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NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH m–POINT INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we consider the existence and uniqueness of solution for a fractional order differential equation involving the Riemann-Liouville fractional derivative. By applying some standard fixed point theorems, we obtain new results on the existence and uniqueness of solution.

1. Introduction

In this paper, we focus on the existence and uniqueness of solutions for nonlinear fractional differential equation given by

$$\begin{cases} -D^p x(t) = A_1 f_1(t,x(t)) + A_2 I^q f_2(t,x(t)), \quad n-1 < p \leqslant n, n \geqslant 2, \quad t \in (0,1), \\ D^{\gamma+k}x(0) = 0,\ 0 \leqslant k \leqslant n-2, \quad D^{\gamma}x(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} D^{\gamma}x(s)dA(s) \end{cases} \tag{1}$$

where $0 < \gamma < 1$, $p-\gamma > n-1$, $0 < q < 1$, $n,k \in \mathbb{N}$ and $0 = \eta_0 < \eta_1 < \ldots < \eta_{m-2} < \eta_{m-1} = 1$, $\alpha_i \geqslant 0$ for $i \in \{1,2,\ldots,m-1\}$. $\int_{\eta_{i-1}}^{\eta_i} D^{\gamma}x(s)dA(s)$ is the Riemann-Stieltjes integral with positive measure. A is a function of bounded variation with $\sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} s^{p-\gamma-1}dA(s) \neq 1$. Here, D^p denotes the Riemann-Liouville fractional derivative of order p and f_1, f_2 are given continuous functions, A_1, A_2 are real constants such that A_1 or A_2 is different from zero.

Recently, boundary value problems for fractional differential equations are of great importance for the researchers due to their applications such as economics, engineering and other fields. Also, this topic has been developed very quickly on the existence results for nonlinear fractional differential equations with local/nonlocal boundary conditions; for example, see [8, 18, 1, 6, 11, 16, 17, 10, 4, 3, 14, 9, 5, 13] and the references

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therein. For instance, Agarwal *et al.* [1] discussed the following integro-differential equation

$$\begin{cases} -D^{\alpha}x(t) = Af(t,x(t)) + BI^{\beta}g(t,x(t)), \quad 2 < \alpha \leqslant 3, \quad t \in [0,1], \\ D^{\delta}x(0) = 0, \quad D^{\delta+1}x(0) = 0, \quad D^{\delta}x(1) - D^{\delta}x(\eta) = a, \end{cases}$$

where $0 < \delta \leqslant 1$, $\alpha - \delta > 3$, $0 < \beta < 1$, $0 < \eta < 1$, $D^{(.)}$ denotes the Riemann-Liouville fractional derivative of order $(.)$, f, g are given continuous functions, and A, B, a are real constants. Here, they studied the existence of solutions for a boundary value problem of integro-differential equations via Sadovskii's fixed point theorem for condensing maps.

The paper is structured as follows. After introducing the basic definitions and lemmas which are required to prove our main results, we prove an existence and uniqueness results by means of the Leray-Schauder's nonlinear alternative theorem, the Banach's fixed point theorem and the Boyd-Wong Contraction Principle.

2. Preliminaries

In this section, we give some basic definitions and lemmas which are useful for the presentation of our main results.

DEFINITION 1. [15, 12] The Riemann Liouville fractional integral of order $p \in \mathbb{R}^+$ for a function $h : (0,\infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{p}h(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1}h(s)ds,$$

provided that the right hand side is pointwise defined on $(0,+\infty)$.

DEFINITION 2. [15, 12] The Riemann-Liouville fractional derivative of order $p > 0$ for a function $h : (0,\infty) \to \mathbb{R}$ is defined by

$$D_{0^+}^{p}h(t) = \left(\frac{d}{dt}\right)^n I_{0^+}^{n-p}h(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-p-1}h(s)ds,$$

where n is the smallest integer greater than or equal to p, provided that the right-hand side is defined pointwise.

LEMMA 1. [12] *Let $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order p ($p > 0$) that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0^+}^{p}D_{0^+}^{p}u(t) = u(t) + c_1t^{p-1} + c_2t^{\alpha-2} + \ldots + c_nt^{p-n},$$

for some $c_i \in \mathbb{R}$, $i = 1,\ldots,n$, where n is the smallest integer greater than or equal to p.

