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GENERALIZED OSTROWSKI–TYPE INEQUALITIES FOR s–CONVEX FUNCTIONS ON THE COORDINATES VIA FRACTIONAL INTEGRALS

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Abstract. We established some new fractional integral inequalities of Ostrowski-type for functions of two independent variables whose second order mixed partial derivatives in absolute value to some powers are s-convex on the coordinates. These results are obtained by using generalized Katugampola-type fractional integrals for functions of two independent variables. Our results generalizes some results in the literature.

1. Introduction

In 1938, the Ukranian mathematician, Alexander Ostrowski [23], obtained the following inequality which is known in the literature as Ostrowski inequality.

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f':(a,b)\to\mathbb{R}$ is bounded in (a,b). If $M:=\sup_{t\in(a,b)}|f'(t)|<\infty$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant\left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)M$$

for all $x\in[a,b]$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Many authors have studied and generalized the Ostrowski inequality in several different ways. For more information about the Ostrowski inequality and its associates, we refer the interested reader to the papers [2,3,6,7,8,9,10,11,12,16,25]. The authors in [2, 10, 11, 12, 16, 25] provided some Ostrowski-type inequalities for some classes of convex functions.

Recall that given an interval I in $\mathbb{R}$, a function $f:I\to\mathbb{R}$ is said to be convex, if

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y)$$

for all $x,y\in I$ and $t\in[0,1]$. The concept of convexity was generalized to the concept of s-convexity in [13] as follows.

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DEFINITION 1. (See [13]) A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be s-convex in the second sense for $s \in [0,1]$, if

$$f(tx + (1-t)y) \leqslant t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

In 2001, Dragomir extended the concept of convex functions of a single variable to functions of two independent variables in [5] as follows.

DEFINITION 2. (See [5]) A function $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ is said to be convex on Δ, if

$$f(tx + (1-t)z, ty + (1-t)w) \leqslant tf(x,y) + (1-t)f(z,w)$$

for all $(x,y), (z,w) \in \Delta$ with $t \in [0,1]$. On the other hand, $f : \Delta \to \mathbb{R}$ is said to be convex on the coordinates on Δ, if

$$\begin{aligned} f(tx + (1-t)z, ry + (1-r)w) \leqslant\ & rtf(x,y) + t(1-r)f(x,w) \\ & + r(1-t)f(z,y) + (1-r)(1-t)f(z,w) \end{aligned}$$

for all $(x,y), (x,w), (z,y), (z,w) \in \Delta$ and $(r,t) \in [0,1] \times [0,1]$.

In similar way, s-convexity in the second sense for functions of two variables is defined as follows.

DEFINITION 3. (See [1]) A function $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ is said to be s-convex in the second sense on Δ for $s \in [0,1]$, if

$$f(tx + (1-t)z, ty + (1-t)w) \leqslant t^s f(x,y) + (1-t)^s f(z,w),$$

for all $(x,y), (z,w) \in \Delta$ with $t \in [0,1]$. On the other hand, $f : \Delta \to \mathbb{R}$ is said to be s-convex in the second sense on the coordinates on Δ for $s \in [0,1]$, if

$$\begin{aligned} f(tx + (1-t)z, ry + (1-r)w) \leqslant\ & r^s t^s f(x,y) + t^s(1-r)^s f(x,w) \\ & + r^s(1-t)^s f(z,y) + (1-r)^s(1-t)^s f(z,w), \end{aligned}$$

for all $(x,y), (x,w), (z,y), (z,w) \in \Delta$ and $(r,t) \in [0,1] \times [0,1]$.

REMARK 1. If we take $s = 1$ in Definition 3, then we obtained the class of convex functions on Δ and functions that convex on the coordinates on Δ as introduced by Dragomir in [5].

In 2013, Latif et al. [21] established the following results for functions of two variables whose second order mixed partial derivatives in absolute value to some powers are s-convex on the coordinates.

THEOREM 2. (See [21]) *Let* $\beta_1, \beta_2 > 0$ *and* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be a twice partially differentiable mapping on* $(a,b) \times (c,d)$ *with* $0 \leqslant a < b$, $0 \leqslant c < d$, *and* $\frac{\partial^2 f}{\partial r \partial t} \in L_1\left([a,b] \times [c,d]\right)$. *If* $\left|\frac{\partial^2 f}{\partial r \partial t}\right|$ *is* s*-convex in the second sense on the coordinates and* $\left|\frac{\partial^2 f}{\partial r \partial t}\right| \leqslant M$ *on* $[a,b] \times [c,d]$, *then the following inequality holds:*

$$\begin{aligned}
&\Bigg| \frac{\left[(b-x)^{\beta_1} + (x-a)^{\beta_1}\right]\left[(d-y)^{\beta_2} + (y-c)^{\beta_2}\right]}{(b-a)(d-c)} f(x,y) \\
&\quad - \frac{\left[(b-x)^{\beta_1} + (x-a)^{\beta_1}\right]\Gamma(\beta_2+1)}{(b-a)(d-c)} \left[I_{y-}^{\beta_2} f(x,c) + I_{y+}^{\beta_2} f(x,d)\right] \\
&\quad - \frac{\left[(d-y)^{\beta_2} + (y-c)^{\beta_2}\right]\Gamma(\beta_1+1)}{(b-a)(d-c)} \left[I_{x-}^{\beta_1} f(a,y) + I_{x+}^{\beta_1} f(b,y)\right] \\
&\quad + \frac{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b-a)(d-c)} \left[I_{x-,y-}^{\beta_1,\beta_2} f(a,c) + I_{x-,y+}^{\beta_1,\beta_2} f(a,d) + I_{x+,y-}^{\beta_1,\beta_2} f(b,c) + I_{x+,y+}^{\beta_1,\beta_2} f(b,d)\right] \Bigg| \\
&\leqslant \left[\frac{M}{(\beta_1+s+1)(\beta_2+s+1)} + \frac{M\mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)} \right. \\
&\quad \left. + \frac{M\mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)} + M\mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)\right] \\
&\quad \times \left[\frac{(x-a)^{\beta_1+1} + (b-x)^{\beta_1+1}}{b-a}\right]\left[\frac{(y-c)^{\beta_2+1} + (d-y)^{\beta_2+1}}{d-c}\right],
\end{aligned}$$

for all $(x,y) \in [a,b] \times [c,d]$ *where* $I_{x-}^{\beta_1} f, I_{x+}^{\beta_1} f, I_{y-}^{\beta_2} f, I_{y+}^{\beta_2} f, I_{x+,y+}^{\beta_1,\beta_2} f, I_{x+,y-}^{\beta_1,\beta_2} f, I_{x-,y+}^{\beta_1,\beta_2} f$ *and* $I_{x-,y-}^{\beta_1,\beta_2} f$ *denotes the Riemann–Liouville fractional integrals of* f *which are special cases of Definition 5 with* $\rho_1 = \rho_2 = 1$ *(see Remark 2) and* B *denotes the beta function defined by*

