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SUCCESSIVE APPROXIMATIONS OF SOLUTIONS TO THE CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We consider an initial value problem involving a single term Caputo differential equation of fractional order strictly greater than one. For those with right hand sides that satisfy an Osgood type condition, we show that there exist successive approximations which converge to the solution at an exponential rate. As an application of this result, we study the Ulam-Hyers stability of these problems.

1. Introduction

The problem under consideration consists of the following single-term Caputo fractional differential equation:

$$ {}^{C}D^{q}[x](t) = f(t,x(t)) \tag{1} $$

coupled with the following initial conditions:

$$ x^{(i)}(0) = A_i, i = 0,1,\dots,\lceil q \rceil - 1, \tag{2} $$

where $\lceil q \rceil$ denotes the smallest integer greater than or equal to q.

Above, f is a real-valued continuous function defined on a rectangular region $[0,a] \times [-b,b]$ and the $A_i \in \mathbb{R}$ are constants.

For certain subclasses for the function f (e.g. linear in the second variable) the above problem can be solved explicitly [3, Section 7.1]. However for even slightly less amenable right hand sides, there are no existing methods for constructing explicit solutions. In such cases it is of interest to study approximations to the solution of the initial value problem (1)-(2).

For ordinary differential equations with quite general right-hand side (namely satisfying the Osgood-type condition defined in Preliminaries below) such results were proved by A. Wintner [15]. More recently, under some more restrictive assumptions on the right-hand side, it was shown that the rate of convergence is exponentially fast [2].

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For fractional initial value problems, there have been a number of results regarding the case $0 < q < 1$. The IVPs with a Krasnoselskii-Krein type condition on f were investigated in [10, 8] and shown to have uniform convergence of Picard style approximations. The paper [9] also looked at Picard approximations for f satisfying a condition involving Osgood's criterion (similar to the one we consider below) and proved uniform convergence (see Remark 1 below for a rigorous explanation).

Recently, the present authors with C. C. Tisdell proved that the IVP (1)-(2) for $q > 1$ has a unique solution when f satisfies the Osgood-type condition [13].

In the present paper we show that for such initial value problems, we can construct a sequence of successive approximations which converge to the solution in the uniform norm of the space of continuous functions on an interval. This result generalises the result of [15] to the fractional case and extends that of [9] to the case of $q > 1$. Also, under the same conditions on the right-hand side as in [2], we estimate the rate of this convergence. In Section 4 we employ the convergence result to study the stability of the initial value problem (1)-(2). For $0 < q < 1$ and Lipschitz right-hand-side, this was studied in [14]. For arbitrary $q > 0$ and Lipschitz right-hand-side this was investigated in [7] for implicit differential equations. Our consideration of the Osgood-type right-hand side broadly complements these results.

2. Preliminaries

Let $\mathbb{N} := \{1,2,3,\ldots\}$ be the set of all natural numbers. For $n \in \mathbb{N}$, let $A^n[0,a]$ be the space of functions on $[0,a]$ with an absolutely continuous $(n-1)$-st derivative.

The Riemann–Liouville fractional integral of order $q > 0$ of $f \in L_1[0,a]$ is defined by the formula

$$I^q[f](t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\, ds,\ t \in [0,a],$$

where Γ is the Gamma function. For $q = 0$, we set I^0 to be an identity operator. Throughout the paper we denote $m := \lceil q \rceil$.

The Caputo fractional derivative of order $q > 0$ of $f \in A^m[0,a]$ is defined by

$${}^C D^q[f](t) = I^{m-q} \left[\frac{d^m f}{dt^m}\right](t).$$

Throughout the paper we fix $q \geqslant 1$.

We define the class of functions which are central to the problem under consideration. Initially this class was introduced in [12] (see also [1]).

DEFINITION 1. A continuous, non-decreasing function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$, $g(z) > 0$ if $z > 0$ is said to satisfy Osgood's condition if

$$\int_0^1 \frac{dz}{g(z)} = \infty. \tag{3}$$

The following simple Gronwall-type result was established in [13].

LEMMA 1. *Let g be a function satisfying Osgood's condition, $a > 0$ and let $(t,s) \mapsto k(t,s)$ be a real-valued function bounded on a triangular region $0 \leqslant t \leqslant a$, $0 \leqslant s \leqslant t$. Let $\phi : [0,a] \to [0,\infty)$ be continuous. If*

$$\phi(t) \leqslant \int_0^t k(t,s)g(\phi(s))\,ds \tag{4}$$

for $0 \leqslant t \leqslant a$, then $\phi \equiv 0$ on $[0,a]$.