By using the substitution $x(t) = I^{\gamma}y(t) = D^{-\gamma}y(t)$, one can transform the fractional BVP (1) to the following form:

$$\begin{cases} -D^{p-\gamma}y(t) = A_1 f_1(t, I^{\gamma}y(t)) + A_2 I^q f_2(t, I^{\gamma}y(t)), \quad t \in (0,1), \\ y^{(k)}(0) = 0,\ 0 \leqslant k \leqslant n-2, \quad y(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} y(s)dA(s). \end{cases} \tag{2}$$

To obtain the solution of the fractional BVP (2), the following lemma is essential.

LEMMA 2. *For any $h \in C[0,1]$, the unique solution of the linear fractional BVP*

$$\begin{cases} -D^{p-\gamma}y(t) = h(t), \quad t \in (0,1), \\ y^{(k)}(0) = 0,\ 0 \leqslant k \leqslant n-2, \quad y(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} y(s)dA(s) \end{cases} \tag{3}$$

is

$$\begin{aligned} y(t) =& -I^{p-\gamma}h(t) \\ &+ \frac{t^{p-\gamma-1}}{1 - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} s^{p-\gamma-1} dA(s)} \Big(I^{p-\gamma}h(1) - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{p-\gamma}h(s)dA(s) \Big). \end{aligned}$$

Proof. By Lemma 1, the solutions of equation (3) are

$$y(t) = -I^{p-\gamma}h(t) - c_1 t^{p-\gamma-1} - c_2 t^{p-\gamma-2} - \ldots - c_n t^{p-\gamma-n},$$

where $c_i\ (i = 1,2,\ldots n) \in \mathbb{R}$ are arbitrary constants. By the conditions $y^{(k)}(0) = 0,\ 0 \leqslant k \leqslant n-2$, we obtain $c_2 = \ldots = c_n = 0$. Then we conclude that

$$y(t) = -I^{p-\gamma}h(t) - c_1 t^{p-\gamma-1}. \tag{4}$$

Now, by the condition $y(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} y(s)dA(s)$, we can get

$$c_1 = \frac{1}{1 - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} s^{p-\gamma-1} dA(s)} \Big[-I^{p-\gamma}h(1) + \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{p-\gamma}h(s)dA(s) \Big].$$

Combining this value with (4), we obtain

$$\begin{aligned} y(t) =& -I^{p-\gamma}h(t) \\ &+ \frac{t^{p-\gamma-1}}{1 - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} s^{p-\gamma-1} dA(s)} \Big(I^{p-\gamma}h(1) - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{p-\gamma}h(s)dA(s) \Big). \end{aligned}$$

The proof is complete. $\square$

Notice that, the solution of the equation $-D^p x(t) = h(t)$ depends on the boundary conditions given by (1) can be expressed as

$$\begin{aligned}
x(t) &= I^\gamma y(t) \\
&= I^\gamma \Big[- I^{p-\gamma} h(t) \\
&\quad + \frac{t^{p-\gamma-1}}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} s^{p-\gamma-1} dA(s)} \Big(I^{p-\gamma} h(1) - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{p-\gamma} h(s) dA(s) \Big) \Big] \\
&= -I^p h(t) + \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} s^{p-\gamma-1} dA(s)} \Big(I^{p-\gamma} h(1) - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{p-\gamma} h(s) dA(s) \Big) \\
&\quad \times \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} s^{p-\gamma-1} ds \\
&= -I^p h(t) + \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} s^{p-\gamma-1} dA(s)} \Big(I^{p-\gamma} h(1) - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{p-\gamma} h(s) dA(s) \Big) \\
&\quad \times \Big\{ \frac{t^{p-1}}{\Gamma(\gamma)} \int_0^1 (1-v)^{\gamma-1} v^{p-\gamma-1} dv \Big\} \\
&= -I^p h(t) \\
&\quad + \frac{t^{p-1}\Gamma(p-\gamma)}{\Gamma(p)(1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} s^{p-\gamma-1} dA(s))} \Big(I^{p-\gamma} h(1) - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} I^{p-\gamma} h(s) dA(s) \Big).
\end{aligned}$$

Assume that $\mathscr{C} = C([0,1],\mathbb{R})$ denotes the Banach space endowed with the norm defined by $\|u\| = \sup_{t\in[0,1]} |u(t)|$.