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

which satisfies the properties $\mathrm{B}(x,y) = \mathrm{B}(y,x)$ *and* $\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, *and* Γ *denotes the gamma function defined in Definition 4.*

THEOREM 3. (See [21]) *Let* $\beta_1, \beta_2 > 0$ *and* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be a twice partially differentiable mapping on* $(a,b) \times (c,d)$ *with* $0 \leqslant a < b$, $0 \leqslant c < d$, *and* $\frac{\partial^2 f}{\partial r \partial t} \in L_1\left([a,b] \times [c,d]\right)$. *If* $\left|\frac{\partial^2 f}{\partial r \partial t}\right|^q$ *is* s*-convex in the second sense on the coordi-*

nates for $q \geqslant 1$ *and* $\left|\frac{\partial^2 f}{\partial r \partial t}\right| \leqslant M$ *on* $[a,b]\times[c,d]$*, then the following inequality holds:*

$$\Bigg| \frac{\left[(b-x)^{\beta_1}+(x-a)^{\beta_1}\right]\left[(d-y)^{\beta_2}+(y-c)^{\beta_2}\right]}{(b-a)(d-c)} f(x,y)$$

$$-\frac{\left[(b-x)^{\beta_1}+(x-a)^{\beta_1}\right]\Gamma(\beta_2+1)}{(b-a)(d-c)}\left[I_{y-}^{\beta_2} f(x,c)+I_{y+}^{\beta_2} f(x,d)\right]$$

$$-\frac{\left[(d-y)^{\beta_2}+(y-c)^{\beta_2}\right]\Gamma(\beta_1+1)}{(b-a)(d-c)}\left[I_{x-}^{\beta_1} f(a,y)+I_{x+}^{\beta_1} f(b,y)\right]$$

$$+\frac{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b-a)(d-c)}\left[I_{x-,y-}^{\beta_1,\beta_2} f(a,c)+I_{x-,y+}^{\beta_1,\beta_2} f(a,d)+I_{x+,y-}^{\beta_1,\beta_2} f(b,c)+I_{x+,y+}^{\beta_1,\beta_2} f(b,d)\right]\Bigg|$$

$$\leqslant M\left(\frac{1}{(\beta_1+1)(\beta_2+1)}\right)^{1-\frac{1}{q}}\left(\frac{1}{(\beta_1+s+1)(\beta_2+s+1)}+\frac{\mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)}\right.$$

$$\left.+\frac{\mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)}+\mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)\right)^{\frac{1}{q}}$$

$$\times\left[\frac{(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}}{b-a}\right]\left[\frac{(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}}{d-c}\right],$$

for all $(x,y)\in[a,b]\times[c,d]$.

THEOREM 4. (See [21]) *Let* $\beta_1,\beta_2>0$ *and* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be a twice partially differentiable mapping on* $(a,b)\times(c,d)$ *with* $0\leqslant a<b,\ 0\leqslant c<d$*, and* $\frac{\partial^2 f}{\partial r\partial t}\in L_1\Big([a,b]\times[c,d]\Big)$*. If* $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ *is* s*-convex in the second sense on the coordinates for* $q>1$ *and* $\left|\frac{\partial^2 f}{\partial r\partial t}\right|\leqslant M$ *on* $[a,b]\times[c,d]$*, then the following inequality holds:*

$$\Bigg| \frac{\left[(b-x)^{\beta_1}+(x-a)^{\beta_1}\right]\left[(d-y)^{\beta_2}+(y-c)^{\beta_2}\right]}{(b-a)(d-c)} f(x,y)$$

$$-\frac{\left[(b-x)^{\beta_1}+(x-a)^{\beta_1}\right]\Gamma(\beta_2+1)}{(b-a)(d-c)}\left[I_{y-}^{\beta_2} f(x,c)+I_{y+}^{\beta_2} f(x,d)\right]$$

$$-\frac{\left[(d-y)^{\beta_2}+(y-c)^{\beta_2}\right]\Gamma(\beta_1+1)}{(b-a)(d-c)}\left[I_{x-}^{\beta_1} f(a,y)+I_{x+}^{\beta_1} f(b,y)\right]$$

$$+\frac{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b-a)(d-c)}\left[I_{x-,y-}^{\beta_1,\beta_2} f(a,c)+I_{x-,y+}^{\beta_1,\beta_2} f(a,d)+I_{x+,y-}^{\beta_1,\beta_2} f(b,c)+I_{x+,y+}^{\beta_1,\beta_2} f(b,d)\right]\Bigg|$$

$$\leqslant M\left(\frac{2}{s+1}\right)^{\frac{2}{q}}\left[\frac{(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}}{(b-a)(\beta_1 p+1)^{\frac{1}{p}}}\right]\left[\frac{(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}}{(d-c)(\beta_2 p+1)^{\frac{1}{p}}}\right],$$

for all $(x,y)\in[a,b]\times[c,d]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Our goal in this paper is to generalize Theorems 2, 3 and 4 by using a generalized Katugampola-type fractional integrals of functions of two independent variables. We conclude this section with the definitions of the Katugampola fractional integrals and its extension to functions of two variables.

DEFINITION 4. (See [14]) Let $\beta,\rho>0$ and f be a real-valued function of a single variable. The Katugampola fractional integrals of f are defined as follows:

$${}^{\rho}I^{\beta}_{a+}f(x)=\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_a^x\left(x^{\rho}-t^{\rho}\right)^{\beta-1}t^{\rho-1}f(t)dt$$

and

$${}^{\rho}I^{\beta}_{b-}f(x)=\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_x^b\left(t^{\rho}-x^{\rho}\right)^{\beta-1}t^{\rho-1}f(t)dt$$

where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt$$

with the property $\Gamma(x+1)=x\Gamma(x)$.

For some recent results related to the Katugampola fractional integrals, we refer the interested reader to the papers [4,14,15,17,18,19,20]. The following fractional integrals for functions of two independent variables are natural extensions of the Katugampola fractional integrals in Definition 4.