For any $a > 0$ we denote $C[0,a]$ to be the space of all functions $f : [0,a] \to \mathbb{R}$ continuous on $[0,a]$, equipped with the uniform norm

$$\|f\| := \max_{0 \leqslant t \leqslant a} |f(t)|.$$

Now fix $a > 0$ and

$$l > \sum_{i=0}^{m-1} \frac{|A_i|}{i!} a^i.$$

Define the interval $I = [-l,l]$ and denote $C([0,a] \times I)$ to be the space of all continuous real valued functions defined on $[0,a] \times I$, equipped with the uniform norm. Throughout the paper we fix $f \in C([0,a] \times I)$ such that for all $t \in [0,a]$ and $x,y \in I$

$$|f(t,x) - f(t,y)| \leqslant g(|x-y|) \tag{5}$$

for some g satisfying Osgood's condition and say that f satisfies the Osgood type condition.

3. Convergence of successive approximations

Define

$$z = \min\left\{ a, \left[\frac{\Gamma(q+1)}{\|f\|_{C([0,a]\times I)}} \left(l - \sum_{i=0}^{m-1} \frac{|A_i|}{i!} a^i \right) \right]^{1/q} \right\} > 0 \tag{6}$$

and the operator F on $C[0,z]$ by setting

$$F[x](t) = \sum_{i=0}^{m-1} \frac{A_i}{i!} t^i + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\,ds. \tag{7}$$

Due to the choice of z it is straightforward to see that $\|F[x]\|_{C[0,z]} \leqslant l$ provided $\|x\|_{C[0,z]} \leqslant l$. Thus we introduce a well defined sequence of successive approximations as follows:

$$x_0 \in C[0,z] \text{ such that } \|x_0\|_{C[0,z]} \leqslant l \text{ and } x(0) = A_0;\ x_{k+1} = F[x_k], k \geqslant 0. \tag{8}$$

THEOREM 1. *Consider the IVP (1)-(2). Let* $\{x_k\}_{k=0}^{\infty}$ *be the sequence of successive approximations defined by (8). If the right-hand side of the IVP satisfies the Osgood type condition, then the sequence* $\{x_k\}_{k=0}^{\infty}$ *converges uniformly on* $[0,z]$ *(where* z *is as in (6)) to the unique continuous solution of the the IVP (1)-(2).*

Proof.
First we show that $\{x_k\}_{k=0}^{\infty}$ is pointwise convergent. For all $t \in [0,z]$, let

$$\mu(t) = \lim_{k\to\infty} \sup_{m\geqslant k} |x_k(t) - x_m(t)|.$$

By definition of our successive approximations, we obtain

$$\begin{aligned}
\mu(t) &= \lim_{k\to\infty} \sup_{m\geqslant k} \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(f(s,x_{k-1}(s)) - f(s,x_{m-1}(s)\right) \, ds \right| \\
&\leqslant \frac{1}{\Gamma(q)} \lim_{k\to\infty} \sup_{m\geqslant k} \int_0^t (t-s)^{q-1} g(|x_{k-1}(s) - x_{m-1}(s)|) \, ds \\
&\leqslant \frac{1}{\Gamma(q)} \lim_{k\to\infty} \int_0^t (t-s)^{q-1} \sup_{m\geqslant k} g(|x_{k-1}(s) - x_{m-1}(s)|) \, ds \\
&= \frac{1}{\Gamma(q)} \lim_{k\to\infty} \int_0^t (t-s)^{q-1} \; g(\sup_{m\geqslant k} |x_{k-1}(s) - x_{m-1}(s)|) \, ds,
\end{aligned}$$

since g is non-decreasing. Next the use of the Dominated convergence theorem yields

$$\begin{aligned}
\mu(t) &\leqslant \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \; \lim_{k\to\infty} g(\sup_{m\geqslant k} |x_{k-1}(s) - x_{m-1}(s)|) \, ds \\
&= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \; g(\lim_{k\to\infty} \sup_{m\geqslant k} |x_{k-1}(s) - x_{m-1}(s)|) \, ds \;\; \text{(since } g \text{ is continuous)} \\
&= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(\mu(s)) \, ds.
\end{aligned}$$

By Lemma 1, $\mu(t)=0$, and it follows that for each $t\in[0,z]$ the sequence $\{x_k(t)\}_{k=1}^{\infty}$ is Cauchy and hence convergent. Consequently, we can define the pointwise limit function $x^*(t) = \lim_{k\to\infty} x_k(t)$ on $[0,z]$.

Next we show the sequence $\{x_k\}_{k=0}^{\infty}$ is uniformly equicontinuous. Let $t_1, t_2 \in [0,z]$ with $t_1 \leqslant t_2$. For every $k \in \mathbb{N}$, using (8), we obtain

$$\begin{aligned}
|x_k(t_1) - x_k(t_2)| = \Bigg| &\left[\sum_{i=0}^{m-1} \frac{A_i}{i!} {t_2}^i + \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} f(s,x_{k-1}(s)) \, ds \right] \\
&- \left[\sum_{i=0}^{m-1} \frac{A_i}{i!} {t_1}^i + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} f(s,x_{k-1}(s)) \, ds \right] \Bigg|
\end{aligned}$$

$$\leqslant \left|\sum_{i=1}^{m-1}\frac{A_i}{i!}({t_2}^i-{t_1}^i)\right|+\frac{1}{\Gamma(q)}\left|\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]f(s,x_{k-1}(s))\,ds\right.$$
$$\left.+\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x_{k-1}(s))\,ds\,\right|$$
$$\leqslant \sum_{i=1}^{m-1}\frac{A_i}{i!}\left|{t_2}^i-{t_1}^i\right|+\frac{1}{\Gamma(q)}\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]\left|f(s,x_{k-1}(s))\right|\,ds$$
$$+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\left|f(s,x_{k-1}(s))\right|\,ds.$$

