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HILFER AND HADAMARD COUPLED VOLTERRA FRACTIONAL INTEGRO–DIFFERENTIAL SYSTEMS WITH RANDOM EFFECTS

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Abstract. This paper deals with some existence results for two classes of coupled systems of Hilfer and Hilfer-Hadamard random fractional integro-differential equations. The main tool used to carry out our results is Itoh's random fixed point theorem.

1. Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [22, 36]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas et al. [2, 6, 7], Ahmad et al. [8], Samko et al. [35], Kilbas et al. [28] and Zhou [42], the papers by Abbas et al. [1, 3, 4, 5], and the references therein. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; see [18, 19, 22, 26, 37, 39].

Coupled differential and integro-differential equations appear in mathematical modeling of many biological phenomena and environmental issues. Lotka-Volterra models for competitive species are probably the most well-known examples of such coupled equations [13, 21]. A particular case of the Lotka–Volterra model is the famous predator-prey problem for two competing species. The Wilson-Cowan [40, 41] model describes the dynamics of interactions between populations of very simple excitatory and inhibitory model neurons. This model has been widely used in modeling neuronal populations [24, 30]. For further details on the utility of coupled systems, see [25, 34]. In [16, 17], the authors studied the existence of asymptotically periodic solutions of linear systems of Volterra difference equations. Recent results on coupled systems of fractional differential equations can be found in [9, 10, 11].

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The nature of a dynamic system in engineering or natural sciences depends on the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the monographs [12, 29, 38], and the papers [14, 15, 31].

In this paper we discuss the existence of solutions for the following coupled system of random Hilfer fractional integro-differential equations

$$\begin{cases} (D_0^{\alpha_1,\beta_1}u_1)(t,w) = f_1(t,u_1(t,w),u_2(t,w),w) \\ +\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}g_1(s,u_1(s,w),u_2(s,w),w)ds \\ \\ (D_0^{\alpha_2,\beta_2}u_2)(t,w) = f_2(t,u_1(t,w),u_2(t,w),w) \\ +\frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}g_2(s,u_1(s,w),u_2(s,w),w)ds \end{cases} ;\ t \in I := [0,T],\ w \in \Omega, \quad (1)$$

supplemented with the initial conditions:

$$\begin{cases} (I_0^{1-\gamma_1}u_1)(0,w) = \phi_1(w) \\ \\ (I_0^{1-\gamma_2}u_2)(0,w) = \phi_2(w) \end{cases} ;\ w \in \Omega, \quad (2)$$

where $T>0$, $\alpha_i \in (0,1)$, $\beta_i \in [0,1]$, $\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$; $i=1,2$; $(\Omega,\mathscr{A})$ is a measurable space, $\phi_i : \Omega \to \mathbb{R}$ is a measurable function, $f_i, g_i : I\times\mathbb{R}\times\mathbb{R}\times\Omega \to \mathbb{R}$ are given functions, $I_0^{1-\gamma_i}$ is the left-sided mixed Riemann–Liouville integral of order $1-\gamma_i$, and $D_0^{\alpha_i,\beta_i}$ is the Hilfer fractional derivative of order α_i and type β_i. Next, we consider the following coupled system of random Hilfer–Hadamard fractional integro-differential equations

$$\begin{cases} ({}^HD_1^{\alpha_1,\beta_1}u_1)(t,w) = f_1(t,u_1(t,w),u_2(t,w),w) \\ +\frac{1}{\Gamma(\alpha_1)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha_1-1} g_1(s,u_1(s,w),u_2(s,w),w)\frac{ds}{s} \\ \\ ({}^HD_1^{\alpha_2,\beta_2}u_2)(t,w) = f_2(t,u_1(t,w),u_2(t,w),w) \\ +\frac{1}{\Gamma(\alpha_2)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha_2-1} g_2(s,u_1(s,w),u_2(s,w),w)\frac{ds}{s} \end{cases} ;\ t \in [1,T],\ w \in \Omega, \quad (3)$$

equipped with the initial conditions:

$$\begin{cases} ({}^HI_1^{1-\gamma_1}u_1)(1,w) = \psi_1(w) \\ \\ ({}^HI_1^{1-\gamma_2}u_2)(1,w) = \psi_2(w) \end{cases} ;\ w \in \Omega, \quad (4)$$

where $T > 1$, $\alpha_i \in (0,1)$, $\beta_i \in [0,1]$, $\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, $\psi_i : \Omega \to \mathbb{R}$; $i = 1,2$ is a measurable function, $f_i, g_i : [1,T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ are given function, ${}^H I_1^{1-\gamma_i}$ is the left-sided mixed Hadamard integral of order $1-\gamma_i$, and ${}^H D_1^{\alpha_i,\beta_i}$ is the Hilfer–Hadamard fractional derivative of order α_i and type β_i; $i = 1,2$.