DEFINITION 5. Let $\beta_1,\beta_2,\rho_1,\rho_2>0$ and f be a function of two independent variables. We define, the Katugampola fractional integrals of f on the coordinates as follows:

$${}^{\rho_1}I^{\beta_1}_{a+}f(x,y):=\frac{\rho_1^{1-\beta_1}}{\Gamma(\beta_1)}\int_a^x\left(x^{\rho_1}-u^{\rho_1}\right)^{\beta_1-1}u^{\rho_1-1}f(u,y)du,$$

$${}^{\rho_1}I^{\beta_1}_{b-}f(x,y):=\frac{\rho_1^{1-\beta_1}}{\Gamma(\beta_1)}\int_x^b\left(u^{\rho_1}-x^{\rho_1}\right)^{\beta_1-1}u^{\rho_1-1}f(u,y)du,$$

$${}^{\rho_2}I^{\beta_2}_{c+}f(x,y):=\frac{\rho_2^{1-\beta_2}}{\Gamma(\beta_2)}\int_c^y\left(y^{\rho_2}-v^{\rho_2}\right)^{\beta_2-1}v^{\rho_2-1}f(x,v)dv$$

and

$$^{\rho_2}I_{d-}^{\beta_2}f(x,y) := \frac{\rho_2^{1-\beta_2}}{\Gamma(\beta_2)}\int_y^d (v^{\rho_2}-y^{\rho_2})^{\beta_2-1}\, v^{\rho_2-1}f(x,v)dv.$$

We define the Katugampola fractional integrals of f in the two variables as follows:

$$^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2}f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_a^x\int_c^y \frac{u^{\rho_1-1}v^{\rho_2-1}}{(x^{\rho_1}-u^{\rho_1})^{1-\beta_1}(y^{\rho_2}-v^{\rho_2})^{1-\beta_2}}f(u,v)dvdu,$$

$$^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2}f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_a^x\int_y^d \frac{u^{\rho_1-1}v^{\rho_2-1}}{(x^{\rho_1}-u^{\rho_1})^{1-\beta_1}(v^{\rho_2}-y^{\rho_2})^{1-\beta_2}}f(u,v)dvdu,$$

$$^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2}f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_x^b\int_c^y \frac{u^{\rho_1-1}v^{\rho_2-1}}{(u^{\rho_1}-x^{\rho_1})^{1-\beta_1}(y^{\rho_2}-v^{\rho_2})^{1-\beta_2}}f(u,v)dvdu$$

and

$$^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2}f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_x^b\int_y^d \frac{u^{\rho_1-1}v^{\rho_2-1}}{(u^{\rho_1}-x^{\rho_1})^{1-\beta_1}(v^{\rho_2}-y^{\rho_2})^{1-\beta_2}}f(u,v)dvdu.$$

REMARK 2. If $\rho_1=\rho_2=1$ in Definition 5, then we obtain the generalized Riemann–Liouville fractional integrals of two independent variables (see [24]).

2. Main results

To simplify our computations in the main results, we present the following useful identities in the following lemma.

LEMMA 1. *Let $\beta_1,\beta_2,\rho_1,\rho_2>0$, and $f:[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\to\mathbb{R}$ be a real-valued function of two independent variables. The following identities hold:*

$$\int_0^1 r^{\beta_2\rho_2-1}f(x^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dr = \frac{\rho_2^{\beta_2-1}\Gamma(\beta_2)}{(y^{\rho_2}-c^{\rho_2})^{\beta_2}}\,{}^{\rho_2}I_{y-}^{\beta_2}f(x^{\rho_1},c^{\rho_2}), \tag{1}$$

$$\int_0^1 r^{\beta_2\rho_2-1}f(x^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})d^{\rho_2})dr = \frac{\rho_2^{\beta_2-1}\Gamma(\beta_2)}{(d^{\rho_2}-y^{\rho_2})^{\beta_2}}\,{}^{\rho_2}I_{y+}^{\beta_2}f(x^{\rho_1},d^{\rho_2}), \tag{2}$$

$$\int_0^1 t^{\beta_1\rho_1-1}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},y^{\rho_2})dt = \frac{\rho_1^{\beta_1-1}\Gamma(\beta_1)}{(x^{\rho_1}-a^{\rho_1})^{\beta_1}}\,{}^{\rho_1}I_{x-}^{\beta_1}f(a^{\rho_1},y^{\rho_2}), \tag{3}$$

$$\int_0^1 t^{\beta_1\rho_1-1} f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, y^{\rho_2})dt = \frac{\rho_1^{\beta_1-1}\Gamma(\beta_1)}{(b^{\rho_1}-x^{\rho_1})^{\beta_1}}\, {}^{\rho_1}I^{\beta_1}_{x+} f(b^{\rho_1}, y^{\rho_2}), \tag{4}$$

$$\begin{aligned}&\int_0^1\int_0^1 t^{\beta_1\rho_1-1} r^{\beta_2\rho_2-1} f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})d^{\rho_2})drdt\\ &=\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-x^{\rho_1})^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}}\, {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{x+,y+} f(b^{\rho_1}, d^{\rho_2}),\end{aligned} \tag{5}$$

$$\begin{aligned}&\int_0^1\int_0^1 t^{\beta_1\rho_1-1} r^{\beta_2\rho_2-1} f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})drdt\\ &=\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-x^{\rho_1})^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}}\, {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{x+,y-} f(b^{\rho_1}, c^{\rho_2}),\end{aligned} \tag{6}$$

$$\begin{aligned}&\int_0^1\int_0^1 t^{\beta_1\rho_1-1} r^{\beta_2\rho_2-1} f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})d^{\rho_2})drdt\\ &=\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(x^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}}\, {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{x-,y+} f(a^{\rho_1}, d^{\rho_2})\end{aligned} \tag{7}$$

and

$$\begin{aligned}&\int_0^1\int_0^1 t^{\beta_1\rho_1-1} r^{\beta_2\rho_2-1} f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})drdt\\ &=\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(x^{\rho_1}-a^{\rho_1})^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}}\, {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{x-,y-} f(a^{\rho_1}, c^{\rho_2}).\end{aligned} \tag{8}$$

Proof. The results follows directly by using change of variables and Definition 5.

