X$, resp. $f : [0,\infty) \times X \to X$, is a generalized asymptotically almost automorphic function (cf. [38] for more details). The main goal of Section 5 is to prove several assertions on the existence and uniqueness of generalized almost automorphic solutions of the semilinear Cauchy inclusions (3)–(4) and $(\mathrm{DFP})_{f,\gamma,s}$. This section is written in expository manner, without giving the proofs of our abstract results. The main reason for this lies in the fact that our results given in Section 4 and composition theorems for generalized almost automorphic functions given in Subsection 3.2 enable one to simply deduce the proofs of our results in Section 5 by using an almost verbatim repeating of the argumentation used in almost periodic case. Section 6 is reserved for examples and applications of our abstract theoretical results, which seem to be new even for a class of almost sectorial operators [46], as well. Therefore, besides examples presented in Section 6, we are in a position to analyze the existence and uniqueness of generalized (asymptotically) almost automorphic solutions for certain classes of higher order (semilinear) elliptic differential equations in Hölder spaces; see e.g. W. von Wahl [54].

2. Multivalued linear operators in Banach spaces

The main aim of this section is to present a brief recollection of elementary definitions and results from the theory of multivalued linear operators. For more details about this intriguing topic, we refer the reader to the monographs by R. Cross [15], A. Favini-A. Yagi [27] and M. Kostić [33].

Let X and Y be two Banach spaces over the field of complex numbers. A multivalued map $\mathscr{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) iff the following two conditions hold:

(i) $D(\mathscr{A}) := \{x \in X : \mathscr{A}x \neq \emptyset\}$ is a linear subspace of X;

(ii) $\mathscr{A}x + \mathscr{A}y \subseteq \mathscr{A}(x+y)$, $x,\ y \in D(\mathscr{A})$ and $\lambda \mathscr{A}x \subseteq \mathscr{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathscr{A})$.

If $X = Y$, then it is said that $\mathscr{A}$ is an MLO in X. We know that $\lambda \mathscr{A}x + \eta \mathscr{A}y = \mathscr{A}(\lambda x + \eta y)$ holds for every $x,\ y \in D(\mathscr{A})$ and for every $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$. If $\mathscr{A}$ is an MLO, then $\mathscr{A}0$ is a linear submanifold of Y and $\mathscr{A}x = f + \mathscr{A}0$ for any $x \in D(\mathscr{A})$ and $f \in \mathscr{A}x$. Set $R(\mathscr{A}) := \{\mathscr{A}x : x \in D(\mathscr{A})\}$. Then the set $N(\mathscr{A}) := \mathscr{A}^{-1}0 = \{x \in$

$D(\mathscr{A}) : 0 \in \mathscr{A}x\}$ is called the kernel space of $\mathscr{A}$. The inverse $\mathscr{A}^{-1}$ of an MLO is defined through $D(\mathscr{A}^{-1}) := R(\mathscr{A})$ and $\mathscr{A}^{-1}y := \{x \in D(\mathscr{A}) : y \in \mathscr{A}x\}$. It is checked at once that $\mathscr{A}^{-1}$ is an MLO in X, as well as that $N(\mathscr{A}^{-1}) = \mathscr{A}0$ and $(\mathscr{A}^{-1})^{-1} = \mathscr{A}$. In the case that $N(\mathscr{A}) = \{0\}$, i.e., if $\mathscr{A}^{-1}$ is single-valued, then $\mathscr{A}$ is said to be injective. If $\mathscr{A}$, $\mathscr{B} : X \to P(Y)$ are two MLOs, then we define its sum $\mathscr{A} + \mathscr{B}$ by $D(\mathscr{A} + \mathscr{B}) := D(\mathscr{A}) \cap D(\mathscr{B})$ and $(\mathscr{A} + \mathscr{B})x := \mathscr{A}x + \mathscr{B}x$, $x \in D(\mathscr{A} + \mathscr{B})$. It is clear that $\mathscr{A} + \mathscr{B}$ is likewise an MLO. We write $\mathscr{A} \subseteq \mathscr{B}$ iff $D(\mathscr{A}) \subseteq D(\mathscr{B})$ and $\mathscr{A}x \subseteq \mathscr{B}x$ for all $x \in D(\mathscr{A})$. Products, integer powers and multiplication with scalar constants are well-known operations for MLOs ([27]).

It is said that an MLO $\mathscr{A} : X \to P(Y)$ is closed if for any sequences (x_n) in $D(\mathscr{A})$ and (y_n) in Y such that $y_n \in \mathscr{A}x_n$ for all $n \in \mathbb{N}$ we have that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $x \in D(\mathscr{A})$ and $y \in \mathscr{A}x$.

Suppose now that $\mathscr{A}$ is an MLO in X, as well as that $C \in L(X)$ is injective and $C\mathscr{A} \subseteq \mathscr{A}C$. The C-resolvent set of $\mathscr{A}$, $\rho_C(\mathscr{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathscr{A})$;

(ii) $(\lambda - \mathscr{A})^{-1}C$ is a single-valued linear continuous operator on X.

The operator $\lambda \mapsto (\lambda - \mathscr{A})^{-1}C$ is said to be the C-resolvent of $\mathscr{A}$ $(\lambda \in \rho_C(\mathscr{A}))$; the resolvent set of $\mathscr{A}$ is defined by $\rho(\mathscr{A}) := \rho_I(\mathscr{A})$, $R(\lambda : \mathscr{A}) \equiv (\lambda - \mathscr{A})^{-1}$ $(\lambda \in \rho(\mathscr{A}))$. Any MLO with non-empty resolvent set is closed and the Hilbert resolvent equation holds in our framework.

Assume now that $\mathscr{A}$ is an MLO in X, as well as that $(-\infty, 0] \subseteq \rho(\mathscr{A})$ and there exist finite numbers $M \geqslant 1$ and $\beta \in (0,1]$ such that

$$\|R(\lambda : \mathscr{A})\| \leqslant M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \leqslant 0.$$

Then there exist two positive real numbers $c > 0$ and $M_1 > 0$ such that $\rho(\mathscr{A})$ contains an open region $\Omega = \{\lambda \in \mathbb{C} : |\Im\lambda| \leqslant (2M_1)^{-1}(c - \Re\lambda)^{\beta},\ \Re\lambda \leqslant c\}$ surrounding the half-line $(-\infty, 0]$, where we have the estimate $\|R(\lambda : \mathscr{A})\| = O((1 + |\lambda|)^{-\beta})$, $\lambda \in \Omega$. Let Γ' be the upwards oriented curve $\{\xi \pm i(2M_1)^{-1}(c - \xi)^{\beta} : -\infty < \xi \leqslant c\}$. We define the fractional power

$$\mathscr{A}^{-\theta} := \frac{1}{2\pi i}\int_{\Gamma'} \lambda^{-\theta}\big(\lambda - \mathscr{A}\big)^{-1}\, d\lambda \in L(X)$$

for $\theta > 1 - \beta$. Set $\mathscr{A}^{\theta} := (\mathscr{A}^{-\theta})^{-1}$ $(\theta > 1 - \beta)$. Then the semigroup properties $\mathscr{A}^{-\theta_1}\mathscr{A}^{-\theta_2} = \mathscr{A}^{-(\theta_1+\theta_2)}$ and $\mathscr{A}^{\theta_1}\mathscr{A}^{\theta_2} = \mathscr{A}^{\theta_1+\theta_2}$ hold for θ_1, $\theta_2 > 1 - \beta$ (let us recall that the fractional power $\mathscr{A}^{\theta}$ need not be injective and that the meaning of $\mathscr{A}^{\theta}$ is understood in the MLO sense).

The vector space $D(\mathscr{A})$ equipped with the norm $\|\cdot\|_{[D(\mathscr{A})]} := \inf_{y \in \mathscr{A}\cdot} \|y\|$ becomes a Banach space. It is well known that $0 \in \rho(\mathscr{A}^{\theta})$ and that $(D(\mathscr{A}^{\theta}), \|\cdot\|_{[D(\mathscr{A}^{\theta})]})$ is likewise a Banach space $(\theta > 1 - \beta)$.

For more details about multivalued linear operators and abstract degenerate integro-differential equations, the reader may consult the monographs [12], [15], [27], [33], [44] and [50].

3. Almost automorphic functions, asymptotically almost automorphic functions and their generalizations

The concept of almost periodicity was introduced by Danish mathematician H. Bohr around 1924-1926 and later generalized by many other authors (cf. [17], [29], [38], [42] and [58] for more details on the subject). Let $I=\mathbb{R}$ or $I=[0,\infty)$, and let $f: I\to X$ be continuous. Given $\varepsilon>0$, we call $\tau>0$ an ε-period for $f(\cdot)$ iff $\|f(t+\tau)-f(t)\|\leqslant\varepsilon,\ t\in I$. The set constituted of all ε-periods for $f(\cdot)$ is denoted by $\vartheta(f,\varepsilon)$. It is said that $f(\cdot)$ is almost periodic, a.p. for short, iff for each $\varepsilon>0$ the set $\vartheta(f,\varepsilon)$ is relatively dense in I, which means that there exists $l>0$ such that any subinterval of I of length l meets $\vartheta(f,\varepsilon)$.

Let $f:\mathbb{R}\to X$ be continuous. As it is well known, $f(\cdot)$ is called almost automorphic, a.a. for short, iff for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g:\mathbb{R}\to X$ such that

$$\lim_{n\to\infty} f\big(t+a_n\big)=g(t) \text{ and } \lim_{n\to\infty} g\big(t-a_n\big)=f(t), \tag{5}$$

pointwise for $t\in\mathbb{R}$. If this is the case, then it is well known that $f\in C_b(\mathbb{R}:X)$ and that the limit function $g(\cdot)$ must be bounded on $\mathbb{R}$ but not necessarily continuous on $\mathbb{R}$. Furthermore, it is clear that the uniform convergence of one of the limits appearing in (5) implies the convergenece of the second one in this equation and that, in this case, the function $f(\cdot)$ has to be almost periodic and the function $g(\cdot)$ has to be continuous. If the convergence of limits appearing in (5) is uniform on compact subsets of $\mathbb{R}$, then we say that $f(\cdot)$ is compactly almost automorphic, c.a.a. for short. The vector space consisting of all almost automorphic, resp., compactly almost automorphic functions, is denoted by $AA(\mathbb{R}:X)$, resp., $AA_c(\mathbb{R}:X)$. By Bochner's criterion [17], any almost periodic function has to be compactly almost automorphic. The converse statement is not true, however [17].