2. Preliminaries

Let C be the Banach space of all continuous functions u from I into $\mathbb{R}$ with the supremum (uniform) norm

$$\|u\|_\infty := \sup_{t\in I} |u(t)|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}$. We denote by $AC^1(I)$ the space defined by

$$AC^1(I) := \{u : I \to \mathbb{R} : \frac{d}{dt}u(t) \in AC(I)\}.$$

By $L^1(I)$, we denote the space of Lebesgue-integrable functions $u : I \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_0^T |u(t)|dt.$$

Let $L^\infty(I)$ be the Banach space of measurable functions $u : I \to \mathbb{R}$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : |u(t)| \leqslant c, \ \text{a.e.} \ t \in I\}.$$

By $C_\gamma(I)$ and $C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{u : (0,T] \to \mathbb{R} : t^{1-\gamma}u(t) \in C\},$$

with the norm

$$\|u\|_{C_\gamma} := \sup_{t\in I} |t^{1-\gamma}u(t)|,$$

and

$$C_\gamma^1(I) = \{u \in C : \frac{du}{dt} \in C_\gamma\},$$

with the norm

$$\|u\|_{C_\gamma^1} := \|u\|_\infty + \|u'\|_{C_\gamma}.$$

Also, by $\mathscr{C} := C_{\gamma_1} \times C_{\gamma_2}$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathscr{C}} = \|u\|_{C_{\gamma_1}} + \|v\|_{C_{\gamma_2}}.$$

DEFINITION 2.1. A function $T : \Omega \times \mathbb{R} \to \mathbb{R}$ is called jointly measurable if $T(\cdot,u)$ is measurable for all $u \in \mathbb{R}$ and $T(w,\cdot)$ is continuous for all $w \in \Omega$.

DEFINITION 2.2. A function $f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(t,w) \to f(t,u,w)$ is jointly measurable for all $u \in \mathbb{R}$, and

(ii) The map $u \to f(t,u,w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let E be a Banach space and $T : \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w,u)$ is measurable in w for all $u \in E$ and it expressed as $T(w)u = T(w,u)$. In this case we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w,u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [23].

DEFINITION 2.3. Let $\mathscr{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathscr{P}(Y)$. A mapping $T : \{(w,y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w,y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Now, we give some results and properties of fractional calculus.

DEFINITION 2.4. [6, 28, 35] The left-sided mixed Riemann–Liouville integral of order $r > 0$ of a function $u \in L^1(I)$ is defined by

$$(I_0^r u)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} u(s)ds; \text{ for a.e. } t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt; \ \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $u \in C$, we have $I_0^r u \in C$, and

$$(I_0^{r_1} I_0^{r_2} u)(t) = (I_0^{r_1+r_2} u)(t); \text{ for a.e. } t \in I.$$

DEFINITION 2.5. [6, 28, 35] The Riemann–Liouville fractional derivative of order $r \in (0,1]$ of a function $u \in L^1(I)$ is defined by

$$\begin{aligned}(D_0^r u)(t) &= \left(\frac{d}{dt} I_0^{1-r} u\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} u(s)ds; \text{ for a.e. } t \in I.\end{aligned}$$

Let $r\in(0,1]$, $\gamma\in[0,1)$ and $u\in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows.

$$(D_0^r I_0^r u)(t)=u(t);\ \text{for all } t\in(0,T].$$

Moreover, if $I_0^{1-r}u\in C^1_{1-\gamma}(I)$, then the following composition is proved in [35]

$$(I_0^r D_0^r u)(t)=u(t)-\frac{(I_0^{1-r}u)(0^+)}{\Gamma(r)}t^{r-1};\ \text{for all } t\in(0,T].$$

DEFINITION 2.6. [6, 28, 35] The Caputo fractional derivative of order $r\in(0,1]$ of a function $u\in L^1(I)$ is defined by

$$\begin{aligned}({}^cD_0^r u)(t) &= \left(I_0^{1-r}\frac{d}{dt}u\right)(t)\\ &= \frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r}\frac{d}{ds}u(s)ds;\ \text{ for a.e. } t\in I.\end{aligned}$$

In [22], Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases (see also [26, 37].

DEFINITION 2.7. (Hilfer derivative) Let $\alpha\in(0,1)$, $\beta\in[0,1]$, $u\in L^1(I)$, $I_0^{(1-\alpha)(1-\beta)}u\in AC(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta}u)(t)=\left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{(1-\alpha)(1-\beta)}u\right)(t);\ \text{ for a.e. } t\in I. \tag{5}$$

SOME PROPERTIES. Let $\alpha\in(0,1)$, $\beta\in[0,1]$, $\gamma=\alpha+\beta-\alpha\beta$, and $u\in L^1(I)$.
(P_1). The operator $(D_0^{\alpha,\beta}u)(t)$ can be written as

$$(D_0^{\alpha,\beta}u)(t)=\left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{1-\gamma}u\right)(t)=\left(I_0^{\beta(1-\alpha)}D_0^{\gamma}u\right)(t);\ \text{ for a.e. } t\in I.$$

Moreover, the parameter γ satisfies

$$\gamma\in(0,1],\ \ \gamma\geqslant\alpha,\ \ \gamma>\beta,\ \ 1-\gamma<1-\beta(1-\alpha).$$

(P_2). The generalization (5) for $\beta=0$, coincides with the Riemann–Liouville derivative and for $\beta=1$ with the Caputo derivative.

$$D_0^{\alpha,0}=D_0^{\alpha},\ \text{ and }\ D_0^{\alpha,1}={}^cD_0^{\alpha}.$$

(P_3). If $D_0^{\beta(1-\alpha)}w$ exists and in $L^1(I)$, then

$$(D_0^{\alpha,\beta}I_0^{\alpha}u)(t)=(I_0^{\beta(1-\alpha)}D_0^{\beta(1-\alpha)}u)(t);\ \text{ for a.e. } t\in I.$$

Furthermore, if $u \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)} u \in C_\gamma^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^\alpha u)(t) = u(t); \text{ for a.e. } t \in I.$$