Since the functions $t\mapsto t^i$ are Lipschitz and f is bounded on $[0,a]\times I$, we can write

$$|x_k(t_1)-x_k(t_2)|\leqslant \sum_{i=1}^{m-1}\frac{A_i}{i!}K_i|t_2-t_1|+\frac{\|f\|_{C([0,z]\times I)}}{\Gamma(q)}\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]\,ds$$
$$+\frac{\|f\|_{C([0,z]\times I)}}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\,ds,$$

where the $K_i = iz^{i-1}$, $i=1,\dots,m$ are Lipschitz constants of $t\mapsto t^i$. Then letting $Z=\sum_{i=1}^{m-1}\frac{A_i}{i!}K_i$ and using the fact that

$$\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]\,ds+\int_{t_1}^{t_2}(t_2-s)^{q-1}\,ds=\frac{t_2^q-t_1^q}{q},$$

we have

$$|x_k(t_1)-x_k(t_2)|\leqslant Z|t_2-t_1|+\frac{\|f\|_{C([0,z]\times I)}}{\Gamma(q)}\left(\frac{t_2^q-t_1^q}{q}\right)\leqslant Z|t_1-t_2|+\frac{\|f\|_{C([0,z]\times I)}L\,|t_1-t_2|}{\Gamma(q+1)},$$

since the function $t\mapsto t^q$ is Lipschitz for $q\geqslant 1$ and $L=qz^{q-1}$ is its Lipschitz constant.

This shows that the sequence $\{x_k\}_{k=1}^{\infty}$ is uniformly equicontinuous on $[0,z]$. Together with pointwise convergence this shows that x_k converges to x^* uniformly and that x^* is continuous.

Now showing that x^* is a solution to the IVP is equivalent to showing that x^* is a fixed point of F. To this end, consider the quantity $|x^*(t)-F[x^*](t)|$ for arbitrary t. For every $k=0,1,2,\dots$ we get

$$0\leqslant |x^*(t)-F[x^*](t)|\leqslant |x^*(t)-x_k(t)|+|x_k(t)-F[x^*](t)|. \tag{9}$$

Firstly, it was shown above that

$$\lim_{k\to\infty}|x^*(t)-x_k(t)|=0 \tag{10}$$

uniformly in $t\in[0,z]$. Secondly,

$$\lim_{k\to\infty}|x_k(t)-F[x^*](t)|=\frac{1}{\Gamma(q)}\lim_{k\to\infty}\left|\int_0^t(t-s)^{q-1}\left[f(s,x_{k-1}(s))-f(s,x^*(s))\right]\,ds\right|$$

$$\leqslant \frac{1}{\Gamma(q)} \lim_{k\to\infty} \int_0^t (t-s)^{q-1} g\left(|x_{k-1}(s)) - x^*(s)|\right)\, ds$$
$$= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \lim_{k\to\infty} g\left(|x_{k-1}(s)) - x^*(s)|\right)\, ds,$$

by the Dominated convergence theorem. Since g is continuous and $g(0)=0$, it follows that

$$\lim_{k\to\infty} |x_k(t) - F[x^*](t)| \leqslant \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g\left(\lim_{k\to\infty} |x_{k-1}(s) - x^*(s)|\right)\, ds$$
$$= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(0)\, ds = 0,$$

where the penultimate equality is by (10). Combining this with (9) yields

$$x^*(t) = F[x^*](t),$$

for all t and we conclude that the limit function x^* is a solution to the IVP. This proves the assertion.

REMARK 1. The convergence of successive approximations for $q \in (0,1)$ is proved with somewhat similar and somewhat more restrictive assumptions on g in [9], using a very interesting technique. We state their result rigorously. They consider the IVP (1)-(2) with $0 < q < 1$ and a continuous right-hand side satisfying

$$|f(t,x) - f(t,y)| \leqslant g(t, |x-y|) \tag{11}$$

for a continuous non-decreasing in the second variable function g such that $g(t,0)=0$ and such that the IVP

$${}^{C}D^{q}[x](t) = g(t,x), \quad x(0) = 0$$

has only the trivial solution. Whereas the condition (11) is more general then that considered in this paper, the other condition on the uniqueness of the trivial solution is not very transparent. The paper [11] shows that the latter condition is quite complicated and fairly restrictive.