Next, we introduce an operator $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ as

$$\begin{aligned}
(\mathscr{T}x)(t) = -A_1 \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} f_1(s,x(s)) ds - A_2 \int_0^t \frac{(t-s)^{p+q-1}}{\Gamma(p+q)} f_2(s,x(s)) ds \\
+ t^{p-1}\theta \Big[A_1 \int_0^1 \frac{(1-s)^{p-\gamma-1}}{\Gamma(p-\gamma)} f_1(s,x(s)) ds \\
+ A_2 \int_0^1 \frac{(1-s)^{p-\gamma+q-1}}{\Gamma(p-\gamma+q)} f_2(s,x(s)) ds \\
- A_1 \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^s \frac{(s-\eta)^{p-\gamma-1}}{\Gamma(p-\gamma)} f_1(\eta,x(\eta)) d\eta dA(s) \\
- A_2 \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^s \frac{(s-\eta)^{p-\gamma+q-1}}{\Gamma(p-\gamma+q)} f_2(\eta,x(\eta)) d\eta dA(s) \Big],
\end{aligned} \tag{5}$$

where

$$\theta = \frac{\Gamma(p-\gamma)}{\Gamma(p)[1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} s^{p-\gamma-1} dA(s)]}.$$

It can be said that x is a solution of the fractional BVP (1) if and only if x is a fixed point of the operator $\mathscr{T}$ on $\mathscr{C}$. For easy statement, denote

$$\Lambda_1 = |A_1|\Big[\frac{1}{\Gamma(p+1)} + \theta\Big(\frac{1}{\Gamma(p-\gamma+1)} + \sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i}\int_0^s \frac{(s-\eta)^{p-\gamma+1}}{\Gamma(p-\gamma)} d\eta dA(s)\Big)\Big],$$

$$\Lambda_2 = |A_2|\Big[\frac{1}{\Gamma(p+q+1)} + \theta\Big(\frac{1}{\Gamma(p-\gamma+q+1)} + \sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i}\int_0^s \frac{(s-\eta)^{p-\gamma+q+1}}{\Gamma(p-\gamma+q)} d\eta dA(s)\Big)\Big].$$

3. Main results

By using the Leray-Schauder's nonlinear alternative theorem [2], the Banach's fixed point theorem and Boyd-Wong Contraction Principle [7], we deal with the existence of solution for the fractional BVP (1).

THEOREM 1. *Suppose that $f_1, f_2 : [0,1]\times\mathbb{R} \to \mathbb{R}$ are continuous functions and $f_1(t,0) \not\equiv 0$ or $f_2(t,0) \not\equiv 0$ on $t \in [0,1]$. Assume that:*

(A_1) *There exist functions $w, w_1 \in L^1([0,1],\mathbb{R}^+)$ and nondecreasing functions $\psi, \psi_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|f_1(t,x)| \leqslant w(t)\psi(\|x\|), \quad |f_2(t,x)| \leqslant w_1(t)\psi_1(\|x\|),$$

for all $(t,x) \in [0,1]\times\mathbb{R}$.

(A_2) *There exists a constant $\iota > 0$ such that*

$$\frac{\iota}{\psi(\iota)||p||\Lambda_1 + \psi_1(\iota)||p_1||\Lambda_1} > 1.$$

Then, the fractional BVP (1) has at least one solution on $[0,1]$.

Proof. By taking into account the operator $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ with

$$\begin{aligned}(\mathscr{T}x)(t) = &-A_1\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} f_1(s,x(s))ds - A_2\int_0^t \frac{(t-s)^{p+q-1}}{\Gamma(p+q)} f_2(s,x(s))ds \\ &+ t^{p-1}\theta\Big[A_1\int_0^1 \frac{(1-s)^{p-\gamma-1}}{\Gamma(p-\gamma)} f_1(s,x(s))ds \\ &+ A_2\int_0^1 \frac{(1-s)^{p-\gamma+q-1}}{\Gamma(p-\gamma+q)} f_2(s,x(s))ds \\ &- A_1\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\int_0^s \frac{(s-\eta)^{p-\gamma-1}}{\Gamma(p-\gamma)} f_1(\eta,x(\eta))d\eta dA(s) \\ &- A_2\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\int_0^s \frac{(s-\eta)^{p-\gamma+q-1}}{\Gamma(p-\gamma+q)} f_2(\eta,x(\eta))d\eta dA(s)\Big],\end{aligned}$$

we take the equation $x = \lambda \mathscr{T} x$ for $\lambda \in (0,1)$ and let x be a solution. After that, the following is obtained.