(P_4). If $D_0^\gamma u$ exists and in $L^1(I)$, then

$$(I_0^\alpha D_0^{\alpha,\beta} u)(t) = (I_0^\gamma D_0^\gamma u)(t) = u(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1}; \text{ for a.e. } t \in I.$$

LEMMA 2.8. *Let $h \in C_\gamma(I)$. Then the Cauchy problem*

$$\begin{cases} (D_0^{\alpha,\beta} u)(t) = h(t);\ t \in I, \\ (I_0^{1-\gamma} u)(t)|_{t=0} = \phi, \end{cases} \tag{6}$$

has a unique solution given by

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha h)(t). \tag{7}$$

Proof. Let $u(\cdot)$ be a solution of problem (6). Then, we have

$$(I_0^\alpha D_0^{\alpha,\beta} u)(t) = (I_0^\alpha h)(t).$$

Thus, from the property (P_4) we get

$$u(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1} = (I_0^\alpha h)(t).$$

Hence, the solution $u(\cdot)$ is given by (7). □

We need the following Itoh's random fixed point theorem.

THEOREM 2.9. [23] *Let X be a non-empty, closed convex bounded subset of the separable Banach space E and let $N : \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $N(w)u = u$ has a random solution.*

3. Coupled systems of Hilfer fractional random integro-differential equations

In this section, we are concerned with the existence of solutions for the system (1). Let us start by defining what we mean by a random solution of the system (1).

DEFINITION 3.1. By a random solution of the problem (1) we mean a coupled measurable functions $(u_1, u_2) : \Omega \to C_{\gamma_1} \times C_{\gamma_2}$ that satisfies the conditions $(I_0^{1-\gamma_i} u_i)(0^+, w) = \phi_i(w);\ i = 1, 2,$ and the equations $(D_0^{\alpha_i,\beta_i} u_i)(t, w) = f_i(t, u_1(t, w), u_2(t, w), w);\ i = 1, 2$ on $I \times \Omega$.

The following hypotheses will be used in the sequel.

(H_1) The functions f_i, g_i; $i=1,2$ are random Carathéodory on $I\times\mathbb{R}\times\mathbb{R}\times\Omega$.

(H_2) There exist measurable and bounded functions $p_i, q_i:\Omega\to L^\infty(I,[0,\infty))$; $i=1,2$, such that

$$|f_i(t,u_1,u_2,w)|\leqslant\frac{p_i(t,w)\max\{|u_1|,|u_2|\}}{1+|u_1|+|u_2|};\ \text{for a.e. } t\in I,\ \text{and each } u_i\in\mathbb{R},\ w\in\Omega,$$

and

$$|g_i(t,u_1,u_2,w)|\leqslant\frac{q_i(t,w)\max\{|u_1|,|u_2|\}}{1+|u_1|+|u_2|};\ \text{for a.e. } t\in I,\ \text{and each } u_i\in\mathbb{R},\ w\in\Omega.$$

Now, we shall prove the following theorem concerning the existence of random solutions of the system (1).

THEOREM 3.2. *Assume that the hypotheses (H_1) and (H_2) hold. Then the system (1) has at least one random solution defined on $I\times\Omega$.*

Proof. Define the operators $N_i:\Omega\times C_{\gamma_i}\to C_{\gamma_i}$; $i=1,2$ by

$$\begin{aligned}(N_i(w)u_i)(t) &= \frac{\phi_i(w)}{\Gamma(\gamma_i)}t^{\gamma_i-1}+\int_0^t(t-s)^{\alpha_i-1}\frac{f_i(s,u_1(s,w),u_2(s,w),w)}{\Gamma(\alpha_i)}ds\\ &\quad+\frac{1}{\Gamma^2(\alpha_i)}\int_0^t\int_0^s(t-s)^{\alpha_i-1}(s-y)^{\alpha_i-1}g_i(y,u_1(y,w),u_2(y,w),w)dyds, \quad (8)\end{aligned}$$

and consider the continuous operator $N:\Omega\times\mathscr{C}\to\mathscr{C}$ defined by

$$N(w)(u_1,u_2)=(N_1(w)u_1,N_2(w)u_2). \quad (9)$$

Set

$$p_i^*=\sup_{w\in\Omega}\|p_i(w)\|_{L^\infty},\ q_i^*=\sup_{w\in\Omega}\|q_i(w)\|_{L^\infty},\ \phi_i^*=\sup_{w\in\Omega}|\phi_i(w)|;\ i=1,2.$$

For each $i=1,2$, the map ϕ_i is measurable for all $w\in\Omega$. Again, as the indefinite integral is continuous on I, then $N_i(w)$ defines a mapping $N_i:\Omega\times C_{\gamma_i}\to C_{\gamma_i}$. Thus (u_1,u_2) is a random solution for the system (1) if and only if $(u_1,u_2)=N(w)(u_1,u_2)$.