3.1. Rate of convergence

Throughout this subsection we impose a stronger condition on g than the stated Osgood condition. This is a fairly natural condition, as it was assumed even in the case when q is an integer [2].

DEFINITION 2. A continuous, non-decreasing function $g : [0,\infty) \to [0,\infty)$ such that $g(0)=0$, $g(t) > 0$ for $t > 0$, is said to satisfy the modified Osgood's condition if g satisfies the Osgood condition and is of the form

$$g(t) = t \int_t^b \frac{w(s)}{s}\, ds, \tag{12}$$

where $b > 0$ and $w : [0,\infty) \to \mathbb{R}$ is a non-negative, non-decreasing and continuous function such that

$$\int_0^b \frac{w(s)}{s}\, ds = \infty. \tag{13}$$

From now on we assume that $b \leqslant 1$.

For example, consider the functions g_k given by

$$g_k(0) = 0, \quad g_k(t) = t\prod_{i=1}^{k}\Bigg(\underbrace{\log\log\ldots\log}_{i\ iterations}\left(\frac{1}{t}\right)\Bigg)^{\beta_i} \tag{14}$$

for some $0 \leqslant \beta_i < 1, i = 1,...,k-1, 0 < \beta_k \leqslant 1, k = 1,2,\ldots$. A direct verification shows that all g_k satisfy the modified Osgood condition with $b \leqslant 1$ on a sufficiently small interval $[0,a]$.

Now we list some properties of functions satisfying the modified Osgood condition. Parts 1, 3, 4 and 5 of the following lemma appeared and were proved in [2]. Part 2 follows directly from the definition of the modified Osgood condition, in particular from the condition (13). The last part is given with proof since it is slightly modified for our situation. Recall that z is defined by the expression (6).

LEMMA 2. *Let g satisfy the modified Osgood's condition with $b \leqslant 1$. There exists $0 < \delta \leqslant z$ (depending only on g) such that for all $0 < t \leqslant \delta$ the following hold:*

1. *The quantity*

$$C_\delta := \max_{0<t\leqslant\delta} \frac{g(t)}{tg'(t)}$$

   is finite.

2. *There exists a constant $\tilde{C}_\delta$ such that $t \leqslant \tilde{C}_\delta g(t)$.*

3. *For $\lambda \geqslant 1$, $k = 1,2,3,\ldots$ we have*

$$g(\lambda g(t)^k) \leqslant k\frac{\lambda}{b}\frac{g(t)^{k+1}}{t}. \tag{15}$$

4. *For $k = 1,2,3,\ldots$ we have*

$$\int_0^t g(s)^k\, ds \leqslant \frac{g(t)^{k+1}}{k+1}. \tag{16}$$

5. *For $k = 1,2,3,\ldots$ we have*

$$\int_0^t \frac{g(s)^k}{s}\, ds \leqslant \frac{C_\delta}{k-1}\, g(t)^k. \tag{17}$$

6. *Let* $y_0 : [0,\delta] \to [0,\infty)$ *such that* $g(\|y_0\|_{C[0,\delta]}) \leqslant 1$. *If the sequence of functions* $y_k : [0,\delta] \to [0,\infty)$, $k = 1,2,3,\ldots$ *is defined by*

$$y_{k+1}(t) = \int_0^t g(y_k(s))\, ds, \tag{18}$$

for $k = 0,1,2,3,\ldots$ *then*

$$y_1(t) \leqslant t \tag{19}$$

and

$$y_k(t) \leqslant \left(\frac{C_\delta}{b}\right)^{k-2} g(t)^k, \tag{20}$$

for $k = 2,3,4,\ldots$.

Proof of part 6. The first assertion follows immediately from definitions. The second is proved by induction: For the base case, since g is non-decreasing,

$$y_2(t) \leqslant \int_0^t g(s)\, ds \leqslant \frac{g(t)^2}{2} \leqslant g(t)^2$$

by part 4. Using the inductive hypothesis and parts 3 and 5, we obtain

$$y_{k+1}(t) \leqslant \int_0^t g\left(\left(\frac{C_\delta}{b}\right)^{k-2} g(s)^k\right) ds \leqslant \frac{k}{b}\left(\frac{C_\delta}{b}\right)^{k-2} \int_0^t \frac{g(s)^{k+1}}{s} ds \leqslant \left(\frac{C_\delta}{b}\right)^{k-1} g(t)^{k+1},$$

as required.

The following theorem shows that the successive approximations converge at an exponential rate on some sufficiently small interval.