It is well-known that the reflexion at zero keeps the spaces $AA(\mathbb{R}:X)$ and $AA_c(\mathbb{R}:X)$ unchanged, as well as that the function $g(\cdot)$ from (5) satisfies $\|f\|_\infty=\|g\|_\infty$ and $R(g)\subseteq\overline{R(f)}$, later needed to be a compact subset of X.

A continuous function $f:\mathbb{R}\to X$ is called asymptotically (compact) almost automorphic, a.(c.)a.a. for short, iff there exist a function $h\in C_0([0,\infty):X)$ and a (compact) almost automorphic function $q:\mathbb{R}\to X$ such that $f(t)=h(t)+q(t),\ t\geqslant 0$. Using Bochner's criterion again, it readily follows that any asymptotically almost periodic function $[0,\infty)\mapsto X$ is asymptotically (compact) almost automorphic. It is well known that the spaces of almost periodic, almost automorphic, compactly almost automorphic functions, and asymptotically (compact) almost automorphic functions are closed subspaces of $C_b(\mathbb{R}:X)$ when equipped with the sup-norm.

3.1. Stepanov and Weyl generalizations

Assume $1\leqslant p<\infty$, $l>0$ and $f,\ g\in L^p_{loc}(I:X)$, where $I=\mathbb{R}$ or $I=[0,\infty)$. Define the Stepanov 'metric' by

$$D^p_{S_l}\big[f(\cdot),g(\cdot)\big]:=\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\big\|f(t)-g(t)\big\|^p\,dt\right]^{1/p}.$$

Then there exists (see e.g. [8, pp. 72–73] for scalar-valued case)

$$D_W^p\left[f(\cdot),g(\cdot)\right] := \lim_{l\to\infty} D_{S_l}^p\left[f(\cdot),g(\cdot)\right] \tag{6}$$

in $[0,\infty]$. The distance appearing in (6) is called the Weyl distance of $f(\cdot)$ and $g(\cdot)$. The Stepanov and Weyl 'norm' of $f(\cdot)$ are defined by

$$\|f\|_{S_l^p} := D_{S_l}^p\left[f(\cdot),0\right] \text{ and } \|f\|_{W^p} := D_W^p\left[f(\cdot),0\right],$$

respectively.

A function $f\in L_{loc}^p(I:X)$ is said to be Stepanov p-bounded, S^p-bounded shortly, iff

$$\|f\|_{S^p} := \sup_{t\in I}\left(\int_t^{t+1}\|f(s)\|^p\,ds\right)^{1/p} < \infty.$$

The above norm turns the space $L_S^p(I:X)$ consisting of all S^p-bounded functions into a Banach space. We say that a function $f\in L_S^p(I:X)$ is Stepanov p-almost periodic, S^p-almost periodic or S^p-a.p. shortly, iff the function $\hat{f}:I\to L^p([0,1]:X)$, defined by $\hat{f}(t)(s):=f(t+s)$, $t\in I$, $s\in[0,1]$ is almost periodic. It is said that $f\in L_S^p([0,\infty):X)$ is asymptotically Stepanov p-almost periodic, asymptotically S^p-a.p. shortly, iff $\hat{f}:[0,\infty)\to L^p([0,1]:X)$ is asymptotically almost periodic.

In [14], V. Casarino has introduced the notions of a Stepanov almost automorphic function and a Stepanov asymptotically almost automorphic function. In this paper, we will use the following notions (see e.g. [26]): A function $f\in L_{loc}^p(\mathbb{R}:X)$ is called Stepanov p-almost automorphic, S^p-almost automorphic or S^p-a.a. shortly, iff for every real sequence (a_n), there exists a subsequence (a_{n_k}) and a function $g\in L_{loc}^p(\mathbb{R}:X)$ such that

$$\lim_{k\to\infty}\int_t^{t+1}\left\|f\left(a_{n_k}+s\right)-g(s)\right\|^p ds=0 \tag{7}$$

and

$$\lim_{k\to\infty}\int_t^{t+1}\left\|g\left(s-a_{n_k}\right)-f(s)\right\|^p ds=0 \tag{8}$$

for each $t\in\mathbb{R}$; a function $f\in L_{loc}^p([0,\infty):X)$ is called asymptotically Stepanov p-almost automorphic, asymptotically S^p-almost automorphic or asymptotically S^p-a.a. shortly, iff there exists an S^p-almost automorphic function $g(\cdot)$ and a function $q\in L_S^p([0,\infty):X)$ such that $f(t)=g(t)+q(t)$, $t\geqslant 0$ and $\hat{q}\in C_0([0,\infty):L^p([0,1]:X))$. It can be easily verified that the S^p-almost automorphy of $f(\cdot)$ implies the compact almost automorphy of the mapping $\hat{f}:I\to L^p([0,1]:X)$ defined above, with the limit function being $g(\cdot)(s):=g(s+\cdot)$ for a.e. $s\in[0,1]$, so that any S^p-almost automorphic function $f(\cdot)$ has to be S^p-bounded ($1\leqslant p<\infty$). The vector space consisting of all S^p-almost automorphic functions is closed under translations and reflexions at zero of argument, and any limit of S^p-almost automorphic functions (f_k) converging to some p-locally integrable X-valued function $f(\cdot)$ in S^p-norm has the property that $f(\cdot)$ is S^p-almost automorphic. The vector space consisting of all S^p-almost automorphic functions, resp., asymptotically S^p-almost automorphic functions, will be denoted by $AAS^p(\mathbb{R}:X)$, resp., $AAAS^p([0,\infty):X)$.

If $1 \leqslant p < q < \infty$ and $f(\cdot)$ is Stepanov q-almost automorphic, resp., Stepanov q-almost periodic, then $f(\cdot)$ is Stepanov p-almost automorphic, resp., Stepanov p-almost periodic (see e.g. [20, Remark 2.15]). Furthermore, the (asymptotical) Stepanov p-almost periodicity of $f(\cdot)$ for some $p \in [1,\infty)$ implies the (asymptotical) Stepanov p-almost automorphy of $f(\cdot)$. It is a well-known fact that if $f(\cdot)$ is an almost periodic (respectively, a.a.p., a.a., a.a.a.) function then $f(\cdot)$ is also S^p-almost periodic (resp., asymptotically S^p-a.p., S^p-a.a., asymptotically S^p-a.a.) for $1 \leqslant p < \infty$. The converse statement is false, however.

The notion of an (equi-)Weyl almost periodic function is given as follows (cf. [4] for scalar-valued case):

DEFINITION 1. Let $1 \leqslant p < \infty$ and $f \in L^p_{loc}(I:X)$.

(i) We say that the function $f(\cdot)$ is equi-Weyl-p-almost periodic, $f \in e-W^p_{ap}(I:X)$ for short, iff for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|f(t+\tau)-f(t)\right\|^p dt\right]^{1/p} \leqslant \varepsilon,$$

$$\text{i.e., } D^p_{S_l}\left[f(\cdot+\tau),f(\cdot)\right] \leqslant \varepsilon.$$

(ii) We say that the function $f(\cdot)$ is Weyl-p-almost periodic, $f \in W^p_{ap}(I:X)$ for short, iff for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l\to\infty}\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|f(t+\tau)-f(t)\right\|^p dt\right]^{1/p} \leqslant \varepsilon,$$

$$\text{i.e., } \lim_{l\to\infty} D^p_{S_l}\left[f(\cdot+\tau),f(\cdot)\right] \leqslant \varepsilon.$$

Let us recall that $APS^p(I:X) \subseteq e-W^p_{ap}(I:X) \subseteq W^p_{ap}(I:X)$ in the set theoretical sense and that any of these two inclusions can be strict ([4]).

For the sequel, we need the following notion from [39].

DEFINITION 2. We say that $q \in L^p_{loc}([0,\infty):X)$ is Weyl-p-vanishing iff

$$\lim_{t\to\infty}\lim_{l\to\infty}\sup_{x\geqslant 0}\left[\frac{1}{l}\int_x^{x+l}\left\|q(t+s)\right\|^p ds\right]^{1/p} = 0. \tag{9}$$

It is clear that for any function $q \in L^p_{loc}([0,\infty):X)$ we can replace the limits in (9). It is said that $q \in L^p_{loc}([0,\infty):X)$ is equi-Weyl-p-vanishing iff

$$\lim_{l\to\infty}\lim_{t\to\infty}\sup_{x\geqslant 0}\left[\frac{1}{l}\int_x^{x+l}\left\|q(t+s)\right\|^p ds\right]^{1/p} = 0.$$

Denote by $W_0^p([0,\infty):X)$ and $e-W_0^p([0,\infty):X)$ the sets consisting of all Weyl-p-vanishing functions and equi-Weyl-p-vanishing functions, respectively.

The concepts of Weyl almost automorphy and Weyl pseudo almost automorphy, more general than those of Stepanov almost automorphy and Stepanov pseudo almost automorphy, were introduced by S. Abass [1] in 2012:

DEFINITION 3. Let $p \geqslant 1$. Then we say that a function $f \in L^p_{loc}(\mathbb{R}:X)$ is Weyl p-almost automorphic iff for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R}:X)$ such that

$$\lim_{k\to\infty}\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Big\|f\big(t+s_{n_k}+x\big)-f^*(t+x)\Big\|^p\,dx=0 \tag{10}$$

and

$$\lim_{k\to\infty}\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Big\|f^*\big(t-s_{n_k}+x\big)-f(t+x)\Big\|^p\,dx=0 \tag{11}$$

for each $t \in \mathbb{R}$. The set of all such functions are denoted by $W^pAA(\mathbb{R}:X)$.