$$\begin{aligned}
\|x\| &= \|\lambda(\mathscr{T}x)\| \\
&\leqslant |A_1|\psi(||x||)||w||\Big[\frac{1}{\Gamma(p+1)} + \theta\Big(\frac{1}{\Gamma(p-\gamma+1)} \\
&\quad + \sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\int_0^s \frac{(s-\eta)^{p-\gamma+1}}{\Gamma(p-\gamma)}d\eta dA(s)\Big)\Big] \\
&\quad + |A_2|\psi_1(||x||)||w_1||\Big[\frac{1}{\Gamma(p+q+1)} + \theta\Big(\frac{1}{\Gamma(p-\gamma+q+1)} \\
&\quad + \sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\int_0^s \frac{(s-\eta)^{p-\gamma+q+1}}{\Gamma(p-\gamma+q)}d\eta dA(s)\Big)\Big] \\
&\leqslant \psi(||x||)||w||\Lambda_1 + \psi_1(||x||)||w_1||\Lambda_2,
\end{aligned}$$

and consequently

$$\frac{\|x\|}{\psi(||x||)||w||\Lambda_1 + \psi_1(||x||)||w_1||\Lambda_2} \leqslant 1.$$

From (A_2), there exists ι such that $\|x\| \neq \iota$. Let us set

$$K = \{x \in C([0,1],\mathbb{R}) : \|x\| < \iota\}.$$

Obviously, the operator $\mathscr{T} : \bar{K} \to C([0,1],\mathbb{R})$ is completely continuous. From the choice of K, there is no $x \in \partial K$ such that $x = \lambda \mathscr{T}(x)$ for some $\lambda \in (0,1)$. As a result, by the Leray-Schauder's nonlinear alternative theorem, $\mathscr{T}$ has a fixed point $x \in \bar{\mathrm{K}}$ which is a solution of the fractional BVP (1). The proof is completed. □

THEOREM 2. *Assume that* $f_1, f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and* $f_1(t,0) \neq 0$ *or* $f_2(t,0) \neq 0$ *on* $t \in [0,1]$ *satisfying the condition*

(A_3) $|f_1(t,x) - f_1(t,y)| \leqslant L_1|x-y|$, $|f_2(t,x) - f_2(t,y)| \leqslant L_2|x-y|$, *for* $t \in [0,1]$, $L_1, L_2 > 0$, $x, y \in \mathbb{R}$.

Then the fractional BVP (1) *has a unique solution if* $L < \frac{1}{\Lambda_1+\Lambda_2}$, *where* $L = \max\{L_1, L_2\}$.

Proof. Let $\sup_{t\in[0,1]}|f_1(t,0)| = M_1$ and $\sup_{t\in[0,1]}|f_2(t,0)| = M_2$. Assume that $M = \max\{M_1, M_2\}$, Choosing $r > \dfrac{(\Lambda_1+\Lambda_2)M}{1-L(\Lambda_1+\Lambda_2)}$, we indicate that $\mathscr{T}K_r \subset K_r$, where $K_r = \{x \in \mathscr{C} : \|x\| \leqslant r\}$. For $x \in K_r$, from (A_3) $|f_1(s,x(s))| \leqslant |f_1(s,x(s)) - f_1(s,0)| + |f_1(s,0)| \leqslant L_1 r + M_1$, $|f_2(s,x(s))| \leqslant |f_2(s,x(s)) - f_2(s,0)| + |f_2(s,0)| \leqslant L_2 r + M_2$. By

(A_3), for $x \in K_r$, we obtain that

$$
\begin{aligned}
&\|(\mathscr{T}x)\| \\
&\leqslant (Lr+M) \sup_{t\in[0,1]} \Big\{ |A_1| \Big[\frac{1}{\Gamma(p+1)} \\
&\quad + \theta \Big(\frac{1}{\Gamma(p-\gamma+1)} + \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^s \frac{(s-\eta)^{p-\gamma+1}}{\Gamma(p-\gamma)} d\eta dA(s) \Big) \Big] \\
&\quad + |A_2| \Big[\frac{1}{\Gamma(p+q+1)} + \theta \Big(\frac{1}{\Gamma(p-\gamma+q+1)} + \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^s \frac{(s-\eta)^{p-\gamma+q+1}}{\Gamma(p-\gamma+q)} d\eta dA(s) \Big) \Big] \Big\} \\
&\leqslant (Lr+M)(\Lambda_1+\Lambda_2) < r.
\end{aligned}
$$