THEOREM 2. *Consider a subclass of the initial value problem* (1)-(2) *for which the right-hand side satisfying* (5) *with the function* g *satisfying the modified Osgood condition with* $b \leqslant 1$. *Let* $r \in (0,1)$ *and* $\delta > 0$ *such that* $g(\delta) \leqslant \dfrac{br}{\max(C_\delta, 1)}$ *and*

$$g\left(\sum_{k=1}^{m} \frac{A_k \delta^k}{k!} + \frac{\|f\|_{C([0,z]\times I)} \cdot \delta^q}{\Gamma(1+q)}\right) \leqslant 1.$$

For $k \in \mathbb{N}$, *we have*

$$|x_{k+1}(t) - x_k(t)| \leqslant r^k,\ t \in \left[0, \min(\delta, z, \Gamma(q)^{\frac{1}{q-1}})\right]. \tag{21}$$

REMARK 2. The choice of δ in the assertion is always possible, since by Lemma 2 part 1 the quantity C_δ is finite and so, $\max(C_\delta, 1) \cdot g(\delta) \to 0$ as $\delta \to 0$.

Proof. For $k = 0,1,2,...$ set $\mu_k(t) = |x_{k+1}(t) - x_k(t)|$. Using the definition of successive approximations and the fact that f satisfies (5), we obtain

$$\mu_k(t) \leqslant \frac{t^{q-1}}{\Gamma(q)} \int_0^t g(\mu_{k-1}(s))\, ds \leqslant \int_0^t g(\mu_{k-1}(s))\, ds,\ k = 1,2,...,$$

since $t \leqslant \Gamma(q)^{\frac{1}{q-1}}$.

Let $\{y_k\}_{k=0}^{\infty}$ be the sequence from Lemma 2 part 6 with $y_0 = \|x_1 - x_0\|$ in the space $C([0, \min(\delta, z, \Gamma(q)^{\frac{1}{q-1}})])$. By the definition of successive approximations

$$\|x_1 - x_0\| \leqslant \sum_{k=1}^{m} \frac{A_k \delta^k}{k!} + \frac{\|f\|_{C([0,z]\times I)} \cdot \delta^q}{\Gamma(1+q)}.$$

So, due to the choice of δ, we have that $g(y_0) \leqslant 1$.

Since $\mu_0(t) \leqslant y_0(t)$, it follows inductively that $\mu_k(t) \leqslant y_k(t)$ for $k = 1,2,...$. Hence, Lemma 2 part 6 yields

$$\mu_k(t) \leqslant \left(\frac{C_\delta}{b}\right)^{k-2} g(t)^k \leqslant \left(\frac{b}{\max(C_\delta, 1)}\right)^2 r^k \leqslant r^k,\ t \in \left[0, \min(\delta, z, \Gamma(q)^{\frac{1}{q-1}})\right],$$

where the penultimate inequality is since $t \leqslant \delta$. This proves the assertion.

3.2. Examples

In this subsection we consider two examples, which numerically demonstrate the theoretical results obtained above.

EXAMPLE 1. Consider the following initial value problem:

$$^{C}D^{3/2}[x](t) = x(t),\ x(0) = x'(0) = 1. \tag{22}$$

The right-hand side satisfies the Osgood type condition (5) with $g(t) = t$.

By [6, Theorem 4.3] the solution to this IVP is $x(t) = E_{3/2,1}(t^{3/2}) + E_{3/2,2}(t^{3/2})$, where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^\alpha}{\Gamma(\alpha k + \beta)}, \alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0$$

are Mittag-Lefler functions defined for every $z \in \mathbb{C}$.

We take the zeroth approximation $x_0(t) = t + 1$ and construct the sequence of successive approximations by the formula (8). Figure 1 demonstrates the convergence of these successive approximations to the theoretical solution on the interval $[0,2]$. The norm of the difference between the exact solution and approximations are 2.73, 0.72, 0.13 and 0.035, respectively for each of the four graphs.

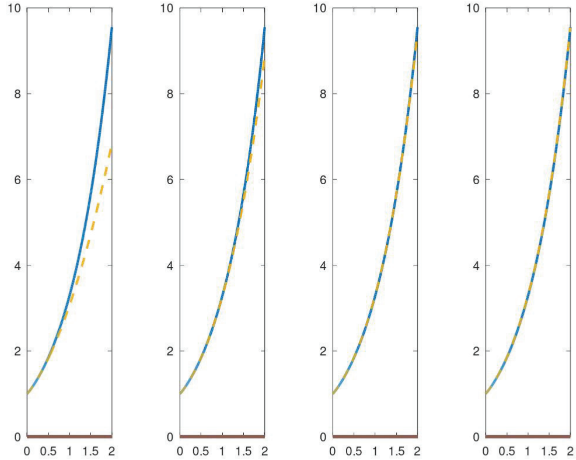

Figure 1: The exact solution to (22) (solid line) and the first four successive approximations (dashed lines).

EXAMPLE 2. Consider the following initial value problem:

$$ {}^{C}D^{5/4}[x](t) = |\sin(t)| \cdot g(x(t)),\ x(0) = x'(0) = 1, \tag{23} $$

where the function $g : [0,\infty) \to [0,\infty)$ is defined as follows:

$$ g(z) = \begin{cases} 0, & z = 0, \\ -z\log z, & 0 < z \leqslant 1/e, \\ 1/e, & z > 1/e. \end{cases} $$

The right-hand side satisfies the Osgood type condition (5) with the function g itself [1, Example 1.4.2]. Moreover, the function g satisfies the modified Osgood condition with $b = 1$.