To prove our main results, we need the following lemma. For convenience, we introduce the following notation which will be used though out the rest of the paper.

$$\begin{aligned}&T_f(\beta_1,\beta_2,\rho_1,\rho_2;a,b,c,d,x,y)\\ :=&\frac{\left[(b^{\rho_1}-x^{\rho_1})^{\beta_1}+(x^{\rho_1}-a^{\rho_1})^{\beta_1}\right]\left[(d^{\rho_2}-y^{\rho_2})^{\beta_2}+(y^{\rho_2}-c^{\rho_2})^{\beta_2}\right]}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} f(x^{\rho_1},y^{\rho_2})\\ &-\frac{\rho_2^{\beta_2}\left[(b^{\rho_1}-x^{\rho_1})^{\beta_1}+(x^{\rho_1}-a^{\rho_1})^{\beta_1}\right]\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\left[{}^{\rho_2}I^{\beta_2}_{y-}f(x^{\rho_1},c^{\rho_2})+{}^{\rho_2}I^{\beta_2}_{y+}f(x^{\rho_1},d^{\rho_2})\right]\\ &-\frac{\rho_1^{\beta_1}\left[(d^{\rho_2}-y^{\rho_2})^{\beta_2}+(y^{\rho_2}-c^{\rho_2})^{\beta_2}\right]\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\left[{}^{\rho_1}I^{\beta_1}_{x-}f(a^{\rho_1},y^{\rho_2})+{}^{\rho_1}I^{\beta_1}_{x+}f(b^{\rho_1},y^{\rho_2})\right]\\ &+\frac{\rho_1^{\beta_1}\rho_2^{\rho_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\Big[{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{x-,y-}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{x-,y+}f(a^{\rho_1},d^{\rho_2})\\ &\qquad+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{x+,y-}f(b^{\rho_1},c^{\rho_2})+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{x+,y+}f(b^{\rho_1},d^{\rho_2})\Big].\end{aligned}$$

LEMMA 2. *Let* $\beta_1,\beta_2,\rho_1,\rho_2>0$ *and* $f:[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\to\mathbb{R}$ *be a twice partially differentiable mapping on* $(a^{\rho_1},b^{\rho_1})\times(c^{\rho_2},d^{\rho_2})$ *with* $0\leqslant a<b$, $0\leqslant c<d$, *and* $\dfrac{\partial^2 f}{\partial r\partial t}\in L_1\Big([a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\Big)$. *Then the following equality holds:*

$$
\begin{aligned}
&T_f(\beta_1,\beta_2,\rho_1,\rho_2;a,b,c,d,x,y)\\
&=\frac{\rho_1\rho_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\\
&\quad\times\Bigg((x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}I_1-(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}I_2\\
&\qquad-(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}I_3+(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}I_4\Bigg)
\end{aligned}
$$

where

$$I_1=\int_0^1\int_0^1 r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\frac{\partial^2}{\partial r\partial t}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dtdr,$$

$$I_2=\int_0^1\int_0^1 r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\frac{\partial^2}{\partial r\partial t}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})d^{\rho_2})dtdr,$$

$$I_3=\int_0^1\int_0^1 r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\frac{\partial^2}{\partial r\partial t}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})b^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dtdr$$

and

$$I_4=\int_0^1\int_0^1 r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\frac{\partial^2}{\partial r\partial t}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})b^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})d^{\rho_2})dtdr.$$

Proof. Integrating by parts, we have

$$
\begin{aligned}
I_1&=\int_0^1 r^{(\beta_2+1)\rho_2-1}\left[\int_0^1 t^{\beta_1\rho_1}t^{\rho_1-1}\frac{\partial^2}{\partial r\partial t}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dt\right]dr\\
&=\int_0^1 r^{(\beta_2+1)\rho_2-1}\Bigg[\frac{1}{(x^{\rho_1}-a^{\rho_1})\rho_1}t^{\beta_1\rho_1}\frac{\partial}{\partial r}f(t^{\rho}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})\Bigg|_{t=0}^{t=1}\\
&\quad-\frac{\beta_1}{(x^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\frac{\partial}{\partial r}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dt\Bigg]dr\\
&=\int_0^1 r^{(\beta_2+1)\rho_2-1}\Bigg[\frac{1}{(x^{\rho_1}-a^{\rho_1})\rho_1}\frac{\partial}{\partial r}f(x^{\rho_1},s^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})\\
&\quad-\frac{\beta_1}{(x^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\frac{\partial}{\partial r}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dt\Bigg]dr\\
&=\frac{1}{(x^{\rho_1}-a^{\rho_1})\rho_1}\int_0^1 r^{\beta_2\rho_2}r^{\rho_2-1}\frac{\partial}{\partial r}f(x^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dr
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\beta_1}{(x^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\Bigg[\int_0^1\Bigg(r^{\beta_2\rho_2}r^{\rho_2-1}\\
&\qquad\times\frac{\partial}{\partial r}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})\Bigg)dr\Bigg]dt\\
&=\frac{1}{(x^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})\rho_1\rho_2}r^{\beta_2\rho_2}f(x^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})\Big|_{r=0}^{r=1}\\
&-\frac{\beta_2}{(x^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})\rho_1}\int_0^1 r^{\beta_2\rho_2-1}f(x^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dr\\
&-\frac{\beta_1}{(x^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\\
&\times\Bigg[\frac{1}{(y^{\rho_2}-c^{\rho_2})\rho_2}r^{\beta_2\rho_2}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})\Big|_{r=0}^{r=1}\\
&-\frac{\beta_2}{(y^{\rho_2}-c^{\rho_2})}\int_0^1 r^{\beta_2\rho_2-1}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dr\Bigg]dt.
\end{aligned}
$$

That is,

$$
\begin{aligned}
I_1&=\frac{1}{(x^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})\rho_1\rho_2}f(x^{\rho_1},y^{\rho_2})\\
&-\frac{\beta_2}{(x^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})\rho_1}\int_0^1 r^{\beta_2\rho_2-1}f(x^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})dr\\
&-\frac{\beta_1}{(x^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})\rho_2}\int_0^1 t^{\beta_1\rho_1-1}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},y^{\rho_2})dt\\
&+\frac{\beta_1\beta_2}{(x^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})}\\
&\times\int_0^1\int_0^1 t^{\beta_1\rho_1-1}r^{\beta_2\rho_2-1}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})drdt. \qquad (9)
\end{aligned}
$$