The set $W^pAA(\mathbb{R}:X)$, equipped with the usual operations of pointwise addition of functions and multiplication of functions with scalars, has a linear vector structure. We can simply prove this fact in the following way. Let (s_n) be an arbitrary real sequence. Then there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R}:X)$ such that (10)–(11) holds. By the Weyl p-almost automorphy of $g(\cdot)$, we get the existence of subsequence $(s_{n_{k_m}})$ of (s_{n_k}) and a function $g^* \in L^p_{loc}(\mathbb{R}:X)$ such that

$$\lim_{m\to\infty}\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Big\|g\big(t+s_{n_{k_m}}+x\big)-g^*(t+x)\Big\|^p\,dx=0 \tag{12}$$

and

$$\lim_{m\to\infty}\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Big\|g^*\big(t-s_{n_{k_m}}+x\big)-g(t+x)\Big\|^p\,dx=0. \tag{13}$$

Since (12)–(13) holds with g and g^* replaced therein with f and f^*, we get that, for a linear combination $\alpha f+\beta g$, we can choose a subsequence $(s_{n_{k_m}})$ of (s_n) and a limit function $\alpha f^*+\beta g^*$ satisfying all the requirements from Definition 3 ($\alpha,\ \beta \in \mathbb{C}$). As stated by S. Abass [1, p. 5, l. 2-3], without a corresponding proof, Weyl-p-almost periodic functions forms a linear submanifold of $W^pAA(\mathbb{R}:X)$.

We continue by providing the following illustrative example.

EXAMPLE 1. Let $f(x):=\chi_{(0,1/2)}(x)$, $x\in\mathbb{R}$, where $\chi_{(0,1/2)}(\cdot)$ denotes the characteristic function of $(0,1/2)$. Then we already know that this function is equi-Weyl-1-almost periodic and not Stepanov p-almost periodic for $1\leqslant p<\infty$; see [4] and [38]. A very simple analysis shows that $f(\cdot)$ is not Stepanov p-almost automorphic for $1\leqslant p<\infty$ as well as that $f(\cdot)$ is Weyl 1-almost automorphic. We will prove here only that $f(\cdot)$ cannot be Stepanov p-almost automorphic for $1\leqslant p<\infty$. If this is the case, then there exist a subsequence of $(a_n:=n^2)$ and a function $g\in L^p_{loc}(\mathbb{R}:X)$ such that (7)–(8) holds good, pointwise for each $t\in\mathbb{R}$. But, for any $t\in\mathbb{R}$ and for any $k_0\in\mathbb{N}$

sufficiently large we have that $\int_t^{t+1} \|f(a_{n_k}+s)-g(s)\|^p\,ds = \int_t^{t+1} \|g(s)\|^p\,ds$. Due to (7), we get that $g(s)=0$ for a.e. $s\in\mathbb{R}$. Coming back to (8), we get that $f(s)=0$ for a.e. $s\in\mathbb{R}$, which is a contradiction.

Furthermore, we have the following:

EXAMPLE 2. Define

$$R := \{ f\in L^\infty(\mathbb{R}:X) : \operatorname{supp}(f) \text{ is compact} \}.$$

Then the computation used in [38, Example 2.11.8] shows that $R\subseteq e-W^1_{ap}(\mathbb{R}:X)$. Any non-trivial function $f(\cdot)$ from R cannot be Stepanov p-almost automorphic for $1\leqslant p<\infty$ and we can prove this fact as follows. If we suppose the contrary, then supremum formula for almost automorphic functions ($g\in AA(\mathbb{R}:X)$ and $t_0\in\mathbb{R}$ imply $\|g\|_\infty = \sup_{t\geqslant t_0}\|g(t)\|$; see [38] for more details) yields that

$$\sup_{t\in\mathbb{R}}\left[\int_t^{t+1}\|f(s)\|\,ds\right]^{1/p} = \sup_{t\geqslant t_0}\left[\int_t^{t+1}\|f(s)\|\,ds\right]^{1/p},\quad t_0\in\mathbb{R},\ p\geqslant 1;$$

by choosing t_0 arbitrarily large, the above would imply $\sup_{t\in\mathbb{R}}[\int_t^{t+1}\|f(s)\|\,ds]^{1/p}=0$ for all $t\in\mathbb{R}$, $p\geqslant 1$ and therefore $f(s)=0$ a.e. $s\in\mathbb{R}$.

The class of Besicovitch almost automorphic functions has been analyzed by F. Bedouhene, N. Challali, O. Mellah, P. Raynaud de Fitte and M. Smaali in [7]. This class extends the class of Weyl almost automorphic functions and its full importance lies in the fact that we do allow now the possible non-existence of limit

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f\left(t+s_{n_k}+x\right)-f^*(t+x)\right\|^p dx,$$

resp.,

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f^*\left(t-s_{n_k}+x\right)-f(t+x)\right\|^p dx$$

in (10), resp., (11). As it is well-known, the limit superiors of these functions always exist and this will be very important for the proofs of Proposition 1 and Proposition 7 below to work:

DEFINITION 4. Let $p\geqslant 1$. Then we say that a function $f\in L^p_{loc}(\mathbb{R}:X)$ is Besicovitch p-almost automorphic iff for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^*\in L^p_{loc}(\mathbb{R}:X)$ such that

$$\lim_{k\to\infty}\limsup_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f\left(t+s_{n_k}+x\right)-f^*(t+x)\right\|^p dx=0 \tag{14}$$

and

$$\lim_{k\to\infty}\limsup_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f^*\left(t-s_{n_k}+x\right)-f(t+x)\right\|^p dx=0 \tag{15}$$

for each $t\in\mathbb{R}$. The set of all such functions are denoted by $B^pAA(\mathbb{R}:X)$.

As in the case of Weyl almost automorphic functions, we can prove that the set $B^pAA(\mathbb{R}:X),$ equipped with the usual operations, has a linear vector structure. In the present situation, the author does not know whether a Besicovitch p-almost periodic function is necessarily Besicovitch p-almost automorphic.

For the sequel, let us recall that, if $f \in AA(\mathbb{R}:X)$ and $g \in L^1(\mathbb{R}),$ then the infinite convolution product $t \mapsto (g*f)(t) := \int_{-\infty}^{\infty} g(t-s)f(s)\,ds,\ t \in \mathbb{R}$ is almost automorphic, as well ([17]). As [26, Theorem 3.1] shows, a similar statement holds for the spaces of compactly almost automorphic functions and Stepanov p-almost automorphic functions ($1 \leqslant p < \infty$). Now we will prove the following simple assertion concerning the invariance of infinite convolution product for the class of Besicovitch 1-almost automorphic functions:

PROPOSITION 1. *Let* $f \in B^1AA(\mathbb{R}:X)$ *and let* $g \in L^1(\mathbb{R})$ *be a scalar-valued function with compact support. Then the function* $F(\cdot) := (g*f)(\cdot)$ *belongs to the class* $B^1AA(\mathbb{R}:X),$ *as well.*

Proof. Let $-\infty < a < b < \infty$, and let $\operatorname{supp}(g) \subseteq [a,b]$. Let $[-l-r,l-r] \subseteq [-2l,2l]$ for all $r \in [a,b]$ and $l \geqslant l_0$. Our assumptions on $g(\cdot)$ imply that $(g*h)(\cdot)$ is a well-defined X-valued locally integrable function for any function $h \in L^1_{loc}(\mathbb{R}:X)$. Let (s_n) be a given sequence. Then we can extract a subsequence (s_{n_k}) of (s_n) and a function $f^* \in L^1_{loc}(\mathbb{R}:X)$ such that (14)–(15) hold with $p=1$. Set $F^*(\cdot) := (g*f^*)(\cdot)$. Then (14) for $F(\cdot)$ and $F^*(\cdot)$ follows from its validity for $f(\cdot)$ and $f^*(\cdot)$, and the following simple integral calculation with Fubini theorem ($l \geqslant l_0,\ k \in \mathbb{N},\ t \in \mathbb{R}$):

$$
\begin{aligned}
&\frac{1}{2l}\int_{-l-t}^{l+t}\Big\|F\big(s_{n_k}+x\big)-F^*(x)\Big\|\,dx \\
&\quad \leqslant \frac{1}{2l}\int_{-l-t}^{l+t}\int_{-\infty}^{\infty}\Big\|f\big(s_{n_k}+x-r\big)-f^*(x-r)\Big\|\|g(r)\|\,dr\,dx \\
&\quad = \int_{-\infty}^{\infty}\|g(r)\|\left[\frac{1}{2l}\int_{-l+t}^{l+t}\Big\|f\big(s_{n_k}+x-r\big)-f^*(x-r)\Big\|\,dx\right]dr \\
&\quad = \int_{-\infty}^{\infty}\|g(r)\|\left[\frac{1}{2l}\int_{-l+t-r}^{l+t-r}\Big\|f\big(s_{n_k}+x\big)-f^*(x)\Big\|\,dx\right]dr \\
&\quad = \int_{a}^{b}\|g(r)\|\left[\frac{1}{2l}\int_{-l-r}^{l-r}\Big\|f\big(s_{n_k}+x+t\big)-f^*(x+t)\Big\|\,dx\right]dr \\
&\quad \leqslant \int_{a}^{b}\|g(r)\|\left[\frac{1}{2l}\int_{-2l}^{2l}\Big\|f\big(s_{n_k}+x\big)-f^*(x)\Big\|\,dx\right]dr.
\end{aligned}
$$

The proof of (15) for $F(\cdot)$ and $F^*(\cdot)$ is similar and therefore omitted, finishing the proof of proposition. $\square$

In the present situation, we do not know whether the assumption that $g(\cdot)$ has a compact support can be relaxed and whether we can consider the case $p > 1$ here.