If $x, y \in \mathscr{C}$, and $t \in [0,1]$, then

$$
\begin{aligned}
&\|\mathscr{T}x - \mathscr{T}y\| \\
&\leqslant \sup_{t\in[0,1]} \Big\{ |A_1| \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} |f_1(s,x(s)) - f_1(s,y(s))| ds \\
&\quad + |A_2| \int_0^t \frac{(t-s)^{p+q-1}}{\Gamma(p+q)} |f_2(s,x(s)) - f_2(s,y(s))| ds \\
&\quad + \theta t^{p-1} \Big[|A_1| \int_0^1 \frac{(1-s)^{p-\gamma-1}}{\Gamma(p-\gamma)} |f_1(s,x(s)) - f_2(s,y(s))| ds \\
&\quad + |A_2| \int_0^1 \frac{(1-s)^{p-\gamma+q-1}}{\Gamma(p-\gamma+q)} |f_2(s,x(s)) - f_2(s,y(s))| ds \\
&\quad + |A_1| \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^s \frac{(s-\eta)^{p-\gamma-1}}{\Gamma(p-\gamma)} |f_1(\eta,x(\eta)) - f_1(\eta,y(\eta))| d\eta dA(s) \\
&\quad + |A_2| \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^s \frac{(s-\eta)^{p-\gamma+q-1}}{\Gamma(p-\gamma+q)} |f_2(\eta,x(\eta)) - f_2(\eta,y(\eta))| d\eta dA(s) \Big] \Big\} \\
&\leqslant L \Big\{ |A_1| \Big[\frac{1}{\Gamma(p+1)} + \theta \Big(\frac{1}{\Gamma(p-\gamma+1)} + \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^s \frac{(s-\eta)^{p-\gamma+1}}{\Gamma(p-\gamma)} d\eta dA(s) \Big) \Big] \\
&\quad + |A_2| \Big[\frac{1}{\Gamma(p+q+1)} + \theta \Big(\frac{1}{\Gamma(p-\gamma+q+1)} \\
&\quad + \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^s \frac{(s-\eta)^{p-\gamma+q+1}}{\Gamma(p-\gamma+q)} d\eta dA(s) \Big) \Big] \Big\} \|x-y\| \\
&= L(\Lambda_1+\Lambda_2)\|x-y\|.
\end{aligned}
$$

As $L < 1/\Lambda_1 + \Lambda_2$, $\mathscr{T}$ is a contraction. Hence, by the Banach's fixed point theorem, the fractional BVP (1) has a unique solution. The proof is completed. □

EXAMPLE 1. Consider the following fractional boundary value problem

$$\begin{cases} -D^{3/2}x(t) = f_1(t,x(t)) + I^{1/3}f_2(t,x(t)), \quad 1 < p \leqslant 2, \quad t \in (0,1), \\ D^{1/4}x(0) = 0, \quad D^{1/4}x(1) = \frac{1}{2}\int_0^{\frac{1}{4}} D^{1/4}x(s)ds + \frac{1}{2}\int_{\frac{1}{3}}^{1} D^{1/4}x(s)ds. \end{cases} \tag{6}$$

Here $n = 2,\ p = 3/2,\ \gamma = 1/4,\ q = 1/3,\ a_1 = 1/2,\ a_2 = 0,\ a_3 = 1/2,\ \eta_0 = 0,\ \eta_1 = 1/4,\ \eta_2 = 1/3,\ \eta_3 = 1$, and $A(s) = s,\ f_1(t,x) = \dfrac{1}{t^2+10}\cos x,\ f_2(t,x) = \dfrac{1}{t^2+12}\sin x$. As $|f_1(t,x) - f_1(t,y)| \leqslant \dfrac{1}{10}|x-y|$, and $|f_2(t,x) - f_2(t,y)| \leqslant \dfrac{1}{12}|x-y|$. Then, (A_3) is satisfied with $L = \max\{L_1, L_2\} = \dfrac{1}{10}$. Further, $\Lambda_1 = 2.122,\ \Lambda_2 = 1.659$ and

$$L(\Lambda_1 + \Lambda_2) \simeq 0.3781 < 1.$$

Therefore, by the conclusion of Theorem 2, the fractional BVP (6) has a unique solution.

Now we present another variant of existence-uniqueness result. This result is based on Boyd-Wong Contraction Principle.