Now, by using Lemma 1 and (9), we have

$$
\begin{aligned}
I_1&=\frac{1}{(x^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})\rho_1\rho_2}f(x^{\rho_1},y^{\rho_2})\\
&-\frac{\rho_2^{\beta_2-1}\Gamma(\beta_2+1)}{(x^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}\rho_1}\,{}^{\rho_2}I_{y-}^{\beta_2}f(x^{\rho_1},c^{\rho_2})\\
&-\frac{\rho_1^{\beta_1-1}\Gamma(\beta_1+1)}{(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})\rho_2}\,{}^{\rho_1}I_{x-}^{\beta_1}f(a^{\rho_1},y^{\rho_2})\\
&+\frac{\rho_1^{\beta_1-1}\rho_2^{\rho_2-1}\Gamma(\beta_1+1)(\beta_2+1)}{(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}}\,{}^{\rho_1,\rho_2}I_{x-,y-}^{\beta_1,\beta_2}f(a^{\rho_1},c^{\rho_2}).
\end{aligned}
$$

So, it follows that

$$
\frac{(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}\rho_1\rho_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}I_1
$$

$$
\begin{aligned}
&= \frac{(x^{\rho_1}-a^{\rho_1})^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} f(x^{\rho_1},y^{\rho_2}) \\
&\quad - \frac{\rho_2^{\beta_2}(x^{\rho_1}-a^{\rho_1})^{\beta_1}\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\, {}^{\rho_2}I_{y-}^{\beta_2} f(x^{\rho_1},c^{\rho_2}) \\
&\quad - \frac{\rho_1^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\, {}^{\rho_1}I_{x-}^{\beta_1} f(a^{\rho_1},y^{\rho_2}) \\
&\quad + \frac{\rho_1^{\beta_1}\rho_2^{\rho_2}\Gamma(\beta_1+1)(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\, {}^{\rho_1,\rho_2}I_{x-,y-}^{\beta_1,\beta_2} f(a^{\rho_1},c^{\rho_2}). \qquad (10)
\end{aligned}
$$

Using similar arguments as in the above, we have

$$
\begin{aligned}
&\frac{(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}\rho_1\rho_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} I_2 \\
&= - \frac{(x^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} f(x^{\rho_1},y^{\rho_2}) \\
&\quad + \frac{\rho_2^{\beta_2}(x^{\rho_1}-a^{\rho_1})^{\beta_1}\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\, {}^{\rho_2}I_{y+}^{\beta_2} f(x^{\rho_1},d^{\rho_2}) \\
&\quad + \frac{\rho_1^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\, {}^{\rho_1}I_{x-}^{\beta_1} f(a^{\rho_1},y^{\rho_2}) \\
&\quad - \frac{\rho_1^{\beta_1}\rho_2^{\rho_2}\Gamma(\beta_1+1)(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\, {}^{\rho_1,\rho_2}I_{x-,y+}^{\beta_1,\beta_2} f(a^{\rho_1},d^{\rho_2}), \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}\rho_1\rho_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} I_3 \\
&= - \frac{(b^{\rho_1}-x^{\rho_1})^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} f(x^{\rho_1},y^{\rho_2}) \\
&\quad + \frac{\rho_2^{\beta_2}(b^{\rho_1}-x^{\rho_1})^{\beta_1}\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\, {}^{\rho_2}I_{y-}^{\beta_2} f(x^{\rho_1},c^{\rho_2}) \\
&\quad + \frac{\rho_1^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\, {}^{\rho_1}I_{x+}^{\beta_1} f(b^{\rho_1},y^{\rho_2}) \\
&\quad - \frac{\rho_1^{\beta_1}\rho_2^{\rho_2}\Gamma(\beta_1+1)(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\, {}^{\rho_1,\rho_2}I_{x+,y-}^{\beta_1,\beta_2} f(b^{\rho_1},c^{\rho_2}) \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}\rho_1\rho_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} I_4 \\
&= \frac{(b^{\rho_1}-x^{\rho_1})^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} f(x^{\rho_1},y^{\rho_2})
\end{aligned}
$$

$$- \frac{\rho_2^{\beta_2}(b^{\rho_1}-x^{\rho_1})^{\beta_1}\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} \, {}^{\rho_2}I_{y+}^{\beta_2} f(x^{\rho_1},d^{\rho_2})$$
$$- \frac{\rho_1^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} \, {}^{\rho_1}I_{x+}^{\beta_1} f(b^{\rho_1},y^{\rho_2})$$
$$+ \frac{\rho_1^{\beta_1}\rho_2^{\rho_2}\Gamma(\beta_1+1)(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} \, {}^{\rho_1,\rho_2}I_{x+,y+}^{\beta_1,\beta_2} f(b^{\rho_1},d^{\rho_2}). \tag{13}$$

The desired identity follows from (10), (11), (12) and (13).

REMARK 3. If we take $\rho_1=\rho_2=1$ in Lemma 2, then we obtain [22, Lemma 1].

THEOREM 5. *Let $\beta_1,\beta_2,\rho_1,\rho_2>0$ and $f:[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\to\mathbb{R}$ be a twice partially differentiable mapping on $(a^{\rho_1},b^{\rho_1})\times(c^{\rho_2},d^{\rho_2})$ with $0\leqslant a<b$, $0\leqslant c<d$, and $\frac{\partial^2 f}{\partial r\partial t}\in L_1\left([a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\right)$. If $\left|\frac{\partial^2 f}{\partial r\partial t}\right|$ is s-convex in the second sense on the coordinates and $\left|\frac{\partial^2 f}{\partial r\partial t}\right|\leqslant M$ on $[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]$, then the following inequality holds:*

$$\begin{aligned}
&\left|T_f(\beta_1,\beta_2,\rho_1,\rho_2;a,b,c,d,x,y)\right|\\
\leqslant & M\left[\frac{1}{(\beta_1+s+1)(\beta_2+s+1)}+\frac{\mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)}\right.\\
&\left.+\frac{\mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)}+\mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)\right]\\
&\times\left[\frac{(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}+(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}}{b^{\rho_1}-a^{\rho_1}}\right]\left[\frac{(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}+(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}}{d^{\rho_2}-c^{\rho_2}}\right],
\end{aligned}$$

for all $(x^{\rho_1},y^{\rho_2})\in[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]$.