Next, for any $u_i\in C_{\gamma_i}$; $i=1,2$, and each $t\in I$ and $w\in\omega$, we have

$$\begin{aligned}&|t^{1-\gamma_i}(N_i(w)u_i)(t)|\\ &\leqslant\frac{|\phi_i(w)|}{\Gamma(\gamma_i)}+\frac{t^{1-\gamma_I}}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}|f_i(s,u_1(s,w),u_2(s,w),w)|ds\\ &\quad+\frac{t^{1-\gamma_i}}{\Gamma^2(\alpha_i)}\int_0^t\int_0^s(t-s)^{\alpha_i-1}(s-y)^{\alpha_i-1}|g_i(y,u_1(y,w),u_2(y,w),w)|dyds\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{|\phi_i(w)|}{\Gamma(\gamma_i)} + \frac{t^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} p_i(s,w)ds \\
&\quad + \frac{t^{1-\gamma_i}}{\Gamma^2(\alpha_i)} \int_0^t \int_0^s (t-s)^{\alpha_i-1}(s-y)^{\alpha_i-1} q_i(s,w)dyds \\
&\leqslant \frac{\phi_I^*}{\Gamma(\gamma_i)} + \frac{p_i^* T^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} ds + \frac{q_i^* T^{1-\gamma_i}}{\Gamma^2(\alpha_i)} \int_0^t \int_0^s (t-s)^{\alpha_i-1}(s-y)^{\alpha_i-1}) dyds \\
&\leqslant \frac{\phi_i^*}{\Gamma(\gamma_i)} + \frac{p_i^* T^{1-\gamma_i+\alpha_i}}{\Gamma(1+\alpha_i)} + \frac{q_i^* T^{1-\gamma_i+2\alpha_i}}{\Gamma^2(1+\alpha_i)}.
\end{aligned}
$$

Thus

$$
\|N(w)(u_1,u_2)\|_{\mathscr{C}} \leqslant \sum_{i=1}^{2} \frac{\phi_i^*}{\Gamma(\gamma_i)} + \frac{p_i^* T^{1-\gamma_i+\alpha_i}}{\Gamma(1+\alpha_i)} + \frac{q_i^* T^{1-\gamma_i+2\alpha_i}}{\Gamma^2(1+\alpha_i)} := R. \tag{10}
$$

This proves that $N(w)$ transforms the ball

$$
B_R := B(0,R) = \{(u_1,u_2) \in \mathscr{C} : \|(u_1,u_2)\|_{\mathscr{C}} \leqslant R\}
$$

into itself. We shall show that the operator $N : \Omega \times B_R \to B_R$ satisfies all the assumptions of Theorem 2.9. The proof will be given in several steps.

Step 1. $N(w)$ is a random operator on $\Omega \times B_R$ into B_R.

Since for each $i = 1,2,$ $f_i(t,u_1,u_2,w)$ is random Carathéodory, the maps $w \to f_i(t,u_1,u_2,w)$ and $w \to g_i(t,u_1,u_2,w)$ are measurable in view of Definition 2.1. Similarly, the product $(t-s)^{\alpha_i-1} f_i(s,u_1(s,w),u_2(s,w),w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$
\begin{aligned}
w \mapsto &\frac{\phi_i(w)}{\Gamma(\gamma_i)} t^{\gamma_i-1} + \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_i(s,u_1(s,w),u_2(s,w),w)ds \\
&+ \frac{1}{\Gamma^2(\alpha_i)} \int_0^t \int_0^s (t-s)^{\alpha_i-1}(s-y)^{\alpha_i-1} g_i(y,u_1(y,w),u_2(y,w),w)dyds
\end{aligned}
$$

is measurable. As a result, $N(w)$ is a random operator on $\Omega \times B_R$ into B_R.

Step 2. $N(w)$ *is continuous.*

Let $\{(u_{1n},u_{2n})\}_{n\in\mathbb{N}}$ be a sequence such that $(u_{1n},u_{2n}) \to (u_1,u_2)$ in B_R. Then, for each $i = 1,2,$ $t \in I,$ and $w \in \Omega,$ we have

$$
\begin{aligned}
&|t^{1-\gamma_i}(N_i(w)u_{in})(t) - t^{1-\gamma_i}(N_i(w)u_i)(t)| \\
&\leqslant \frac{t^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} |f_i(s,u_{1n}(s,w),u_{2n}(s,w),w) - f_i(s,u_1(s,w),u_2(s,w),w)| ds \\
&\quad + \frac{1}{\Gamma^2(\alpha_i)} \int_0^t \int_0^s (t-s)^{\alpha_i-1}(s-y)^{\alpha_i-1} \\
&\quad \times |g_i(y,u_{1n}(y,w),u_{2n}(y,w),w) - g_i(y,u_1(y,w),u_2(y,w),w)| dyds.
\end{aligned} \tag{11}
$$

Since $(u_{1n},u_{2n}) \to (u_1,u_2)$ as $n \to \infty$ and f_i and g_i are random Carathéodory, then by the Lebesgue dominated convergence theorem, equation (11) implies

$$
\|N(w)(u_{1n},u_{2n}) - N(w)(u_1,u_2)\|_{\mathscr{C}} \to 0 \quad \text{as } n \to \infty.
$$

Step 3. $N(w)B_R$ *is uniformly bounded.*

This is clear since $N(w)B_R \subset B_R$ and B_R is bounded.