3.2. Generalized two-parameter almost automorphic functions

For our later purposes, we need to introduce another pivot space over the field of complex numbers, say $(Y, \|\cdot\|_Y)$. By $C_0([0,\infty) \times Y : X)$ we designate the space of all continuous functions $h : [0,\infty) \times Y \to X$ such that $\lim_{t\to\infty} h(t,y) = 0$ uniformly for y in any compact subset of Y. A jointly continuous function $F : \mathbb{R} \times Y \to X$ is said to be almost automorphic iff for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that

$$G(t,y) := \lim_{n\to\infty} F\big(t + s_n, y\big)$$

is well defined for each $t \in \mathbb{R}$ and $y \in Y$, and

$$\lim_{n\to\infty} G\big(t - s_n, y\big) = F(t,y)$$

for each $t \in \mathbb{R}$ and $y \in Y$. The vector space consisting of such functions will be denoted by $AA(\mathbb{R} \times Y : X)$.

The notion of a pseudo almost-automorphic function was introduced by T.-J. Xiao, J. Liang and J. Zhang in [57] (2008). Let us recall that the space of pseudo-almost automorphic functions, denoted shortly by $PAA(\mathbb{R} : X)$, is defined as the direct sum of spaces $AA(\mathbb{R} : X)$ and $PAP_0(\mathbb{R} : X)$, where $PAP_0(\mathbb{R} : X)$ denotes the space consisting of all bounded continuous functions $\Phi : \mathbb{R} \to X$ such that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\Phi(s)\|\, ds = 0.$$

Equipped with the sup-norm, the space $PAA(\mathbb{R} : X)$ becomes one of Banach's. A bounded continuous function $f : \mathbb{R} \times Y \to X$ is said to be pseudo-almost automorphic iff $F = G + \Phi$, where $G \in AA(\mathbb{R} \times Y : X)$ and $\Phi \in PAP_0(\mathbb{R} \times Y : X)$; here, $PAP_0(\mathbb{R} \times Y : X)$ denotes the space consisting of all continuous functions $\Phi : \mathbb{R} \times Y \to X$ such that $\{\Phi(t,y) : t \in \mathbb{R}\}$ is bounded for all $y \in Y$, and

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\Phi(s,y)\|\, ds = 0,$$

uniformly in $y \in Y$. The collection of such functions will be denoted henceforth by $PAA(\mathbb{R} \times Y : X)$.

The notion of a Stepanov two-parameter p-almost automorphic function has been already introduced in the existing literature. Definition goes as follows:

DEFINITION 5. Let $1 \leqslant p < \infty$, and let $f : \mathbb{R} \times Y \to X$ satisfy that for each $y \in Y$ we have $f(\cdot,y) \in L^p_{loc}(\mathbb{R} : X)$. Then it is said that $f(\cdot,\cdot)$ is Stepanov p-almost automorphic iff for every $y \in Y$ the mapping $f(\cdot,y)$ is S^p-almost automorphic; that is, for any real sequence (a_n) there exist a subsequence (a_{n_k}) of (a_n) and a map $g : \mathbb{R} \times Y \to X$ such that $g(\cdot,y) \in L^p_{loc}(\mathbb{R} : X)$ for all $y \in Y$ as well as that:

$$\lim_{k\to\infty} \int_0^1 \Big\| f\big(t + a_{n_k} + s, y\big) - g(t+s,y)\Big\|^p\, ds = 0$$

and

$$\lim_{k\to\infty} \int_0^1 \Big\| g\big(t + s - a_{n_k}, y\big) - f(t+s,y)\Big\|^p\, ds = 0$$

for each $t \in \mathbb{R}$ and for each $y \in Y$. We denote by $AAS^p(\mathbb{R} \times Y : X)$ the vector space consisting of all such functions.

We start our work by observing that the well-known results of Fan et al. [25] and Ding et al. [22] (see e.g. [17, pp. 134–138]) continue to hold in the case that the pivot spaces X and Y are mutually different:

THEOREM 1. *Assume that* $1 \leqslant p < \infty$, *and* $f \in AAS^p(\mathbb{R} \times Y : X)$. *If there exists a constant* $L > 0$ *such that for all* $x,\ y \in L^p_{loc}(\mathbb{R} : Y)$

$$\int_0^1 \Big\| f(t+s,x(s)) - f(t+s,y(s)) \Big\|^p ds \leqslant L \int_0^1 \|x(s) - y(s)\|_Y^p \, ds, \tag{16}$$

then for each $x \in AAS^p(\mathbb{R} : Y)$ *with relatively compact range in* Y *one has that* $f(\cdot, x(\cdot)) \in AAS^p(\mathbb{R} : X)$.

THEOREM 2. *Suppose that the following conditions hold:*

(i) $f \in AAS^p(\mathbb{R} \times Y : X)$ *with* $p > 1$, *and there exist a number* $r \geqslant \max(p, p/p-1)$ *and a function* $L_f \in L^r_S(\mathbb{R})$ *such that*

$$\|f(t,x) - f(t,y)\| \leqslant L_f(t)\|x-y\|_Y, \quad t \in \mathbb{R},\ x,\ y \in Y; \tag{17}$$

(ii) $x \in AAS^p(\mathbb{R} : Y)$, *and there exists a set* $E \subseteq \mathbb{R}$ *with* $m(E) = 0$ *such that* $K := \{x(t) : t \in \mathbb{R} \setminus E\}$ *is relatively compact in* Y.

Then $q := pr/p + r \in [1, p)$ *and* $f(\cdot, x(\cdot)) \in AAS^q(\mathbb{R} : X)$.

The following composition principle is basically due to Liang et al [43]. Its validity for class $PAA(\mathbb{R} \times Y : X)$, where $Y \neq X$, can be proved similarly.

THEOREM 3. *Suppose that* $f = g + \phi \in PAA(\mathbb{R} \times Y : X)$ *with* $g \in AA(\mathbb{R} \times Y : X)$, $\phi \in PAP_0(\mathbb{R} \times Y : X)$ *and the following holds:*

(i) *the mapping* $(t,y) \mapsto g(t,y)$ *is uniformly continuous in any bounded subset* $K \subseteq Y$ *uniformly for* $t \in \mathbb{R}$;

(ii) *the mapping* $(t,y) \mapsto \phi(t,y)$ *is uniformly continuous in any bounded subset* $K \subseteq Y$ *uniformly for* $t \in \mathbb{R}$;

Then for each $y \in PAA(\mathbb{R} : Y)$ *one has* $f(\cdot, y(\cdot)) \in PAA(\mathbb{R} : X)$.

We continue by stating two composition principle for asymptotically Stepanov almost automorphic functions. Keeping in mind Theorem 2 and the proof of [38, Proposition 2.7.3], where we have examined almost periodic case, we can immediately state the following result:

PROPOSITION 2. *Let* $I = [0, \infty)$. *Suppose that the following conditions hold:*

(i) $g \in AAS^p(\mathbb{R} \times Y : X)$ *with* $p > 1$, *and there exist a number* $r \geqslant \max(p, p/p-1)$ *and a function* $L_g \in L_S^r(I)$ *such that* (17) *holds with the function* $f(\cdot,\cdot)$ *replaced by the function* $g(\cdot,\cdot)$ *therein.*

(ii) $y \in AAS^p(\mathbb{R} : Y)$, *and there exists a set* $E \subseteq \mathbb{R}$ *with* $m(E) = 0$ *such that* $K = \{y(t) : t \in \mathbb{R} \setminus E\}$ *is relatively compact in* Y.

(iii) $f(t,y) = g(t,y) + q(t,y)$ *for all* $t \geqslant 0$ *and* $y \in Y$, *where* $\hat{q} \in C_0([0,\infty) \times Y : L^q([0,1] : X))$ *and* $q := pr/p + r$.

(iv) $x(t) = y(t) + z(t)$ *for all* $t \geqslant 0$, *where* $\hat{z} \in C_0([0,\infty) : L^p([0,1] : Y))$.

(v) *There exists a set* $E' \subseteq I$ *with* $m(E') = 0$ *such that* $K' = \{x(t) : t \in I \setminus E'\}$ *is relatively compact in* Y.

Then $q \in [1, p)$ *and* $f(\cdot, x(\cdot)) \in AAAS^q(I : X)$.

Appealing to Theorem 1 in place of Theorem 2, we can similarly prove the following result:

PROPOSITION 3. *Let* $I = [0,\infty)$. *Suppose that the following conditions hold:*

(i) $g \in AAS^p(\mathbb{R} \times Y : X)$ *with* $p \geqslant 1$, *and there exist a constant* $L > 0$ *such that for all* $x,\ y \in L^p_{loc}(\mathbb{R} : Y)$ *we have that* (16) *holds.*

(ii) $y \in AAS^p(\mathbb{R} : Y)$, *and there exists a set* $E \subseteq \mathbb{R}$ *with* $m(E) = 0$ *such that* $K = \{y(t) : t \in \mathbb{R} \setminus E\}$ *is relatively compact in* Y.

(iii) $f(t,y) = g(t,y) + q(t,y)$ *for all* $t \geqslant 0$ *and* $y \in Y$, *where* $\hat{q} \in C_0([0,\infty) \times Y : L^q([0,1] : X))$ *and* $q := pr/p + r$.

(iv) $x(t) = y(t) + z(t)$ *for all* $t \geqslant 0$, *where* $\hat{z} \in C_0([0,\infty) : L^p([0,1] : Y))$.

(v) *There exists a set* $E' \subseteq I$ *with* $m(E') = 0$ *such that* $K' = \{x(t) : t \in I \setminus E'\}$ *is relatively compact in* Y.

Then $f(\cdot, x(\cdot)) \in AAAS^p(I : X)$.