DEFINITION 3. Assume that $\mathtt{E}$ is a Banach space and $T : \mathtt{E} \to \mathtt{E}$ is a mapping. If there exists a continuous nondecreasing function $\psi : \mathtt{R}^+ \to \mathtt{R}^+$ such that $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:

$$||Tx - Ty|| \leqslant \psi(||x-y||), \qquad \forall x, y \in \mathtt{F}.$$

then, we say that T is a nonlinear contraction.

THEOREM 3. (Boyd-Wong Contraction Principle) [7] *Suppose that $\mathtt{B}$ is a Banach space and $T : \mathtt{B} \to \mathtt{B}$ is a nonlinear contraction. Then T has a unique fixed point in $\mathtt{B}$.*

THEOREM 4. *Assume that $f_1, f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and H_1, $H_2 > 0$ satisfying the condition*

(A_4) $|f_1(t,x) - f_1(t,y)| \leqslant \dfrac{|x-y|}{H_1 + |x-y|}$, $|f_2(t,x) - f_2(t,y)| \leqslant \dfrac{|x-y|}{H_2 + |x-y|}$, *for* $t \in [0,1]$, $x, y \in \mathbb{R}$.

Then the fractional BVP (1) has a unique solution on $[0,1]$.

Proof. We define an operator $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ as in (5) and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(\varepsilon) = \frac{H\varepsilon}{H+\varepsilon}, \forall \varepsilon \geqslant 0,$$

where $\Lambda_1+\Lambda_2 \leqslant H \leqslant \min\{H_1,\ H_2\}$. We notice that the function ψ satisfies $\psi(0)=0$ and $\psi(\varepsilon)<\varepsilon$ for all $\varepsilon>0$. For any $x,y\in\mathscr{C}$, and for each $t\in[0,1]$, we obtain

$$\begin{aligned}
&\|\mathscr{T}x-\mathscr{T}y\| \\
&\leqslant \sup_{t\in[0,1]}\Big\{|A_1|\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)}|f_1(s,x(s))-f_1(s,y(s))|ds \\
&\quad +|A_2|\int_0^t \frac{(t-s)^{p+q-1}}{\Gamma(p+q)}|f_2(s,x(s))-f_2(s,y(s))|ds \\
&\quad +\theta t^{p-1}\Big[|A_1|\int_0^1 \frac{(1-s)^{p-\gamma-1}}{\Gamma(p-\gamma)}|f_1(s,x(s))-f_2(s,y(s))|ds \\
&\quad +|A_2|\int_0^1 \frac{(1-s)^{p-\gamma+q-1}}{\Gamma(p-\gamma+q)}|f_2(s,x(s))-f_2(s,y(s))|ds \\
&\quad +|A_1|\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\int_0^s \frac{(s-\eta)^{p-\gamma-1}}{\Gamma(p-\gamma)}|f_1(\eta,x(\eta))-f_1(\eta,y(\eta))|d\eta dA(s) \\
&\quad +|A_2|\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\int_0^s \frac{(s-\eta)^{p-\gamma+q-1}}{\Gamma(p-\gamma+q)}|f_2(\eta,x(\eta))-f_2(\eta,y(\eta))|d\eta dA(s)\Big]\Big\} \\
&\leqslant \frac{|x-y|}{H_1+|x-y|}\Lambda_1+\frac{|x-y|}{H_2+|x-y|}\Lambda_2 \\
&\leqslant \psi(||x-y||).
\end{aligned}$$

Then, we get $||Tx-Ty||\leqslant\psi(||x-y||)$. Hence, T is a nonlinear contraction. Thus, by Theorem 3 the operator T has a unique fixed point which is the unique solution of the fractional BVP (1). The proof is completed. □

EXAMPLE 2. Consider the following fractional boundary value problem

$$\begin{cases} -D^{5/2}x(t)=f_1(t,x(t))+I^{1/2}f_2(t,x(t)), \quad 2<p\leqslant 3, \quad t\in(0,1), \\ D^{1/4}x(0)=D^{5/4}x(0)=0, \quad D^{1/4}x(1)=\frac{1}{2}\int_0^{\frac{1}{8}}D^{1/4}x(s)ds+\frac{1}{2}\int_{\frac{1}{8}}^{\frac{1}{4}}D^{1/4}x(s)ds. \end{cases} \tag{7}$$