Step 4. $N(w)B_R$ *is equicontinuous.*

Let $t_1,t_2 \in I,\ t_1 < t_2$ and let $(u_1,u_2) \in B_R$. Then, for each $i = 1,2,$ and $w \in \Omega$, we have

$$\begin{aligned}
&|t_2^{1-\gamma_i}(N_i(w)u_i)(t_2) - t_1^{1-\gamma_i}(N_i(w)u_i)(t_1)|\\
\leqslant& \Bigg| t_2^{1-\gamma_i}\int_0^{t_2}(t_2-s)^{\alpha_i-1}\frac{f_i(s,u_1(s,w),u_2(s,w),w)}{\Gamma(\alpha_i)}ds\\
&-t_1^{1-\gamma_i}\int_0^{t_1}(t_1-s)^{\alpha_i-1}\frac{f_i(s,u_1(s,w),u_2(s,w),w)}{\Gamma(\alpha_i)}ds\\
&+t_2^{1-\gamma_i}\int_0^{t_2}\int_0^{s}(t_2-s)^{\alpha_i-1}(s-y)^{\alpha_i-1}\frac{g_i(s,u_1(s,w),u_2(s,w),w)}{\Gamma^2(\alpha_i)}dyds\\
&-t_1^{1-\gamma_i}\int_0^{t_1}\int_0^{s}(t_1-s)^{\alpha_i-1}(s-y)^{\alpha_i-1}\frac{g_i(s,u_1(s,w),u_2(s,w),w)}{\Gamma^2(\alpha_i)}dyds\Bigg|\\
\leqslant& t_2^{1-\gamma_i}\int_{t_1}^{t_2}(t_2-s)^{\alpha_i-1}\frac{|f_i(s,u_1(s,w),u_2(s,w),w)|}{\Gamma(\alpha_i)}ds\\
&+\int_0^{t_1}|t_2^{1-\gamma_i}(t_2-s)^{\alpha_i-1}-t_1^{1-\gamma_i}(t_1-s)^{\alpha_i-1}|\frac{|f_i(s,u_1(s,w),u_2(s,w),w)|}{\Gamma(\alpha_i)}ds\\
&+t_2^{1-\gamma_i}\int_{t_1}^{t_2}\int_0^{s}(t_2-s)^{\alpha_i-1}(s-y)^{\alpha_i-1}\frac{|g_i(s,u_1(s,w),u_2(s,w),w)|}{\Gamma^2(\alpha_i)}dyds\\
&+\int_0^{t_1}\int_0^{s}|t_2^{1-\gamma_i}(t_2-s)^{\alpha_i-1}-t_1^{1-\gamma_i}(t_1-s)^{\alpha_i-1}|(s-y)^{\alpha_i-1}\\
&\times\frac{|g_i(s,u_1(s,w),u_2(s,w),w)|}{\Gamma^2(\alpha_i)}dyds.
\end{aligned}$$

Thus,

$$\begin{aligned}
&|t_2^{1-\gamma_i}(N_i(w)u_i)(t_2) - t_1^{1-\gamma_i}(N_i(w)u_i)(t_1)|\\
\leqslant& t_2^{1-\gamma_i}\int_{t_1}^{t_2}(t_2-s)^{\alpha_i-1}\frac{p_i(s,w)}{\Gamma(\alpha_i)}ds+\int_0^{t_1}|t_2^{1-\gamma_i}(t_2-s)^{\alpha_i-1}-t_1^{1-\gamma_i}(t_1-s)^{\alpha_i-1}|\frac{p_i(s,w)}{\Gamma(\alpha_i)}ds\\
&+t_2^{1-\gamma_i}\int_{t_1}^{t_2}\int_0^{s}(t_2-s)^{\alpha_i-1}(s-y)^{\alpha_i-1}\frac{q_i(s,w)}{\Gamma^2(\alpha_i)}dyds\\
&+\int_0^{t_1}\int_0^{s}|t_2^{1-\gamma_i}(t_2-s)^{\alpha_i-1}-t_1^{1-\gamma_i}(t_1-s)^{\alpha_i-1}|(s-y)^{\alpha_i-1}\frac{q_i(s,w)}{\Gamma^2(\alpha_i)}dyds.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&|t_2^{1-\gamma_i}(N_i(w)u_i)(t_2) - t_1^{1-\gamma_i}(N_i(w)u_i)(t_1)|\\
\leqslant& \left(p_i^*+q_i^*\frac{T^{\alpha_i-1}}{\Gamma(\alpha_i)}\right)\left[\frac{T^{1-\gamma_i+\alpha_i}}{\Gamma(1+\alpha_i)}(t_2-t_1)^{\alpha_i}\right.\\
&\left.+\frac{1}{\Gamma(\alpha_i)}\int_0^{t_1}|t_2^{1-\gamma_i}(t_2-s)^{\alpha_i-1}-t_1^{1-\gamma_i}(t_1-s)^{\alpha_i-1}|ds\right].
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 4 together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \times B_R \to B_R$ is continuous and compact. From an application of Theorem 2.9, we deduce that the operator equation $N(w)(u_1,u_2) = (u_1,u_2)$ has a random solution. This implies that the random system (1) has a random solution. □

4. Hilfer–Hadamard fractional random integro-differential equations

Now, we are concerned with some existence results for the coupled system (3). Set $C := C([1,T])$. Denote the weighted space of continuous functions defined by

$$C_{\gamma,\ln}([1,T]) = \{w(t) : (\ln t)^{1-\gamma} w(t) \in C\},$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in[1,T]} |(\ln t)^{1-r} w(t)|.$$

By $\mathscr{C}_{\ln} := C_{\gamma_1,\ln} \times C_{\gamma_2,\ln}$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathscr{C}_{\ln}} = \|u\|_{C_{\gamma_1,\ln}} + \|v\|_{C_{\gamma_2,\ln}}.$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [28] for a more detailed analysis.