Various classes of weighted pseudo-almost automorphic solutions, as well as $C^{(n)}$-Stepanov and $C^{(n)}$-Weyl almost automorphic solutions of abstract Volterra integro-differential inclusions will be considered somewhere else (cf. [21] and references cited therein for further information in this direction). Our main results, providing a stable base for applications and further study of semilinear Cauchy inclusions, will be clarified in the following section.

4. Generalized (asymptotically) almost autmorphic properties of convolution products

In this section, we investigate the generalized (asymptotically) almost automorphic properties of various types of convolution products. We start by observing that the assertion of [16, Lemma 3.1] can be formulated for strongly continuous operator families which do have integrable singularities at zero (see also [11, Theorem 2.1]):

PROPOSITION 4. *Suppose that $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying that $\int_0^\infty \|R(t)\|\,dt < \infty$. If $f : \mathbb{R} \to X$ is almost automorphic, then the function $F(\cdot)$, given by*

$$F(t) := \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \geqslant 0, \tag{18}$$

is well-defined and almost automorphic.

REMARK 1. In [24, Lemma 2.2], H.-S. Ding, J. Liang and T.-J. Xiao have proved, under the assumption on strong continuity of $(R(t))_{t\geqslant 0}$ and the existence of nonincreasing continuous function $\phi \in L^1([0,\infty))$ satisfying $\|R(t)\| \leqslant \phi(t)$, $t \geqslant 0$ that the function $F(\cdot)$ is almost automorphic provided only the Stepanov 1-almost automorphy of $f : \mathbb{R} \to X$. Here we would like to observe that their result holds provided that the strong continuity of $(R(t))_{t\geqslant 0}$ is replaced by the strong continuity of $(R(t))_{t>0}$ and the boundedness of $\sup_{t\in(0,1]} \|R(t)\|$. Possible applications can be made, e.g., in the qualitative analysis of the Poisson heat equation in the space $H^{-1}(\Omega)$, where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is an open bounded domain with smooth boundary (see [27, Theorem 3.1, Proposition 3.2, p. 48; Remark, p. 52; Example 3.3, pp. 74–75] with $\beta = 1$, and Example 3 for further information in this direction).

Our first original contribution in this section reads as follows (see [37, Proposition 2.11] for almost periodic case).

PROPOSITION 5. *Suppose that $1 \leqslant p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$. If $f : \mathbb{R} \to X$ is S^p-almost automorphic, then the function $F(\cdot)$, given by (18), is well-defined and almost automorphic.*

Proof. It is clear that, for every $t \geqslant 0$, we have $F(t) = \int_0^\infty R(s)f(t-s)\,ds$. The measurability of integrand is a consequence of the proof of [5, Proposition 1.3.4], while the absolute convergence of integral follows from the Hölder inequality and S^p-boundedness of function $f(\cdot)$:

$$\begin{aligned}\int_0^\infty \|R(s)\|\|f(t-s)\|\,ds &= \sum_{k=0}^{\infty} \int_k^{k+1} \|R(s)\|\|f(t-s)\|\,ds \\ &\leqslant \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \|f\|_{S^p} = M\|f\|_{S^p}, \quad t \geqslant 0.\end{aligned}$$

Define $F_k(t) := \int_k^{k+1} R(s)f(t-s)\,ds,\ t\in\mathbb{R}\ (k\in\mathbb{N}_0)$. We claim that $F_k(\cdot)$ is continuous. Let numbers $\varepsilon>0$ and $t\in\mathbb{R}$ be given in advance, and let (t_n) be a real sequence converging to t. Then the Hölder inequality yields that:

$$\begin{aligned}\big\|F_k(t_n)-F_k(t)\big\| &\leqslant \int_k^{k+1}\|R(\sigma)\|\|f(t_n-\sigma)-f(t-\sigma)\|\,d\sigma\\ &\leqslant \|R(\cdot)\|_{L^q[k,k+1]}\left(\int_k^{k+1}\big\|f(t_n-\sigma)-f(t-\sigma)\big\|^p\,d\sigma\right)^{1/p}\\ &= \|R(\cdot)\|_{L^q[k,k+1]}\left(\int_{t-k-1}^{t-k}\big\|f(t_n-t+\sigma)-f(\sigma)\big\|^p\,d\sigma\right)^{1/p},\quad k\in\mathbb{N}_0. \qquad (19)\end{aligned}$$

Since $f\in L^p_{loc}(\mathbb{R}:X)$, the last term in brackets tends to zero as $n\to\infty$; see [38] for a direct proof of this fact. Since we have assumed that $\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^q[k,k+1]}<\infty$, the Weierstrass criterion implies that $\sum_{k=0}^{\infty}F_k(t)=F(t)$ uniformly in $t\in\mathbb{R}$, so that $F(\cdot)$ is continuous on $\mathbb{R}$, as well. Since $AA(\mathbb{R}:X)$ is closed in $C_b(\mathbb{R}:X)$, it suffices to show that $F_k\in AA(\mathbb{R}:X)$ for all $k\in\mathbb{N}_0$. Fix an integer $k\in\mathbb{N}_0$. Then, for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g:\mathbb{R}\to X$ such that (7) and (8) hold pointwise for $t\in\mathbb{R}$. Define $g_{k,c}:\mathbb{R}\to X$ by $g_{k,c}(t):=\int_k^{k+1}R(\sigma)g(t-\sigma)\,d\sigma$, $t\in\mathbb{R}$. Due to the Hölder inequality, we have

$$\begin{aligned}\big\|F_k(t+t_n)-g_{k,c}(t)\big\| &\leqslant \int_k^{k+1}\Big\|R(\sigma)\big[f(t+t_n-\sigma)-g(t-\sigma)\big]\Big\|\,d\sigma\\ &\leqslant \|R(\cdot)\|_{L^q[k,k+1]}\left(\int_k^{k+1}\big\|f(t+t_n-\sigma)-g(t-\sigma)\big\|^p\,d\sigma\right)^{1/p}\\ &= \|R(\cdot)\|_{L^q[k,k+1]}\left(\int_{t-k-1}^{t-k}\big\|f(\sigma+t_n)-g(\sigma)\big\|^p\,d\sigma\right)^{1/p},\quad t\in\mathbb{R}.\end{aligned}$$

This in combination with (7) implies $\lim_{n\to\infty}\|F_k(t+t_n)-g_{k,c}(t)\|=0$ pointwise in $t\in\mathbb{R}$. We can similarly prove that $\lim_{n\to\infty}\|g_{k,c}(t-t_n)-F_k(t)\|=0$ pointwise in $t\in\mathbb{R}$, finishing the proof of theorem. □

Keeping in mind Proposition 5 and the proof of [37, Proposition 2.13], we can immediately state the following assertion:

PROPOSITION 6. *Suppose that* $1\leqslant p<\infty,\ 1/p+1/q=1$ *and* $(R(t))_{t>0}\subseteq L(X)$ *is a strongly continuous operator family satisfying that, for every* $s\geqslant 0$, *we have that* $m_s:=\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^q[s+k,s+k+1]}<\infty$. *Suppose, further, that* $g:\mathbb{R}\to X$ *is* S^p*-almost automorphic, as well as that the locally* p*-integrable function* $q:[0,\infty)\to X$ *satisfy* $\hat{q}\in C([0,\infty):L^p([0,1]:X))$ *and* $f(t)=g(t)+q(t),\ t\geqslant 0$. *Let there exist a finite number* $M>0$ *such that the following holds:*

(i) $\lim_{t\to+\infty}\int_t^{t+1}\big[\int_M^s\|R(r)\|\|q(s-r)\|\,dr\big]^p\,ds=0.$

(ii) $\lim_{t\to+\infty}\int_t^{t+1}m_s^p\,ds=0.$

Then the function $H(\cdot)$, *given by*

$$H(t) := \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0, \tag{20}$$

is well-defined, bounded and asymptotically S^p*-almost automorphic.*

Conditions $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$ and $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[s+k,s+k+1]} < \infty$ $(s \geqslant 0)$ have been examined in [37, Remark 2.12, Remark 2.14(ii)]. Briefly speaking, these conditions always hold in the case that $R(\cdot)$ is exponentially decaying or that $p = 1$ and $R(\cdot)$ is polynomially decaying at infinity, having the integrable singularity there.

Concerning the class of Besicovitch p-almost automorphic functions, the following result seems to be satisfactory only for the abstract differential equations with integer order derivatives and nonautonomous differential equations (for fractional resolvent families, the condition (21) stated below does not hold in practical situations; as already mentioned, the case $p > 1$ is much more difficult to deal with):

PROPOSITION 7. *Suppose that* $(R(t))_{t>0} \subseteq L(X)$ *is a strongly continuous operator family satisfying that*

$$\int_0^{\infty} (1+t)\|R(t)\|\,dt < \infty. \tag{21}$$

Let $f \in B^1AA(\mathbb{R} : X)$, *and let* $f(\cdot)$ *be essentially bounded. Then the function* $F(\cdot)$, *given by* (18)*, is bounded and belongs to the class* $B^1AA(\mathbb{R} : X)$.