Proof. By using Lemma 2 and the properties of the absolute value, we have

$$\begin{aligned}
&\left|T_f(\beta_1,\beta_2,\rho_1,\rho_2;a,b,c,d,x,y)\right|\\
\leqslant & \frac{\rho_1\rho_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\\
&\times\Big((x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}|I_1|+(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}|I_2|\\
&+(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}|I_3|+(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}|I_4|\Big).
\end{aligned} \tag{14}$$

Now, by using the s-convexity in the second sense of $\left|\frac{\partial^2 f}{\partial r \partial t}\right|$ on the coordinates, we have

$$
\begin{aligned}
|I_1| \leqslant & \int_0^1 \int_0^1 \Big(r^{(\beta_2+1)\rho_2-1} t^{(\beta_1+1)\rho_1-1} \\
& \qquad \times \left| \frac{\partial^2}{\partial r \partial t} f(t^{\rho_1} x^{\rho_1} + (1-t^{\rho_1}) a^{\rho_1}, r^{\rho_2} y^{\rho_2} + (1-r^{\rho_2}) c^{\rho_2}) \right| \Big) dt dr \\
\leqslant & \left| \frac{\partial^2}{\partial r \partial t} f(x^{\rho_1}, y^{\rho_2}) \right| \int_0^1 \int_0^1 r^{(\beta_2+1)\rho_2-1} t^{(\beta_1+1)\rho_1-1} r^{s\rho_2} t^{s\rho_1} dt dr \qquad (15) \\
& + \left| \frac{\partial^2}{\partial r \partial t} f(x^{\rho_1}, c^{\rho_2}) \right| \int_0^1 \int_0^1 r^{(\beta_2+1)\rho_2-1} t^{(\beta_1+1)\rho_1-1} (1-r^{\rho_2})^s t^{s\rho_1} dt dr \\
& + \left| \frac{\partial^2}{\partial r \partial t} f(a^{\rho_1}, y^{\rho_2}) \right| \int_0^1 \int_0^1 r^{(\beta_2+1)\rho_2-1} t^{(\beta_1+1)\rho_1-1} r^{s\rho_2} (1-t^{\rho_1})^s dt dr \\
& + \left| \frac{\partial^2}{\partial r \partial t} f(a^{\rho_1}, c^{\rho_2}) \right| \int_0^1 \int_0^1 r^{(\beta_2+1)\rho_2-1} t^{(\beta_1+1)\rho_1-1} (1-r^{\rho_2})^s (1-t^{\rho_1})^s dt dr.
\end{aligned}
$$

Now, we observe that

$$
\int_0^1 \int_0^1 r^{(\beta_2+1)\rho_2-1} t^{(\beta_1+1)\rho_1-1} r^{s\rho_2} t^{s\rho_1} dt dr = \frac{1}{(\beta_1+s+1)(\beta_2+s+1)\rho_1\rho_2}, \qquad (16)
$$

$$
\int_0^1 \int_0^1 r^{(\beta_2+1)\rho_2-1} t^{(\beta_1+1)\rho_1-1} (1-r^{\rho_2})^s t^{s\rho_1} dt dr = \frac{\mathrm{B}(\beta_2+1, s+1)}{(\beta_1+s+1)\rho_1\rho_2}, \qquad (17)
$$

$$
\int_0^1 \int_0^1 r^{(\beta_2+1)\rho_2-1} t^{(\beta_1+1)\rho_1-1} (1-t^{\rho_1})^s r^{s\rho_2} dt dr = \frac{\mathrm{B}(\beta_1+1, s+1)}{(\beta_2+s+1)\rho_1\rho_2} \qquad (18)
$$

and

$$
\begin{aligned}
& \int_0^1 \int_0^1 r^{(\beta_2+1)\rho_2-1} t^{(\beta_1+1)\rho_1-1} (1-r^{\rho_2})^s (1-t^{\rho_1})^s dt dr \\
= & \frac{\mathrm{B}(\beta_1+1, s+1)\mathrm{B}(\beta_2+1, s+1)}{\rho_1\rho_2}. \qquad (19)
\end{aligned}
$$

Using (15), (16), (17), (18), (19) and the fact that $\left|\frac{\partial^2 f}{\partial r \partial t}\right| \leqslant M$, we have

$$
\begin{aligned}
|I_1| \leqslant & \frac{M}{(\beta_1+s+1)(\beta_2+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_2+1, s+1)}{(\beta_1+s+1)\rho_1\rho_2} \\
& + \frac{M\mathrm{B}(\beta_1+1, s+1)}{(\beta_2+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_1+1, s+1)\mathrm{B}(\beta_2+1, s+1)}{\rho_1\rho_2}. \qquad (20)
\end{aligned}
$$

By using similarly arguments, we have

$$|I_2| \leqslant \frac{M}{(\beta_1+s+1)(\beta_2+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)}{\rho_1\rho_2}, \tag{21}$$

$$|I_3| \leqslant \frac{M}{(\beta_1+s+1)(\beta_2+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)}{\rho_1\rho_2} \tag{22}$$

and

$$|I_4| \leqslant \frac{M}{(\beta_1+s+1)(\beta_2+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)\rho_1\rho_2} + \frac{M\mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)}{\rho_1\rho_2}. \tag{23}$$

The desired inequality follows from (14) and using (20), (21), (22) and (23).

REMARK 4. If we take $\rho_1 = \rho_2 = 1$ in Theorem 5, then we obtain Theorem 2.

THEOREM 6. *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$ and $f : [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \to \mathbb{R}$ be a twice partially differentiable mapping on $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$ with $0 \leqslant a < b$, $0 \leqslant c < d$, and $\frac{\partial^2 f}{\partial r \partial t} \in L_1\left([a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}]\right)$. If $\left|\frac{\partial^2 f}{\partial r \partial t}\right|^q$ is s-convex in the second sense on the coordinates for $q > 1$ and $\left|\frac{\partial^2 f}{\partial r \partial t}\right| \leqslant M$ on $[a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}]$, then the following inequality holds:*

$$\begin{aligned}
&\left|T_f(\beta_1, \beta_2, \rho_1, \rho_2; a, b, c, d, x, y)\right| \\
\leqslant & M\left(\frac{1}{(\beta_1+1)(\beta_2+1)}\right)^{1-\frac{1}{q}} \\
&\times \left(\frac{1}{(\beta_1+s+1)(\beta_2+s+1)} + \frac{\mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)} + \frac{\mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)} + \mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)\right)^{\frac{1}{q}} \\
&\times \left[\frac{(x^{\rho_1}-a^{\rho_1})^{\beta_1+1} + (b^{\rho_1}-x^{\rho_1})^{\beta_1+1}}{b^{\rho_1}-a^{\rho_1}}\right]\left[\frac{(y^{\rho_2}-c^{\rho_2})^{\beta_2+1} + (d^{\rho_2}-y^{\rho_2})^{\beta_2+1}}{d^{\rho_2}-c^{\rho_2}}\right],
\end{aligned}$$

for all $(x^{\rho_1}, y^{\rho_2}) \in [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}]$.