DEFINITION 4.1. [28] (Hadamard fractional integral) The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,T])$, is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

provided the integral exists.

EXAMPLE 4.2. Let $0 < q < 1$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q}, \text{ for a.e. } t \in [0,e].$$

Set

$$\delta = x\frac{d}{dx},\ q > 0,\ n = [q] + 1,$$

and

$$AC_\delta^n := \{u : [1,T] \to \mathbb{R} : \delta^{n-1}[u(x)] \in AC(I)\}.$$

Analogous to the Riemann–Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way:

DEFINITION 4.3. [28] (Hadamard fractional derivative). The Hadamard fractional derivative of order $q>0$ applied to the function $w \in AC^n_\delta$ is defined as

$$({}^H D^q_1 w)(x) = \delta^n({}^H I^{n-q}_1 w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^H D^q_1 w)(x) = \delta({}^H I^{1-q}_1 w)(x).$$

EXAMPLE 4.4. Let $0<q<1$. Then

$${}^H D^q_1 \ln t = \frac{1}{\Gamma(2-q)}(\ln t)^{1-q}, \quad \text{for a.e. } t \in [0,e].$$

It has been proved (see e.g. Kilbas [[27], Theorem 4.8]) that in the space $L^1(I,\mathbb{R})$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$({}^H D^q_1)({}^H I^q_1 w)(x) = w(x).$$

From Theorem 2.3 of [28], we have

$$({}^H I^q_1)({}^H D^q_1 w)(x) = w(x) - \frac{({}^H I^{1-q}_1 w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo–Hadamard fractional derivative is defined in the following way:

DEFINITION 4.5. (Caputo–Hadamard fractional derivative) The Caputo-Hadamard fractional derivative of order $q>0$ applied to the function $w \in AC^n_\delta$ is defined as

$$({}^{Hc} D^q_1 w)(x) = ({}^H I^{n-q}_1 \delta^n w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^{Hc} D^q_1 w)(x) = ({}^H I^{1-q}_1 \delta w)(x).$$

From the Hadamard fractional integral, the Hilfer–Hadamard fractional derivative (introduced for the first time in [32]) is defined in the following way:

DEFINITION 4.6. (Hilfer–Hadamard fractional derivative) Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha+\beta-\alpha\beta$, $w \in L^1(I)$, and ${}^H I^{(1-\alpha)(1-\beta)}_1 w \in AC(I)$. The Hilfer–Hadamard fractional derivative of order α and type β applied to the function w is defined as

$$\begin{aligned}({}^H D^{\alpha,\beta}_1 w)(t) &= \left({}^H I^{\beta(1-\alpha)}_1 ({}^H D^\gamma_1 w)\right)(t) \\ &= \left({}^H I^{\beta(1-\alpha)}_1 \delta({}^H I^{1-\gamma}_1 w)\right)(t); \quad \text{for a.e. } t \in [1,T].\end{aligned} \tag{12}$$

This new fractional derivative (4.6) may be viewed as interpolating the Hadamard fractional derivative and the Caputo–Hadamard fractional derivative. Indeed for $\beta = 0$ this derivative reduces to the Hadamard fractional derivative and when $\beta = 1$, we recover the Caputo–Hadamard fractional derivative.

$$^{H}D_1^{\alpha,0} = {}^{H}D_1^{\alpha}, \text{ and } {}^{H}D_1^{\alpha,1} = {}^{Hc}D_1^{\alpha}.$$

Now we give a similar existence result for the system (3). The following hypotheses will be used in the sequel.

(H_1') The functions f_i, g_i; $i=1,2$ are random Carathéeodory on $[1,T]\times\mathbb{R}\times\mathbb{R}\times\Omega$,

(H_2') There exist measurable and bounded functions $p_i, q_i : \Omega \to L^\infty([1,T],[0,\infty))$, such that

$$|f_i(t,u_1,u_2,w)| \leqslant \frac{p_i(t,w)\max\{|u_1|,|u_2|\}}{1+|u_1|+|u_2|}; \text{ for a.e. } t\in[1,T], \text{ and each } u_i\in\mathbb{R},\, w\in\Omega.$$

and

$$|g_i(t,u_1,u_2,w)| \leqslant \frac{q_i(t,w)\max\{|u_1|,|u_2|\}}{1+|u_1|+|u_2|}; \text{ for a.e. } t\in[1,T], \text{ and each } u_i\in\mathbb{R},\, w\in\Omega.$$

THEOREM 4.7. *Assume that the hypotheses (H_1') and (H_2') hold. Then the coupled system (3) has at least one random solution defined on $[1,T]\times\Omega$.*

Proof. Define the operators $\overline{N}_i : \Omega\times C_{\gamma_i,\ln} \to C_{\gamma_i,\ln}$; $i=1,2$ by

$$(\overline{N}_i(w)u_i)(t) = \frac{\psi_i(w)}{\Gamma(\gamma_i)}(\ln t)^{\gamma_i-1} + \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha_i-1} \frac{f_i(s,u_1(s,w),u_2(s,w),w)}{s\Gamma(\alpha_i)}ds$$
$$+\frac{1}{\Gamma^2(\alpha_i)}\int_1^t\int_1^s \left(\ln\frac{t}{s}\right)^{\alpha_i-1}\left(\ln\frac{s}{y}\right)^{\alpha_i-1} \frac{g_i(y,u_1(y,w),u_2(y,w),w)}{sy}dyds, \tag{13}$$

and consider the continuous operator $\overline{N} : \Omega\times\mathscr{C}_{\ln} \to \mathscr{C}_{\ln}$ defined by

$$\overline{N}(w)(u_1,u_2) = (\overline{N}_1(w)u_1, \overline{N}_2(w)u_2). \tag{14}$$