We take the zeroth approximation $x_0(t) = t+1$ and construct the sequence x_k, $k = 1,2,\dots$ of successive approximations by the formula (8). Figure 2 demonstrates the convergence of these successive approximations on the interval $[0,2]$. The solid line on Figure 2 is the 20th approximation to the solution. At this stage the norm difference between two successive approximations is less than $3.6 \cdot 10^{-15}$.

The norms $\|x_{k+1} - x_k\|$ have values of 1.4, 0.44, 0.1 and 0.02 for $k = 1,2,3$ and 4, respectively. Hence, on the interval $[0,2]$ the sequence x_k, $k = 1,2,\dots$ does not satisfy the conclusion of Theorem 2 (even for $k = 1$). However, on a smaller interval, say $[0,1]$ we have the following values for the norms $\|x_{k+1} - x_k\|$:

k	1	2	3	4	5	6
$\|x_{k+1} - x_k\|$	0.077	0.007	0.00049	$2.8 \cdot 10^{-5}$	$1.3 \cdot 10^{-6}$	$5.7 \cdot 10^{-8}$

This suggests that the sequence x_k, $k = 1,2,\dots$ converges exponentially fast and in particular that the conclusion of Theorem 2 holds for $r = 0.1$.

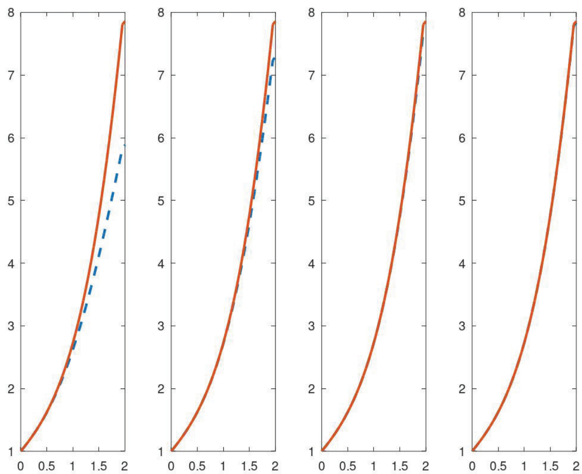

Figure 2: The 20th approximation to the solution to (23) (solid line) and the first four successive approximations (dashed lines).

4. Stability of the fractional differential equations

In this section we employ the convergence result established above to the stability of the Caputo differential equations. First we rigorously define the type of stability we deal with (see e.g. [4, 5]).

DEFINITION 3. An equation (1) is said to be Ulam-Hyers stable on the interval $[0,z]$ if there exists a constant $K_f > 0$ such that for every $\varepsilon > 0$ and $y \in C[0,z]$ satisfying

$$\left|{}^C D^q[y](t) - f(t,y(t))\right| \leqslant \varepsilon,\ \forall\, t \in [0,z], \tag{24}$$

there exists a solution x of (1)-(2) such that

$$|y(t) - x(t)| \leqslant K_f \cdot \varepsilon,\ \forall\, t \in [0,z].$$

It is straightforward to see that if a function y satisfies (24), then there exists a function $\sigma_y : [0,z] \to \mathbb{R}$ such that $|\sigma_y(t)| \leqslant \varepsilon$ for every $t \in [0,a]$ and

$${}^C D^q[y](t) = f(t,y(t)) + \sigma_y(t).$$

Equivalently, y satisfies the following equation:

$$y(t) = \sum_{i=0}^{m-1} \frac{A_i}{i!} t^i + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,y(s))\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sigma_y(s)\, ds. \tag{25}$$

The following result shows that choosing the initial approximation x_0 in Theorem 1 in a special way slightly improves the estimate (21). Below, $\lfloor q \rfloor$ denotes the greatest integer less than or equal to q.

LEMMA 3. *Consider a subclass of the initial value problems (1)-(2) with f satisfying (5), where g satisfies the modified Osgood condition with $b \leqslant 1$. Let y be a function satisfying (24) for some $\varepsilon > 0$. Set $x_0 \equiv y$ and $\{x_k\}_{k=1}^{\infty}$ be the sequence of successive approximations defined by (8). There exists $0 < \delta \leqslant \min(1,z)$ (exactly the same δ as in Lemma 2) such that for every $0 < t \leqslant \delta$ one has*

$$|x_1(t) - x_0(t)| \leqslant \varepsilon \frac{t^q}{\Gamma(q)}$$

and

$$|x_{k+1}(t) - x_k(t)| \leqslant \varepsilon B^k \frac{t^{q-1}}{\Gamma(q)} g(t)^{1+\lfloor q \rfloor k}, k = 1,2,... \tag{26}$$

where

$$B := \frac{C_\delta \cdot \tilde{C}_\delta^{\lfloor q \rfloor}}{b\Gamma(q)} > 0.$$

Here $C_\delta, \tilde{C}_\delta$ are constants from Lemma 2.