Proof. By using Lemma 2 and the properties of the absolute value, we have

$$
\begin{aligned}
&\left|T_f(\beta_1,\beta_2,\rho_1,\rho_2;a,b,c,d,x,y)\right| \\
\leqslant &\frac{\rho_1\rho_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} \\
&\times\Big((x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}|I_1|+(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}|I_2| \\
&+(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}|I_3|+(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}|I_4|\Big).
\end{aligned}
\tag{24}
$$

Now, by using the power mean inequality and the s-convexity in the second sense of $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ on the coordinates, we have

$$
\begin{aligned}
|I_1| \leqslant &\left(\int_0^1\int_0^1 r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}drdt\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1\Big(r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\right. \\
&\left.\times\left|\frac{\partial^2}{\partial r\partial t}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})\right|^q\Big)dtdr\right)^{\frac{1}{q}} \\
\leqslant &\left(\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \\
&\times\left(\left|\frac{\partial^2}{\partial r\partial t}f(x^{\rho_1},y^{\rho_2})\right|^q\int_0^1\int_0^1 r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}r^{s\rho_2}t^{s\rho_1}dtdr\right. \\
&+\left|\frac{\partial^2}{\partial r\partial t}f(x^{\rho_1},c^{\rho_2})\right|^q\int_0^1\int_0^1 r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}(1-r^{\rho_2})^s t^{s\rho_1}dtdr \\
&+\left|\frac{\partial^2}{\partial r\partial t}f(a^{\rho_1},y^{\rho_2})\right|^q\int_0^1\int_0^1 r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}r^{s\rho_2}(1-t^{\rho_1})^s dtdr \\
&\left.+\left|\frac{\partial^2}{\partial r\partial t}f(a^{\rho_1},c^{\rho_2})\right|^q\int_0^1\int_0^1 r^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}(1-r^{\rho_2})^s(1-t^{\rho_1})^s dtdr\right)^{\frac{1}{q}}.
\end{aligned}
\tag{25}
$$

Using (16), (17), (18), (19) in (25) and the fact that $\left|\frac{\partial^2 f}{\partial r\partial t}\right|\leqslant M$, we have

$$
\begin{aligned}
|I_1| \leqslant &\left(\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{M^q}{(\beta_1+s+1)(\beta_2+s+1)\rho_1\rho_2}+\frac{M^q\mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)\rho_1\rho_2}\right. \\
&\left.+\frac{M^q\mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)\rho_1\rho_2}+\frac{M^q\mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)}{\rho_1\rho_2}\right)^{\frac{1}{q}}.
\end{aligned}
\tag{26}
$$

By using similarly arguments, we have

$$\begin{aligned}|I_2| \leqslant & \left(\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \\ & \times\left(\frac{M^q}{(\beta_1+s+1)(\beta_2+s+1)\rho_1\rho_2}+\frac{M^q \mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)\rho_1\rho_2}\right. \\ & \left.+\frac{M^q \mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)\rho_1\rho_2}+\frac{M^q \mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)}{\rho_1\rho_2}\right)^{\frac{1}{q}}, \end{aligned} \tag{27}$$

$$\begin{aligned}|I_3| \leqslant & \left(\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \\ & \times\left(\frac{M^q}{(\beta_1+s+1)(\beta_2+s+1)\rho_1\rho_2}+\frac{M^q \mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)\rho_1\rho_2}\right. \\ & \left.+\frac{M^q \mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)\rho_1\rho_2}+\frac{M^q \mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)}{\rho_1\rho_2}\right)^{\frac{1}{q}} \end{aligned} \tag{28}$$

and

$$\begin{aligned}|I_4| \leqslant & \left(\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \\ & \times\left(\frac{M^q}{(\beta_1+s+1)(\beta_2+s+1)\rho_1\rho_2}+\frac{M^q \mathrm{B}(\beta_2+1,s+1)}{(\beta_1+s+1)\rho_1\rho_2}\right. \\ & \left.+\frac{M^q \mathrm{B}(\beta_1+1,s+1)}{(\beta_2+s+1)\rho_1\rho_2}+\frac{M^q \mathrm{B}(\beta_1+1,s+1)\mathrm{B}(\beta_2+1,s+1)}{\rho_1\rho_2}\right)^{\frac{1}{q}}. \end{aligned} \tag{29}$$

The desired inequality follows from (24) and using (26), (27), (28) and (29).

REMARK 5. If we take $\rho_1=\rho_2=1$ in Theorem 6, then we obtain Theorem 3.

THEOREM 7. *Let* $\beta_1,\beta_2,\rho_1,\rho_2>0$ *and* $f:[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\to\mathbb{R}$ *be a twice partially differentiable mapping on* $(a^{\rho_1},b^{\rho_1})\times(c^{\rho_2},d^{\rho_2})$ *with* $0\leqslant a<b$, $0\leqslant c<d$, *and* $\frac{\partial^2 f}{\partial r\partial t}\in L_1\left([a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\right)$. *If* $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ *is* s*-convex in the second sense on the coordinates for* $q>1$ *and* $\left|\frac{\partial^2 f}{\partial r\partial t}\right|\leqslant M$ *on* $[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]$, *then the following inequality holds:*

$$\begin{aligned}&\left|T_f(\beta_1,\beta_2,\rho_1,\rho_2;a,b,c,d,x,y)\right| \\ \leqslant & M\left(\frac{1}{(\beta_1 w+1)(\beta_2 w+1)}\right)^{\frac{1}{w}}\left(\frac{4}{(s+1)^2}\right)^{\frac{1}{q}}\end{aligned}$$

$$\times \left[\frac{(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}+(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}}{b^{\rho_1}-a^{\rho_1}}\right]\left[\frac{(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}+(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}}{d^{\rho_2}-c^{\rho_2}}\right],$$

for all $(x^{\rho_1},y^{\rho_2})\in[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]$ *and* $\frac{1}{w}+\frac{1}{q}=1$.