Set

$$p_i^* = \sup_{w\in\Omega}\|p_i(w)\|_{L^\infty},\quad q_i^* = \sup_{w\in\Omega}\|q_i(w)\|_{L^\infty},\quad \psi_i^* = \sup_{w\in\Omega}|\psi_i(w)|;\ i=1,2.$$

For each $i=1,2$, the map ψ_i is measurable for all $w\in\Omega$. Again, as the indefinite integral is continuous on $[1,T]$, then $\overline{N}_i(w)$ defines a mapping $\overline{N}_i : \Omega\times C_{\gamma_i,\ln} \to C_{\gamma_i,\ln}$. Thus (u_1,u_2) is a random solution for the system (3) if and only if $(u_1,u_2) = \overline{N}(w)(u_1,u_2)$.

Next, for any $u_i \in C_{\gamma_i,\ln}$; $i=1,2$, and each $t\in[1,T]$ and $w\in\omega$, we get

$$|(\ln t)^{1-\gamma_i}(\overline{N}_i(w)u_i)(t)| \leqslant \frac{\psi_i^*}{\Gamma(\gamma_i)} + \frac{p_i^*(\ln T)^{1-\gamma_i+\alpha_i}}{\Gamma(1+\alpha_i)} + \frac{q_i^*(\ln T)^{1-\gamma_i+2\alpha_i}}{\Gamma^2(1+\alpha_i)}.$$

Thus

$$\|\overline{N}(w)(u_1,u_2)\|_{\mathscr{C}_{\ln}} \leqslant \sum_{i=1}^{2} \frac{\psi_i^*}{\Gamma(\gamma_i)} + \frac{p_i^*(\ln T)^{1-\gamma_i+\alpha_i}}{\Gamma(1+\alpha_i)} + \frac{q_i^*(\ln T)^{1-\gamma_i+2\alpha_i}}{\Gamma^2(1+\alpha_i)} := R'. \tag{15}$$

This proves that $\overline{N}(w)$ transforms the ball

$$B'_R := B'(0,R) = \{(u_1,u_2) \in \mathscr{C}_{\ln} : \|(u_1,u_2)\|_{\mathscr{C}_{\ln}} \leqslant R'\}$$

into itself. We shall show that the operator $\overline{N} : \Omega \times B'_R \to B'_R$ satisfies all the assumptions of Theorem 2.9. The proof will be given in four steps.

Step 1. $\overline{N}(w)$ is a random operator on $\Omega \times B'_R$ into B'_R.

Since for each $i = 1,2$, $f_i(t,u_1,u_2,w)$ is random Carathéodory, the maps $w \to f_i(t,u_1,u_2,w)$ and $w \to g_i(t,u_1,u_2,w)$ are measurable in view of Definition 2.1. Similarly, the product $\left(\ln \frac{t}{s}\right)^{\alpha_i-1} f_i(s,u_1(s,w),u_2(s,w),w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$\begin{aligned} w \mapsto \frac{\psi_i(w)}{\Gamma(\gamma_i)}(\ln t)^{\gamma_i-1} &+ \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha_i-1} \frac{f_i(s,u_1(s,w),u_2(s,w),w)}{s\Gamma(\alpha_i)} ds \\ &+ \frac{1}{\Gamma^2(\alpha_i)} \int_1^t \int_1^s \left(\ln \frac{t}{s}\right)^{\alpha_i-1} \left(\ln \frac{s}{y}\right)^{\alpha_i-1} \frac{g_i(y,u_1(y,w),u_2(y,w),w)}{sy} dyds \end{aligned}$$

is measurable. As a result, $\overline{N}(w)$ is a random operator on $\Omega \times B'_R$ into B'_R.

Step 2. $\overline{N}(w)$ *is continuous.*

Let $\{(u_{1n},u_{2n})\}_{n\in\mathbb{N}}$ be a sequence such that $(u_{1n},u_{2n}) \to (u_1,u_2)$ in B'_R. Then, for each $i = 1,2$, $t \in I$, and $w \in \Omega$, we have

$$\begin{aligned} &|(\ln t)^{1-\gamma_i}(\overline{N}_i(w)u_{in})(t) - (\ln t)^{1-\gamma_i}(\overline{N}_i(w)u_i)(t)| \\ &\leqslant \frac{(\ln t)^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha_i-1} |f_i(s,u_{1n}(s,w),u_{2n}(s,w),w) - f_i(s,u_1(s,w),u_2(s,w),w)| ds \\ &+ \frac{1}{\Gamma^2(\alpha_i)} \int_1^t \int_1^s \left(\ln \frac{t}{s}\right)^{\alpha_i-1} \left(\ln \frac{s}{y}\right)^{\alpha_i-1} \\ &\times |g_i(y,u_{1n}(y,w),u_{2n}(y,w),w) - g_i(y,u_1(y,w),u_2(y,w),w)| dyds. \end{aligned} \tag{16}$$

Since $(u_{1n},u_{2n}) \to (u_1,u_2)$ as $n \to \infty$ and f_i and g_i are random Carathéodory, then by the Lebesgue dominated convergence theorem, equation (16) implies

$$\|\overline{N}(w)(u_{1n},u_{2n}) - \overline{N}(w)(u_1,u_2)\|_{\mathscr{C}_{\ln}} \to 0 \quad \text{as } n \to \infty.$$

Step 3. $\overline{N}(w)B'_R$ *is uniformly bounded.*

This is clear since $\overline{N}(w)B'_R \subset B'_R$ and B'_R is bounded.