Proof. The fact that the function $F(\cdot)$ is bounded and well-defined follows from the proofs of Proposition 5 and [39, Proposition 5.1]. It remains to be proved that $F \in B^1AA(\mathbb{R} : X)$. Towards this end, let (s_n) be an arbitrary real sequence. By definition and elementary changes of variables, we know that there exist a subsequence (s_{n_k}) and a function $f^* \in L^1_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l+t}^{l+t} \left\| f\left(s_{n_k}+x\right) - f^*(x) \right\| dx = 0 \tag{22}$$

and

$$\lim_{k\to\infty} \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l+t}^{l+t} \left\| f^*\left(x-s_{n_k}\right) - f(x) \right\| dx = 0$$

for each $t \in \mathbb{R}$. Set $F^*(x) := \int_{-\infty}^{x} R(x-s)f^*(s)\,ds$, $x \in \mathbb{R}$. Then $F^* \in L^1_{loc}(\mathbb{R} : X)$. To see this, it suffices to observe that, for $-\infty < a < b < \infty$, we have

$$\begin{aligned}
&\int_a^b \left\| \int_{-\infty}^{x} R(x-s)f^*(s)\,ds \right\| dx \\
&\quad \leqslant \int_a^b \int_{-\infty}^{x} \|R(x-s)\| \|f^*(s)\|\,ds\,dx \\
&\quad = \int_a^b \int_0^{\infty} \|R(s)\| \|f^*(x-s)\|\,ds\,dx
\end{aligned}$$

$$= \int_0^{\infty}\int_a^b \big\|R(s)\big\|\big\|f^*(x-s)\big\|\,dx\,ds$$
$$= \int_0^{\infty}(1+s)\big\|R(s)\big\|\left[\frac{1}{s+1}\int_{a-s}^{b-s}\big\|f^*(r)\big\|\,dr\right]ds$$

as well as that the continuous mapping $s\mapsto (s+1)^{-1}\int_{a-s}^{b-s}\big\|f^*(r)\big\|\,dr,\ s\geqslant 0$ is bounded since the condition (22) with $t=0$ and the essential boundedness of function $f(\cdot)$ shows that there exists a number s_0 such that $[a-s,b-s]\subseteq[-2s,2s],\ s\geqslant s_0$ and $s^{-1}\int_{-2s}^{2s}\|f^*(r)\|\,dr\leqslant 4\|f\|_\infty+4,\ s\geqslant s_0$; here we also use (21). Therefore, we need to prove that

$$\lim_{k\to\infty}\limsup_{l\to+\infty}\frac{1}{2l}\int_{-l+t}^{l+t}\Big\|F\big(s_{n_k}+x\big)-F^*(x)\Big\|\,dx=0$$

and

$$\lim_{k\to\infty}\limsup_{l\to+\infty}\frac{1}{2l}\int_{-l+t}^{l+t}\Big\|F^*\big(x-s_{n_k}\big)-F(x)\Big\|\,dx=0 \tag{23}$$

pointwise for $t\in\mathbb{R}$. The first of these equalities follows from the next computation involving the Fubini heorem:

$$\frac{1}{2l}\int_{-l+t}^{l+t}\Big\|F\big(s_{n_k}+x\big)-F^*(x)\Big\|\,dx$$
$$=\frac{1}{2l}\int_{-l+t}^{l+t}\left\|\int_{-\infty}^{x+s_{n_k}}R\big(x-s+s_{n_k}\big)f(s)\,ds-\int_{-\infty}^{x}R(x-s)f^*(s)\,ds\right\|dx$$
$$=\frac{1}{2l}\int_{-l+t}^{l+t}\left\|\int_{-\infty}^{x}R\big(x-s\big)f\big(s+s_{n_k}\big)\,ds-\int_{-\infty}^{x}R(x-s)f^*(s)\,ds\right\|dx$$
$$=\frac{1}{2l}\int_{-l+t}^{l+t}\left\|\int_0^{\infty}R(r)\big[f\big(x-r+s_{n_k}\big)-f^*(x-r)\big]\,dr\right\|dx$$
$$\leqslant\frac{1}{2l}\int_{-l+t}^{l+t}\int_0^{\infty}\|R(r)\|\big\|f\big(x-r+s_{n_k}\big)-f^*(x-r)\big\|\,dr\,dx$$
$$=\int_0^{\infty}\|R(r)\|\frac{1}{2l}\int_{-l+t-r}^{l+t-r}\big\|f\big(x+s_{n_k}\big)-f^*(x)\big\|\,dx\,dr$$
$$=\int_0^{\infty}(1+r)\|R(r)\|$$
$$\times\left[\frac{1}{1+r}\frac{2(l+r)}{2l}\frac{1}{2(l+r)}\int_{(-l+r)+t}^{l+t+r}\big\|f\big(x+s_{n_k}\big)-f^*(x)\big\|\,dx\right]dr.$$

For any $\varepsilon>0$ given in advance, we can find $k_0(\varepsilon)>0$ such that for every $k\geqslant k_0(\varepsilon)$ we can find $y_0(\varepsilon,k)>0$ such that, for every $r\geqslant 0$ and $l\geqslant y_0(\varepsilon,k)$, we have

$$\frac{1}{2(l+r)}\int_{(-l+r)+t}^{l+t+r}\big\|f\big(x+s_{n_k}\big)-f^*(x)\big\|<\varepsilon.$$

Since (21) is assumed, this proves the claimed. The proof of (23) is similar and therefore omitted. $\square$

REMARK 2. Let the requirements of the previous proposition hold, and let the function $q \in L^1_{loc}([0,\infty) : X)$ be Weyl-1-vanishing, resp., equi-Weyl-1-vanishing. Set formally

$$J(t,l) := \sup_{x \geqslant 0} \left\{ \int_0^{x+t} \left[\frac{1}{l} \int_{x+t-r}^{x+t-r+l} \|R(v)\|\,dv \right] \|q(r)\|\,dr \right\}, \quad t > 0,\ l > 0.$$

Assume that the condition

$$\lim_{t\to\infty} \lim_{l\to\infty} J(t,l) = 0,$$

holds provided that $q(\cdot)$ is Weyl-1-vanishing, resp., that the condition

$$\lim_{l\to\infty} \lim_{t\to\infty} J(t,l) = 0$$

holds provided that $q(\cdot)$ is equi-Weyl-1-vanishing (see [39, Example 5.4–Example 5.6] for some concrete situations ensuring the validity of above conditions). By the proof of [39, Proposition 5.1], we have that $\lim_{t\to\infty} \int_t^\infty R(s)g(t-s)\,ds = 0$ as well as that the function $t \mapsto \int_0^t R(t-s)q(s)\,ds,\ t \geqslant 0$ is Weyl-1-vanishing, resp., equi-Weyl-1-vanishing. Hence, the function $t \mapsto \int_0^t R(t-s)[f(s)+q(s)]\,ds,\ t \geqslant 0$ belongs to the class $B^1AA(\mathbb{R} : X) + W_0^1([0,\infty) : X)$, $B^1AA(\mathbb{R} : X) + e - W_0^1([0,\infty) : X)$, with the meaning clear. Here we would like to note only that the condition (21) enables one to estimate the term appearing in definition of $J(t,l)$ for $x \geqslant 0$ in the following way:

$$\begin{aligned}
&\int_0^{x+t} \left[\frac{1}{l} \int_{x+t-r}^{x+t-r+l} \|R(v)\|\,dv \right] \|q(r)\|\,dr \\
&\leqslant \int_0^{x+t} \left[\frac{1}{l(1+x+t-r)} \int_{x+t-r}^{x+t-r+l} (1+x+t-r)\|R(v)\|\,dv \right] \|q(r)\|\,dr \\
&\leqslant \left[\int_0^\infty (1+t)\|R(t)\|\,dt \right] \int_0^{x+t} \frac{1}{l(1+x+t-r)} \|q(r)\|\,dr, \quad t > 0,\ l > 0.
\end{aligned}$$

5. Semilinear Cauchy inclusions

Our aim here is to explain how the already proven statements on almost periodic and pseudo almost-periodic solutions of semilinear (fractional) Cauchy inclusions (see [38]) can be formulated for almost automorphy and pseudo almost-automorphy. We will also provide some results for the abstract Cauchy inclusion $(\mathrm{DFP})_{f,\gamma}$.

Suppose that the multivalued linear operator $\mathscr{A}$ satisfies the condition [27, (P), p. 47] introduced by A. Favini and A. Yagi:

(P) There exist finite constants c, $M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Psi_c := \left\{ \lambda \in \mathbb{C} : \Re\lambda \geqslant -c\big(|\Im\lambda|+1\big) \right\} \subseteq \rho(\mathscr{A})$$

and

$$\|R(\lambda : \mathscr{A})\| \leqslant M\big(1+|\lambda|\big)^{-\beta}, \quad \lambda \in \Psi;$$

then we can define the fractional power $(-\mathscr{A})^{\theta}$ for any $\theta > \beta - 1$ (see Section 2 for more details). Put $Y := [D((-\mathscr{A})^{\theta})]$ and $\|\cdot\|_Y := \|\cdot\|_{[D((-\mathscr{A})^{\theta})]}$; then Y is a Banach space that is continuously embedded in X. Set

$$T_{\nu}(t)x := \frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{\nu}e^{\lambda t}\big(\lambda - \mathscr{A}\big)^{-1}x\,d\lambda, \quad x \in X,\ t > 0\ (\nu > 0),$$

where Γ is the upwards oriented curve $\lambda = -c(|\eta|+1)+i\eta$ $(\eta \in \mathbb{R})$. Then there exists a finite constant $M > 0$ such that:

(A) $\|T_{\nu}(t)\| \leqslant Me^{-ct}t^{\beta-\nu-1}$, $t > 0$, $\nu > 0$.