Proof. By using Lemma 2 and the properties of the absolute value, we have

$$\begin{aligned}
&\left|T_f(\beta_1,\beta_2,\rho_1,\rho_2;a,b,c,d,x,y)\right| \\
\leqslant&\frac{\rho_1\rho_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} \\
&\times\Big((x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}|I_1|+(x^{\rho_1}-a^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}|I_2| \\
&+(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(y^{\rho_2}-c^{\rho_2})^{\beta_2+1}|I_3|+(b^{\rho_1}-x^{\rho_1})^{\beta_1+1}(d^{\rho_2}-y^{\rho_2})^{\beta_2+1}|I_4|\Big).
\end{aligned} \tag{30}$$

Now, by using the Hölder's inequality and the s-convexity in the second sense of $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ on the coordinates, we have

$$\begin{aligned}
|I_1|\leqslant&\left(\int_0^1\int_0^1 r^{\beta_2 w\rho_2}t^{\beta_1 w\rho_1}r^{\rho_2-1}t^{\rho_1-1}drdt\right)^{\frac{1}{w}} \\
&\times\left(\int_0^1\int_0^1 r^{\rho_2-1}t^{\rho_1-1}\left|\frac{\partial^2}{\partial r\partial t}f(t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})a^{\rho_1},r^{\rho_2}y^{\rho_2}+(1-r^{\rho_2})c^{\rho_2})\right|^q drdt\right)^{\frac{1}{q}} \\
\leqslant&\left(\frac{1}{(\beta_1 w+1)(\beta_2 w+1)\rho_1\rho_2}\right)^{\frac{1}{w}}\left(\left|\frac{\partial^2}{\partial r\partial t}f(x^{\rho_1},y^{\rho_2})\right|^q\int_0^1\int_0^1 r^{(s+1)\rho_2-1}t^{(s+1)\rho_1-1}drdt\right. \\
&+\left|\frac{\partial^2}{\partial r\partial t}f(x^{\rho_1},c^{\rho_2})\right|^q\int_0^1\int_0^1 r^{\rho_2-1}(1-r^{\rho_2})^s t^{(s+1)\rho_1-1}drdt \\
&+\left|\frac{\partial^2}{\partial r\partial t}f(a^{\rho_1},y^{\rho_2})\right|^q\int_0^1\int_0^1 r^{(s+1)\rho_2-1}(1-t^{\rho_1})^s t^{\rho_1-1}drdt \\
&\left.+\left|\frac{\partial^2}{\partial r\partial t}f(a^{\rho_1},c^{\rho_2})\right|^q\int_0^1\int_0^1 r^{\rho_2-1}t^{\rho_1-1}(1-r^{\rho_2})^s(1-t^{\rho_1})^s drdt\right)^{\frac{1}{q}} \\
=&\left(\frac{1}{(\beta_1 w+1)(\beta_2 w+1)\rho_1\rho_2}\right)^{\frac{1}{w}}\left(\frac{1}{(s+1)\rho_1(s+1)\rho_2}\left|\frac{\partial^2}{\partial r\partial t}f(x^{\rho_1},y^{\rho_2})\right|^q\right. \\
&+\frac{1}{(s+1)\rho_1(s+1)\rho_2}\left|\frac{\partial^2}{\partial r\partial t}f(x^{\rho_1},c^{\rho_2})\right|^q \\
&+\frac{1}{(s+1)\rho_1(s+1)\rho_2}\left|\frac{\partial^2}{\partial r\partial t}f(a^{\rho_1},y^{\rho_2})\right|^q
\end{aligned}$$

$$+\frac{1}{(s+1)\rho_1(s+1)\rho_2}\left|\frac{\partial^2}{\partial r\partial t}f\left(a^{\rho_1},c^{\rho_2}\right)\right|^q\Bigg)^{\frac{1}{q}}.$$

That is,

$$|I_1|\leqslant\left(\frac{1}{(\beta_1 w+1)(\beta_2 w+1)\rho_1\rho_2}\right)^{\frac{1}{w}}\left(\frac{1}{(s+1)^2\rho_1\rho_2}\right)^{\frac{1}{q}}$$
$$\times\left(\left|\frac{\partial^2}{\partial r\partial t}f(x^{\rho_1},y^{\rho_2})\right|^q+\left|\frac{\partial^2}{\partial r\partial t}f\left(x^{\rho_1},c^{\rho_2}\right)\right|^q+\left|\frac{\partial^2}{\partial r\partial t}f\left(a^{\rho_1},y^{\rho_2}\right)\right|^q\right.$$
$$\left.+\left|\frac{\partial^2}{\partial r\partial t}f\left(a^{\rho_1},c^{\rho_2}\right)\right|^q\right)^{\frac{1}{q}}.$$

Hence, by using the fact that $\left|\frac{\partial^2 f}{\partial r\partial t}\right|\leqslant M$, we have

$$|I_1|\leqslant 4^{\frac{1}{q}}M\left(\frac{1}{(\beta_1 w+1)(\beta_2 w+1)\rho_1\rho_2}\right)^{\frac{1}{w}}\left(\frac{1}{(s+1)^2\rho_1\rho_2}\right)^{\frac{1}{q}}. \tag{31}$$

Using similar arguments, we have

$$|I_2|\leqslant 4^{\frac{1}{q}}M\left(\frac{1}{(\beta_1 w+1)(\beta_2 w+1)\rho_1\rho_2}\right)^{\frac{1}{w}}\left(\frac{1}{(s+1)^2\rho_1\rho_2}\right)^{\frac{1}{q}}, \tag{32}$$

$$|I_3|\leqslant 4^{\frac{1}{q}}M\left(\frac{1}{(\beta_1 w+1)(\beta_2 w+1)\rho_1\rho_2}\right)^{\frac{1}{w}}\left(\frac{1}{(s+1)^2\rho_1\rho_2}\right)^{\frac{1}{q}} \tag{33}$$

and

$$|I_4|\leqslant 4^{\frac{1}{q}}M\left(\frac{1}{(\beta_1 w+1)(\beta_2 w+1)\rho_1\rho_2}\right)^{\frac{1}{w}}\left(\frac{1}{(s+1)^2\rho_1\rho_2}\right)^{\frac{1}{q}}. \tag{34}$$

The desired inequality follows from (30) and using (31)–(34). This completes the proof.

REMARK 6. If we take $\rho_1=\rho_2=1$ in Theorem 7, then we obtain Theorem 4.

3. Conclusion

We introduce three new Ostrowski type integral inequalities for functions of two variables whose mixed second order partial derivatives in absolute value to certain powers are s-convex on the coordinates by using generalized fractional integral operators. We deduce some results in the literature by considering some specific values of some of the parameters (see Remarks 4, 5 and 6). Several other interesting new results could be derived from our results by considering different values of the parameters. It is worth noting that similar results for functions whose second order mixed partial derivatives are convex on the coordinates can be obtained from our results by taking $s = 1$. We believe that these results will inspire further research on fractional integral inequalities and their applications.

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