Proof. From (7) and (25) the following we have the estimate

$$|x_1(t) - x_0(t)| = |F[y](t) - y(t)| = \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sigma_y(s)\, ds \right| \leqslant \varepsilon \frac{t^q}{q\Gamma(q)} \leqslant \varepsilon \frac{t^q}{\Gamma(q)}, \tag{27}$$

since $|\sigma_y(t)| \leqslant \varepsilon$ for every $0 \leqslant t \leqslant z$ and $q > 1$.

We proceed further by induction. For $k = 1$, from (7) and the fact that f satisfies (5) we obtain

$$\begin{aligned} |x_2(t) - x_1(t)| &= \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[f(s, x_1(s) - f(s, x_0(s)) \right] ds \right| \\ &\leqslant \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g\left(|x_1(s) - x_0(s)| \right) ds \\ &\leqslant \frac{t^{q-1}}{\Gamma(q)} \int_0^t g\left(|x_1(s) - x_0(s)| \right) ds. \end{aligned}$$

Using (27) and the fact that g is non-decreasing, we obtain

$$g\left(|x_1(s) - x_0(s)|\right) \leqslant g\left(\varepsilon \frac{s^q}{\Gamma(q)} \right) \leqslant g\left(\varepsilon \frac{s^{\lfloor q \rfloor}}{\Gamma(q)} \right),$$

since $s \leqslant t \leqslant \delta \leqslant 1$.

By Lemma 2 part 2, there exists a constant $\tilde{C}_\delta$ such that $t \leqslant \tilde{C}_\delta g(t)$. Hence $t^n \leqslant \tilde{C}_\delta^n g(t)^n$ for every $n \in \mathbb{N}$ and $0 \leqslant t \leqslant h$. Thus, by Lemma 2 part 3, we have

$$g\left(|x_1(s) - x_0(s)|\right) \leqslant g\left(\varepsilon \frac{\tilde{C}_\delta^{\lfloor q \rfloor} \cdot g(s)^{\lfloor q \rfloor}}{\Gamma(q)} \right) \leqslant \lfloor q \rfloor \frac{\varepsilon \tilde{C}_\delta^{\lfloor q \rfloor}}{b\Gamma(q)} \frac{g(s)^{\lfloor q \rfloor + 1}}{s}.$$

Summing up and using Lemma 2 part 4, we obtain

$$|x_2(t) - x_1(t)| \leqslant \frac{t^{q-1}}{\Gamma(q)} \lfloor q \rfloor \frac{\varepsilon \tilde{C}_\delta^{\lfloor q \rfloor}}{b\Gamma(q)} \int_0^t \frac{g(s)^{\lfloor q \rfloor +1}}{s}\, ds$$
$$\leqslant \varepsilon \frac{t^{q-1}}{\Gamma(q)} \frac{\tilde{C}_\delta^{\lfloor q \rfloor}}{b\Gamma(q)} C_\delta \cdot g(t)^{\lfloor q \rfloor +1},$$

which is exactly (26) for $k = 1$.

Suppose (26) holds for $k-1$. Similarly we obtain

$$|x_{k+1}(t) - x_k(t)| \leqslant \frac{t^{q-1}}{\Gamma(q)} \int_0^t g\left(|x_k(s) - x_{k-1}(s)|\right)\, ds$$

and

$$g\left(|x_k(s) - x_{k-1}(s)|\right) \leqslant g\left(\varepsilon B^{k-1} \frac{t^{q-1}}{\Gamma(q)} g(t)^{1+\lfloor q \rfloor (k-1)}\right)$$
$$\leqslant g\left(\frac{\varepsilon B^{k-1} \tilde{C}_\delta^{\lfloor q \rfloor -1}}{\Gamma(q)} \cdot g(t)^{\lfloor q \rfloor k}\right)$$
$$\leqslant \lfloor q \rfloor k \frac{\varepsilon B^{k-1} \tilde{C}_\delta^{\lfloor q \rfloor -1}}{b\Gamma(q)} \frac{g(t)^{\lfloor q \rfloor k+1}}{t},$$

by Lemma 2 part 3. Using this estimate and Lemma 2 part 5, for every $0 \leqslant t \leqslant \delta$, we obtain

$$|x_{k+1}(t) - x_k(t)| \leqslant \frac{t^{q-1}}{\Gamma(q)} \frac{\varepsilon B^{k-1} \tilde{C}_\delta^{\lfloor q \rfloor -1}}{b\Gamma(q)} \cdot C_\delta \cdot g(t)^{\lfloor q \rfloor k+1} \leqslant \frac{t^{q-1}}{\Gamma(q)} \frac{\varepsilon B^{k-1} \tilde{C}_\delta^{\lfloor q \rfloor}}{b\Gamma(q)} \cdot C_\delta \cdot g(t)^{\lfloor q \rfloor k+1},$$

since without loss of generality $\tilde{C}_\delta \geqslant 1$. This proves the result.