Let $L_f(\cdot)$ be a locally bounded non-negative function, and let M denote the constant from (A), with $\nu = \theta$. Set, for every $n \in \mathbb{N}$,

$$M_n := M^n \sup_{t\in\mathbb{R}} \int_{-\infty}^{t}\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_2} e^{-c(t-x_n)}\big(t-x_n\big)^{\beta-\theta-1} \times \prod_{i=2}^{n} e^{-c(x_i-x_{i-1})}\big(x_i-x_{i-1}\big)^{\beta-\theta-1}\prod_{i=1}^{n}L_f(x_i)\,dx_1\,dx_2\cdots dx_n. \tag{24}$$

Set

$$T_{\gamma,\nu}(t)x := t^{\gamma\nu}\int_0^{\infty} s^{\nu}\Phi_{\gamma}(s)T_0\big(st^{\gamma}\big)x\,ds, \quad t > 0,\ x \in X,\ \nu > -\beta,$$

and following E. Bazhlekova [6], R.-N. Wang, D.-H. Chen, T.-J. Xiao [55],

$$S_{\gamma}(t) := T_{\gamma,0}(t) \text{ and } P_{\gamma}(t) := \gamma T_{\gamma,1}(t)/t^{\gamma}, \quad t > 0.$$

Define also

$$R_{\gamma}(t) := t^{\gamma-1}P_{\gamma}(t),\ t > 0 \text{ and}$$
$$R_{\gamma}^{\theta}(t) := \gamma t^{\gamma-1}\int_0^{\infty} s\Phi_{\gamma}(s)T_{\theta}\big(st^{\gamma}\big)x\,ds,\ t > 0,\ x \in X.$$

Suppose that (17) holds for a.e. $t > 0$, with locally bounded non-negative function $L_f(\cdot)$. Define finally, for every $n \in \mathbb{N}$,

$$B_n := \sup_{t\geqslant 0}\int_{-\infty}^{t}\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_2}\big\|R_{\gamma}^{\theta}(t-x_n)\big\| \times \prod_{i=2}^{n}\big\|R_{\gamma}^{\theta}(x_i-x_{i-1})\big\|\prod_{i=1}^{n}L_f(x_i)\,dx_1\,dx_2\cdots dx_n. \tag{25}$$

Let $(Z, \|\cdot\|_Z)$ be a complex Banach space, and let Z be continuously embedded in X. We will use the following notion of a mild solution of (3), see [34]:

DEFINITION 6. Let $f : I \times Z \to X$. By a mild solution of (3), we mean any Z-continuous function $u(\cdot)$ such that $u(t) = (\Lambda u)(t)$, $t \in \mathbb{R}$, where

$$t \mapsto (\Lambda u)(t) := \int_{-\infty}^{t} T(t-s)f(s,u(s))\,ds,\ t \in \mathbb{R}.$$

Concerning the abstract semilinear Cauchy inclusion (4), we will use the following notion [34]:

DEFINITION 7. Let $f : I \times Z \to X$. By a mild solution of (4), we mean any Z-continuous function $u(\cdot)$ such that $u(t) = (\Lambda_\gamma u)(t)$, $t \in \mathbb{R}$, where

$$t \mapsto (\Lambda_\gamma u)(t) := \int_{-\infty}^{t} (t-s)^{\gamma-1} P_\gamma(t-s) f(s,u(s))\, ds, \ t \in \mathbb{R}.$$

Let $M > 0$ denote the constant from (A), and let the sequence (M_n) be defined through (24). Keeping in mind Proposition 5 and Theorem 1-Theorem 2, it is straightforward to prove the following automorphic versions of [38, Theorem 2.10.3-Theorem 2.10.4] and [38, Theorem 2.10.9-Theorem 2.10.10] (see also [34]–[35]):

THEOREM 4. *Suppose that* (P) *holds,* $\beta > \theta > 1-\beta$ *and the following conditions hold:*

(i) $f \in AAS^p(\mathbb{R} \times Y : X)$ *with* $p > 1$, *and there exist a number* $r \geqslant \max(p, p/p-1)$ *as well as a locally bounded non-negative function* $L_f \in L_S^r(\mathbb{R})$ *such that* $r > p/p-1$ *and* (17) *holds.*

Set $q := pr/p+r$ *and* $q' := \frac{pr}{pr-p-r}$.

Assume also that:

(ii) $q'(\beta - \theta - 1) > -1$.

(iii) $M_n < 1$ *for some* $n \in \mathbb{N}$.

Then there exists an almost automorphic mild solution of inclusion (3). *The uniqueness of mild solutions holds in the case that* $\mathscr{A}$ *is single-valued.*

THEOREM 5. *Suppose that* (P) *holds,* $\beta > \theta > 1-\beta$ *and the following conditions hold:*

(i) $f \in AAS^p(\mathbb{R} \times Y : X)$ *with* $p > 1$, *and there exists a constant* $L > 0$ *such that* (17) *holds.*

(ii) $\frac{p}{p-1}(\beta - \theta - 1) > -1$.

(iii) $M_n < 1$ *for some* $n \in \mathbb{N}$.

Then there exists an almost automorphic mild solution of inclusion (3). *The uniqueness of mild solutions holds provided that, in addition to* (i)-(iii), $\mathscr{A}$ *is single-valued.*

THEOREM 6. *Suppose that* (P) *holds,* $\beta > \theta > 1-\beta$ *and the following conditions hold:*

(i) $f \in AAS^p(\mathbb{R} \times Y : X)$ *with* $p > 1$, *and there exist a number* $r \geqslant \max(p, p/p-1)$ *as well as a locally bounded non-negative function* $L_f \in L^r_S(\mathbb{R})$ *such that* $r > p/p-1$ *and* (17) *holds.*

Set $q := pr/p+r$ *and* $q' := \frac{pr}{pr-p-r}$.

Assume also that:

(ii) $q'(\gamma(\beta-\theta)-1) > -1$.

(iii) $B_n < 1$ *for some* $n \in \mathbb{N}$.

Then there exists an almost automorphic mild solution of inclusion (4). *The uniqueness of mild solutions holds provided that, in addition to* (i)-(iii), $\mathscr{A}$ *is single-valued.*

THEOREM 7. *Suppose that* (P) *holds,* $\beta > \theta > 1-\beta$ *and the following conditions hold:*

(i) $f \in AAS^p(\mathbb{R} \times Y : X)$ *with* $p > 1$, *and there exists a constant* $L > 0$ *such that* (17) *holds.*

(ii) $\frac{p}{p-1}(\gamma(\beta-\theta)-1) > -1$.

(iii) $B_n < 1$ *for some* $n \in \mathbb{N}$.

Then there exists an almost automorphic mild solution of inclusion (4). *The uniqueness of mild solutions holds provided that, in addition to* (i)-(iii), $\mathscr{A}$ *is single-valued.*

Using Theorem 3 and Proposition 5, we can simply clarify the following modification of [38, Theorem 2.12.5], as well (cf. Definition 6 with $Z = X$):

THEOREM 8. *Suppose that the following conditions hold:*

(i) $f \in PAA(\mathbb{R} \times X : X)$ *is pseudo-almost automorphic.*

(ii) The inequality (17) *holds with* $I = \mathbb{R}$, $X = Y$ *and some bounded non-negative function* $L_f(\cdot)$.

(iii) $\sum_{n=1}^{\infty} M_n < \infty$.

Then there exists a unique pseudo-almost automorphic solution of inclusion (3).

As already announced in [38], the existence and uniqueness of pseudo-almost automorphic solutions of semilinear Cauchy inclusion (4) can be analyzed similarly.

We refer the reader to [38, Definition 2.9.2] for the notion of a classical solution of the abstract Cauchy inclusion $(\mathrm{DFP})_{f,\gamma}$, and [38, Definition 2.9.9] for the notion of a mild solution of the abstract semilinear inclusion $(\mathrm{DFP})_{f,\gamma,s}$. We first state the following automorphic versions of [38, Lemma 2.9.3], formulated here as a proposition, and [38, Theorem 2.9.5]; the proofs are similar and therefore omitted (cf. [27] and [33] for the notion of interpolation space $X^{\theta}_{\mathscr{A}}$ used below). It is also worth noting that [38, Theorem 2.9.7] can be formulated for asymptotical almost automorphy.

PROPOSITION 8. *Let* $f \in AAAS^q([0,\infty):X)$ *with some* $q \in (1,\infty)$, *let* $1/q+1/q'=1$, *and let* $q'(\gamma\beta-1)>-1$. *Define*

$$H(t) := \int_0^t R_\gamma(t-s)f(s)\,ds, \quad t \geqslant 0.$$

Then $H \in AAA([0,\infty):X)$.

THEOREM 9. *Suppose that* $1 \geqslant \theta > 1-\beta$ *and* $x_0 \in D((-\mathscr{A})^\theta)$, *resp.* $1>\theta>1-\beta$ *and* $x_0 \in X^\theta_{\mathscr{A}}$, *as well as there exists a constant* $\sigma > \gamma(1-\beta)$ *such that, for every* $T>0$, *there exists a finite constant* $M_T>0$ *such that* $f:[0,\infty)\to X$ *satisfies*

$$\|f(t)-f(s)\| \leqslant M_T|t-s|^\sigma, \quad 0 \leqslant t,\ s \leqslant T.$$

Let $1 \geqslant \theta > 1-\beta$, *resp.* $1>\theta>1-\beta$, *and let*

$$f \in L^\infty_{loc}\Big((0,\infty):\big[D\big((-\mathscr{A})^\theta\big)\big]\Big), \text{ resp. } f \in L^\infty_{loc}\Big((0,\infty):X^\theta_{\mathscr{A}}\Big).$$

Then there exists a unique classical solution $u(\cdot)$ *of problem* $(\mathrm{DFP})_{f,\gamma}$. *If, additionally,* $f \in AAAS^q([0,\infty):X)$ *with some* $q \in (1,\infty)$, $1/q+1/q'=1$ *and* $q'(\gamma\beta-1)>-1$, *then* $u \in AAA([0,\infty):X)$.