Here $n=3,\ p=5/2,\ \gamma=1/4,\ q=1/2,\ a_1=1/2,\ a_2=1/2,\ a_3=0,\ \eta_0=0,$ $\eta_1=1/8,\ \eta_2=1/4,\ \eta_3=1$ and $A(s)=s$, $f_1(t,x)=\dfrac{sint}{t+1}\cdot\dfrac{|x|}{1+|x|},\ f_2(t,x)=\dfrac{1}{t^2+1}\cdot\dfrac{|x|}{2+|x|}$. We choose $H_1=1,\ H_2=2,\ H=0.9$ and we obtain $\Lambda_1=0.6717,\ \Lambda_2=0.167$ and $\Lambda_1+\Lambda_2=0.8387\leqslant H=0.9\leqslant\min\{H_1,\ H_2\}=1$. Clearly

$$|f_1(t,x)-f_1(t,y)|\leqslant\frac{|x-y|}{1+|x-y|},\quad |f_2(t,x)-f_2(t,y)|\leqslant\frac{|x-y|}{2+|x-y|},$$

for $t\in[0,1]$, $x,y\in\mathbb{R}$.

Hence, by Theorem 4, the fractional BVP (7) has a unique solution.

REFERENCES

[1] R. P. AGARWAL, S. K. NTOUYAS, B. AHMAD, M. S. ALHOTHUALI, *Existence of solutions for integro-differential equations of fractional order with nonlocal three-point fractional boundary conditions*, Advances in Difference Equations, (2013), 2013:128.
[2] R. P. AGARWAL, M. MEEHAN, D. OREGAN, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridje, (2001).
[3] B. AHMAD AND J. J. NIETO, *Existence Results for Nonlinear Boundary Value Problems of Fractional Integrodifferential Equations with Integral Boundary Conditions*, Boundary Value Problems, 2009, Article ID 708576, 11 pp.
[4] A. ANBER, S. BELARBI, Z. DAHMANI, *New Existence and Uniqueness Results for Fractional Differential Equations*, Analele Stiintifice ale Universitatii Ovidius Constanta, Vol. 21 (3), 2013, 33–41.
[5] A. BENMEZAI, A. SAADI, *Positive Solutions for Boundary Value Problems of N-Dimension Nonlinear Fractional Differential System with Integral Boundary Conditions*, Fractional Differential Calculus, Vol. 7, No. 2 (2017), 185–197.
[6] A. BOUCHERIF, S. K. NTOUYAS, *Integro-Differential Equations of Fractional order with Nonlocal Fractional Boundary Conditions Associated with Financial Asset Model*, Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 60, pp. 1–10.
[7] D. W. BOYD, J. S. W WONG, *On nonlinear contractions*, Proceedings of the American Mathematical Society **20**, 458–464 (1969).
[8] A. CABADA, Z. HAMDI, *Nonlinear fractional differential equations with integral boundary value conditions*, Applied Mathematics and Computation **228** (2014), 251–257.
[9] A. CABADA, G. WANG, *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, Journal of Mathematical Analysis and Applications **389** (2012), 403–411.
[10] Z. DAHMANIA, S. BELARBI, *New results for fractional evolution equations using Banach fixed point theorem*, International Journal of Nonlinear Analysis and Applications **5** (2014), No. 2, 22–30.
[11] A. GUEZANE-LAKOUDA, R. KHALDI, *Solvability of a fractional boundary value problem with fractional integral condition*, Nonlinear Analysis **75**(2012), 2692–2700.
[12] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics Studies, Vol. 204, Elsevier Science B. V, Amsterdam, 2006.
[13] Y. LIU, *Existence of Global Solutions of Impulsive IVPS of Singular Fractional Differential Systems on Half Line*, Fractional Differential Calculus, Vol. 6, No: 1 (2016), 35–56.
[14] N. NYAMORADI, T. BASHIRI, S. M. VAEZPOUR, D. BALEANU, *Uniqueness and existence of positive solutions for singular fractional differential equations*, Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 130, pp. 1–13.
[15] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego (1999).
[16] W. SUDSUTAD, J. TARIBOON, *Boundary value problems for fractional differential equations with three-point fractional integral boundary conditions*, Advance in Difference Equations, (2012) 2012:93.
[17] W. SUDSUTAD, J. TARIBOON, *Existence results of fractional integro-differential equations with m-point multi-term fractional order integral boundary conditions*, Boundary Value Problems, (2012) 2012:94.
[18] X. ZHANG, *Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions*, Applied Mathematics Letters **39** (2015), 22–27.

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