The following result is the main result of this section.

THEOREM 3. *The equation (1) is Ulam-Hyers stable on the interval $[0,\delta]$, for $\delta = \min(\delta_1, \delta_2, \delta_3)$, where $\delta_1 = \min(\delta_0, z, \Gamma(q)^{\frac{1}{q-1}})$ (δ_0 is the δ from Theorem 2), δ_2 is the δ from Lemma 3, and δ_3 is such that $B \cdot g(\delta_3)^{\lfloor q \rfloor} < 1$ (this choice is possible since g is a continuous non-decreasing function and $g(0) = 0$).*

Proof. By Theorem 2, for every $k \in \mathbb{N}$, we have $|x_{k+1}(t) - x_k(t)| \leqslant r^k$ for some $r \in (0,1)$ and $t \in [0,\delta]$. Thus, the function

$$x(t) = x_0(t) + \sum_{k=0}^{\infty} (x_{k+1}(t) - x_k(t)), 0 \leqslant t \leqslant \delta \tag{28}$$

is well-defined, as the series on the right-hand side converges absolutely and uniformly in the uniform norm.

1. We claim that the function x, defined by (28), solves the IVP (1)-(2) on $[0,\delta]$. We will show that the following quantity is zero:

$$\left|x(t)-\sum_{i=0}^{m-1}\frac{A_i}{i!}t^i+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))\,ds\right|,0\leqslant t\leqslant\delta. \tag{29}$$

Adding and subtracting x_{j+1} and using the definition of our successive approximations gives that (29) is estimated from above by

$$\begin{aligned}&\left|x(t)-x_{j+1}(t)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(f(s,x_j(s))-f(s,x(s)))\,ds\right|\\ \leqslant&|x(t)-x_{j+1}(t)|+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}g(|x_j(s)-x(s)|)\,ds,\end{aligned}$$

by the Osgood type condition on f. Since g is non-decreasing, the second term in the above expression is estimated by

$$\begin{aligned}\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}g(|x_j(s)-x(s)|)\,ds&\leqslant\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\,ds\cdot g\left(\|x_j-x\|_{C[0,\delta]}\right)\\&=\frac{t^q}{\Gamma(q+1)}\cdot g\left(\|x_j-x\|_{C[0,\delta]}\right).\end{aligned}$$

Summing up,

$$\begin{aligned}&\left|x(t)-\sum_{i=0}^{m-1}\frac{A_i}{i!}t^i+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))\,ds\right| \\ \leqslant&|x(t)-x_{j+1}(t)|+\frac{t^q}{\Gamma(q+1)}\cdot g\left(\|x_j-x\|_{C[0,\delta]}\right).\end{aligned} \tag{30}$$

Since the x_j's are merely the partial sums of the right-hand side of (28), it follows that $|x(t)-x_{j+1}(t)|\to 0$ uniformly in $0\leqslant t\leqslant\delta$ as $j\to\infty$. Since $g(0)=0$ and the left-hand side of (30) does not depend on j we conclude that

$$x(t)=\sum_{i=0}^{m-1}\frac{A_i}{i!}t^i+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))\,ds$$

and thus x solves the IVP (1)-(2).

2. Since $x_0\equiv y$, it follows from definition of x and Lemma 3 that for every $0\leqslant t\leqslant\delta$ we have

$$|y(t)-x(t)|\leqslant\sum_{k=0}^{\infty}|x_{k+1}(t)-x_k(t)|\leqslant\varepsilon\left(\frac{\delta^q}{\Gamma(q)}+\frac{\delta^{q-1}g(\delta)}{\Gamma(q)}\sum_{k=0}^{\infty}\left(B\cdot g(\delta)^{\lfloor q\rfloor}\right)^k\right).$$

By the choice of δ, $B\cdot g(h)^{\lfloor q\rfloor}<1$. Hence, the quantity

$$K_f:=\frac{\delta^q}{\Gamma(q)}+\frac{\delta^{q-1}g(\delta)}{\Gamma(q)}\sum_{k=0}^{\infty}\left(B\cdot g(\delta)^{\lfloor q\rfloor}\right)^k$$

is finite. This shows that the equation (1) is Ulam-Hyers stable on the interval $[0,\delta]$.

5. Conclusion

We proved that for the IVP (1)-(2) with right-hand side satisfying the Osgood type condition, we can construct successive approximations that converge uniformly to the unique solution on a sufficiently small interval. Under a more restrictive assumption (the modified Osgood condition) this convergence is exponentially fast. In this case we also proved that the equation ${}^{C}D^{q}[x](t) = f(t, x(t))$ is Ulam-Hyers stable. It is still open as to whether the latter two results hold under the general Osgood type condition.

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