Step 4. $\overline{N}(w)B'_R$ *is equicontinuous.*

Let $t_1,t_2 \in I,\ t_1 < t_2$ and let $(u_1,u_2) \in B'_R$. Then, for each $i = 1,2,$ and $w \in \Omega$, we get

$$\begin{aligned}
&|(\ln t_2)^{1-\gamma_i}(\overline{N}_i(w)u_i)(t_2) - (\ln t_1)^{1-\gamma_i}(\overline{N}_i(w)u_i)(t_1)| \\
\leqslant &\left(p_i^* + q_i^* \frac{(\ln T)^{\alpha_i-1}}{\Gamma(\alpha_i)}\right)\left[\frac{(\ln T)^{1-\gamma_i+\alpha_i}}{\Gamma(1+\alpha_i)}\left(\ln\frac{t_1}{t_2}\right)^{\alpha_i}\right. \\
&\left. + \frac{1}{\Gamma(\alpha_i)}\int_1^{t_1}\left|(\ln t_2)^{1-\gamma_i}\left(\ln\frac{t_2}{s}\right)^{\alpha_i-1} - (\ln t_1)^{1-\gamma_i}\left(\ln\frac{t_1}{s}\right)^{\alpha_i-1}\right|ds\right].
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of the above steps, from the Arzelá-Ascoli theorem, we can conclude that $\overline{N} : \Omega \times B'_R \to B'_R$ is continuous and compact. From an application of Theorem 2.9, we deduce that the operator equation $\overline{N}(w)(u_1,u_2) = (u_1,u_2)$ has a random solution which is a random solution for the random system (3). □

5. An example

Let

$$E = l^1 = \left\{ w = (w_1,\ w_2,\ \ldots,\ w_n,\ \ldots) : \sum_{n=1}^{\infty} |w_n| < \infty \right\},$$

be the Banach space with norm $\|w\|_E = \sum_{n=1}^{\infty}|w_n|$, $\Omega = (-\infty,0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$. As an application of our results we consider the following system of Hilfer random fractional integro-differential equations of the form

$$\begin{cases}
(D_0^{\frac{1}{3},\frac{1}{2}}u)(t,w) = f_1(t,u(t,w),v(t,w),w) \\
+\frac{1}{\Gamma(\frac{1}{3})}\int_0^t (t-s)^{-\frac{2}{3}} g_1(s,u(s,w),v(s,w),w)ds;\ t \in [0,1], \\
\\
(D_0^{\frac{1}{4},\frac{1}{6}}v)(t,w) = f_2(t,u(t,w),v(t,w),w) \\
+\frac{1}{\Gamma(\frac{1}{4})}\int_0^t (t-s)^{-\frac{3}{4}} g_2(s,u(s,w),v(s,w),w)ds;\ t \in [0,1], \\
\\
(I_0^{\frac{1}{3}}u)(t,w)|_{t=0} = (I_0^{\frac{5}{8}}v)(t,w)|_{t=0} = 1+w^2,
\end{cases} ;\ w \in \Omega, \tag{17}$$

where

$$\begin{cases}
f_1(t,u,v,w) = \dfrac{t^{\frac{-1}{4}}|u|\sin t}{(1+w^2+\sqrt{t})(1+|u|+|v|)};\ t \in (0,1]\ u,v \in \mathbb{R}, \\
f_1(0,u,v) = 0; \qquad u,v \in \mathbb{R},
\end{cases}$$

$$g_1(t,u,v,w) = \frac{t^{\frac{1}{4}}|v|}{1+w^2+|u|+|v|};\ t \in [0,1]\ u,v \in \mathbb{R},$$

$$\begin{cases} f_2(t,u,v,w) = \dfrac{t^{\frac{-1}{4}}|v|(1+t\ln t)}{(1+\sqrt{t})+(1+w^2+|u|+|v|)}; \ t\in(0,1] \ u,v\in\mathbb{R}, \\ f_2(0,u,v)=0; \qquad u,v\in\mathbb{R}, \end{cases}$$

and

$$g_2(t,u,v,w) = \frac{t^{\frac{1}{4}}|v|}{1+w^2+|v|}; \ t\in[0,1] \ u,v\in\mathbb{R}.$$

Clearly, the functions f_i and g_i; $i=1,2$ are random Carathéodory.

The hypothesis (H_2) is satisfied with

$$\begin{cases} p_1(t,w) = \dfrac{t^{\frac{-1}{4}}|\sin t|}{1+w^2+\sqrt{t}}; \ t\in(0,1], \\ p_1(0)=0. \end{cases}$$

$$\begin{cases} p_2(t,w) = \dfrac{t^{\frac{-1}{4}}|1+t\ln t|}{1+\sqrt{t}}; \ t\in(0,1], \\ p_2(0)=0. \end{cases}$$

and $q_1(t,w)=q_2(t)=t^{\frac{1}{4}}$; $t\in(0,1]$, Hence, Theorem 3.2 implies that the coupled system (17) has at least one solution defined on $[0,1]$.

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