Keeping in mind Proposition 2-Proposition 3 and our results clarified in the previous section, we can repeat almost literally the proofs of our structural results given in [38, Subsection 2.9.1, Subsection 2.9.2]. In such a way, we can simply state the automorphic versions of [38, Theorem 2.9.10-Theorem 2.9.11, Corollary 2.9.12-Corollary 2.9.13] and [38, Theorem 2.9.15, Theorem 2.9.17-Theorem 2.9.18, Corollary 2.9.19-Corollary 2.9.20], where we have also analyzed applications of C-regularized semigroups in the analysis of existence and uniqueness of generalized (asymptotically) automorphic solutions of abstract Cauchy inclusions $(\mathrm{DFP})_{f,\gamma}$ and $(\mathrm{DFP})_{f,\gamma,s}$. For the sake of completeness, we will reformulate the above-mentioned Theorem 2.9.10 in our new context, only:

THEOREM 10. *Suppose that* $I=[0,\infty)$ *and the following conditions hold:*

(i) $g \in AAS^p(\mathbb{R}\times X:X)$ *with* $p>1$, *and there exist a number* $r \geqslant \max(p, p/p-1)$ *and a function* $L_g \in L^r_S(I)$ *such that* (17) *holds.*

(ii) $f(t,x)=g(t,x)+q(t,x)$ *for all* $t \geqslant 0$ *and* $x \in X$, *where* $\hat{q} \in C_0(I\times X:L^q([0,1]:X))$ *and* $q=pr/p+r$.

Set

$$q' := \infty, \text{ provided } r=p/p-1 \text{ and } q' := \frac{pr}{pr-p-r}, \text{ provided } r>p/p-1.$$

Assume also that:

(iii) $q'(\gamma\beta-1)>-1$,

(iv) (17) *holds for a.e.* $t > 0$, *with* $X = Y$ *and a locally bounded positive function* $L_f(\cdot)$ *satisfying* $A_n < 1$ *for some* $n \in \mathbb{N}$; *here,*

$$A_n := \sup_{t \geqslant 0} \int_0^t \int_0^{x_n} \cdots \int_0^{x_2} \|R_\gamma(t - x_n)\| \times \prod_{i=2}^{n} \|R_\gamma(x_i - x_{i-1})\| \prod_{i=1}^{n} L_f(x_i)\, dx_1\, dx_2 \cdots dx_n,\ n \in \mathbb{N}.$$

Then there exists a unique asymptotically almost automorphic solution of inclusion $(\mathrm{DFP})_{f,\gamma,s}$.

6. Examples and applications

The main aim of this section is to provide some applications of our abstract results in the analysis of existence and uniqueness of various types of generalized (asymptotically) almost automorphic solutions for certain classes of abstract (semilinear) fractional integro-differential inclusions.

EXAMPLE 3. It is well known that the unique solution of (1)–(2), resp. $(\mathrm{DFP})_{f,\gamma}$, is of the form (18), resp. (20), with a suitable operator family $(R(t))_{t>0}$ locally integrable at zero and having polynomially decaying integrable singularity at infinity (cf. [38] for more details). Therefore, our results from Section 4 apply almost directly; concerning Proposition 7, it is worth noting once more that it is susceptible to applications only in the case that $\gamma = 1$, when $(R(t))_{t>0}$ decays exponentially at infinity. It is also clear that our results from Section 5 can be applied in the analysis of semilinear Cauchy inclusions (3)–(4) and $(\mathrm{DFP})_{f,\gamma,s}$.

Arguing so, we can analyze the existence and uniqueness of (asymptotically) almost automorphic solutions of the fractional Poisson heat equations

$$\begin{cases} D_{t,+}^{\gamma}[m(x)v(t,x)] = -(\Delta - b)v(t,x) + f(t,x), \quad t \in \mathbb{R},\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

and

$$\begin{cases} \mathbf{D}_t^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), \quad t \geqslant 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $\gamma \in (0,1)$ and $1 < p < \infty$, as well as their semilinear analogues

$$\begin{cases} D_{t,+}^{\gamma}[m(x)v(t,x)] = -(\Delta - b)v(t,x) + f(t, m(x)v(t,x)),\ t \in \mathbb{R},\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

and

$$\begin{cases} \mathbf{D}_t^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t, m(x)v(t,x)),\ t \geqslant 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

with (asymptotically) Stepanov almost automorphic coefficients (cf. [27] and [33] for more details).

In the following example, we will reexamine some important results established by A. Favini and A. Yagi in [27, Section VI], regarding certain types of abstract degenerate second order differential equations whose solutions can be sought by using the usual matrix reduction to the system of two first order differential equations. Our main aim here is to apply Proposition 6, considering only classical abstract inhomogeneous Cauchy problems, not their semilinear analogues.

EXAMPLE 4. Assume that A, B and C are closed linear operators in X, $D(B) \subseteq D(A) \cap D(C)$, $B^{-1} \in L(X)$ and the conditions [27, (6.4)–(6.5)] hold with certain numbers $c > 0$ and $0 < \beta \leqslant \alpha = 1$. In [27, Chapter VI], the following second order differential equation

$$\frac{d}{dt}\big(Cu'(t)\big) + Bu'(t) + Au(t) = f(t),\ t > 0; \quad u(0) = u_0,\ Cu'(0) = Cu_1$$

has been considered by the usual converting into the first order matricial system

$$\frac{d}{dt}Mz(t) = Lz(t) + F(t),\ t > 0; \quad Mz(0) = Mz_0,$$

where

$$M = \begin{bmatrix} I & O \\ O & C \end{bmatrix},\ L = \begin{bmatrix} O & I \\ -A & -B \end{bmatrix},\ z_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \text{ and } F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}\ (t > 0).$$

By the proof of [27, Theorem 6.1] (see also [27, Theorem 1.14]), we know that the multivalued linear operator $(L_{[D(B)]\times X} - \omega M_{[D(B)]\times X})(M_{[D(B)]\times X})^{-1}$ satisfies the condition (P) for a sufficiently large number $\omega > 0$, in the pivot space $[D(B)] \times X$. Therefore, this MLO generates a degenerate semigroup $(T(t))_{t>0}$ in $[D(B)] \times X$, having an integrable singularity at zero and exponentially decaying growth rate at infinity. This enables one to apply [27, Theorem 3.8, Theorem 3.9] in the analysis of existence and uniqueness of solutions of the problem

$$\frac{d}{dt}Mz(t) = (L - \omega M)z(t) + F(t),\ t > 0; \quad Mz(0) = Mz_0, \tag{26}$$

cf. [27, Section 3.1] for more details, as well as [36, Theorem 4.3] in the analysis of existence and uniqueness of solutions of the fractional problem

$$\mathbf{D}_t^{\gamma}[Mz(t)] = (L - \omega M)z(t) + F(t),\ t > 0; \quad Mz(0) = Mz_0,$$

where the Caputo fractional derivative $\mathbf{D}_t^{\gamma}$ is taken in a slightly weakened sense [36]. Consider first the case of the abstract Cauchy problem (26). Denoting the components of $z(t)$ by $u(t)$ and $v(t)$, from (26) we get that $v(t) = u'(t) + \omega u(t)$, $t \geqslant 0$ and $(d/dt)(Cv(t)) = -Au(t) - (B + \omega C)v(t) + f(t)$, $t \geqslant 0$, so that we are ready to solve the following second order differential equation

$$\frac{d}{dt}\big(Cu'(t)\big) + (2\omega C + B)u'(t) + \big(A + \omega B + \omega^2 C\big)u(t) = f(t),\ t > 0;$$

$$u(0) = u_0,\ C\big[u'(0) + \omega u_0\big] = Cu_1. \tag{27}$$

Roughy speaking, if $M[u_0\ u_1]^T$ belongs to the domain of continuity of $(T(t))_{t>0}$ and $f(\cdot)$ is Hölder continuous with an approporiate Hölder index, then there exists a unique solution $z(t)$ of (26), continuous for $t \geqslant 0$, and moreover,

$$Mz(t) = M\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = T(t)M\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t T(t-s)\begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds, \quad t \geqslant 0.$$

Therefore, if $f(\cdot)$ additionally satisfies the requirements of Proposition 6 (here we can apply a great number of similar assertions known for asymptotical almost periodicity or asymptotical almost automorphy), then the unique solution $u(\cdot)$ of (27) will satisfy that $Mz(\cdot) = [u(\cdot)\ \ C(u'(\cdot) + \omega u(\cdot))]^T$ is bounded and asymptotically S^p-almost automorphic. We can simply apply this result in the analysis of existence and uniqueness of asymptotically S^p-almost automorphic solutions of the following damped Poisson-wave type equation in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$:

$$\begin{cases} \frac{\partial}{\partial t}\big(m(x)\frac{\partial u}{\partial t}\big) + \big(2\omega m(x) - \Delta\big)\frac{\partial u}{\partial t} + \big(A(x;D) - \omega\Delta + \omega^2 m(x)\big)u(x,t) = f(x,t), \\ t \geqslant 0,\ x \in \Omega\ ;\ u = \partial u/\partial t = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ u(0,x) = u_0(x),\ m(x)\big[(\partial u/\partial t)(x,0) + \omega u_0\big] = m(x)u_1(x), \quad x \in \Omega. \end{cases}$$

Here, $\Omega \subseteq \mathbb{R}^n$ is a bounded open domain with smooth boundary, $1 < p < \infty$, $m(x) \in L^\infty(\Omega)$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, Δ is the Dirichlet Laplacian in $L^2(\Omega)$, acting with domain $H_0^1(\Omega) \cap H^2(\Omega)$, and $A(x;D)$ is a second order linear differential operator on Ω with coefficients continuous on $\overline{\Omega}$; see [27, Example 6.1] for more details. In the fractional relaxation case, we can similarly consider the existence and uniqueness of asymptotically S^p-almost automorphic solutions of the following fractional damped Poisson-wave type equation in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$:

$$\begin{cases} \mathbf{D}_t^\gamma\big(m(x)\mathbf{D}_t^\gamma u\big) + \big(2\omega m(x) - \Delta\big)\mathbf{D}_t^\gamma u + \big(A(x;D) - \omega\Delta + \omega^2 m(x)\big)u(x,t) = f(x,t), \\ t \geqslant 0,\ x \in \Omega\ ;\ u = \mathbf{D}_t^\gamma = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ u(0,x) = u_0(x),\ m(x)\big[\mathbf{D}_t^\gamma u(x,0) + \omega u_0\big] = m(x)u_1(x), \quad x \in \Omega. \end{cases}$$

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