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(ω,c)–ASYMPTOTICALLY PERIODIC MILD SOLUTIONS TO SOME ψ–HILFER FRACTIONAL EVOLUTION EQUATIONS

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Abstract. In this paper, we consider the following abstract ψ-Hilfer fractional differential equation

$$D_{0^+}^{\alpha,\beta;\psi}u(t) = Au(t) + f(t,u(t)), \quad 0<\alpha<1, \quad 0<\beta<1; \quad t \geqslant 0 \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(\theta)\}_{\theta>=0}$ on a Banach space $\mathbb{X}$ such that there exist positive constants $M,\lambda \geqslant 0$ with

$$\|T(\theta)\|_{\mathbb{X}} \leqslant Me^{-\lambda\theta}, \; \theta \geqslant 0. \tag{2}$$

We use the Banach fixed point principle to prove the existence and uniqueness of (ω,c)-asymptotically periodic mild solutions to equation (1). Further, Ulam-Hyers stability results are established.

1. Introduction

Fractional differential equations have attracted a growing interest, particularly in modeling complex physical phenomena with memory effects. The existence and uniqueness of periodic solutions for certain fractional differential equations have attracted several researchers. In 2020, G. Mophou and G. M. N'Guérékata [30] studied an existence result of (ω,c)-periodic mild solutions to some fractional differential equations. In 2021 M. Kéré , G. M. N'Guérékata and E. R. Oueama studied an existence result of (w,c)-almost periodic mild solutions to some fractional differential equations. For more results of periodic functions, (see [1, 4, 5, 6, 8, 10, 16, 21, 23, 26, 27, 32, 34, 35]).

It was in 2018 that E. Alvaraz *et al.* [6] have introduced the concept of (ω,c)-periodic functions which includes the class of periodic, antiperiodic and Bloch periodic functions. The following year, i.e. in 2019, E. Alvarez, M. Pinto and S. Castillo [7] extended the concept of (ω,c)-periodic functions to a new class of functions so-called (ω,c)-asymptotically periodic functions. A continuous function f is said to be (ω,c)-asymptotically periodic if it can be written as $f = f_1 + f_2$ where f_1 is a (ω,c)-periodic

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function and f_2 is c-asymptotic function. Following this, several authors became interested in this class of functions.

J. Larrouy and G. M. N'Guérékata [26] studied the existence and uniqueness of (ω, c)-periodic and asymptotically (ω, c)-periodic mild solutions to the following semilinear fractional differential equations:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + {}^C D_t^{\alpha-1} f(t, u(t)), \quad 1 < \alpha < 2, \quad t \in \mathbb{R} \\ u(0) = 0 \end{cases}$$

and

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + {}^C D_t^{\alpha-1} f(t, u(t-h)), \quad 1 < \alpha < 2, \quad t \in \mathbb{R}_+ \\ u(0) = 0 \end{cases}$$

where ${}^C D_t^\alpha(.)$ stands for the Caputo derivative and A is a linear densely defined operator of sectorial type on a complex Banach space $\mathbb{X}$.

In 2023, R. G. Foko Tiomela, G. M. N'Guérékata [18] consider the following fractional evolution equation:

$${}^C D_t^\alpha u(t) = Au(t) + f(t), \quad t \in \mathbb{R}$$

where ${}^C D_t^\alpha(.)$ denotes the Caputo fractional derivative of order $0 < \alpha \leqslant 1$. They proved that, mild solution of this equation are (ω, c)-asymptotically periodic and they established an existence and uniqueness result for optimal (ω, c)-asymptotically periodic mild solution.

Recently, in 2018, J. Vanterler da C. Sousa, E. C. de Oliveira [38], introduced a new fractional derivative with respect to another function the so-called ψ-Hilfer fractional derivative. The same year, J. Vanterler da C. Sousa, K. D. Kucche, E. C. de Oliveira [39] published an article entitled: On the Ulam-Hyers stabilities of the solutions of ψ-Hilfer fractional differential equation with abstract Volterra operator. In this paper, they proved the existence, uniqueness and established Ulam-Hyers stability of the solution.

This new fractional derivative attracted several authors. In 2021, F. Norouzi, G. M. N'Guérékata [33], in their paper, considered the following ψ-Hilfer fractional neutral fractional differential equations with infinite delays:

$$\begin{cases} {}^H D_{0^+}^{\alpha,\beta;\psi}[x(t) - h(t, x_t)] = Ax(t) + f(t, x(t), x_t), \quad t \in [0, b], \quad b > 0 \\ x(t) = \Phi(t), \quad t \in (-\infty, 0] \end{cases}$$

where ${}^H D_{0^+}^{\alpha,\beta;\psi}(.)$ is the ψ-Hilfer fractional derivative of order $0 < \alpha \leqslant 1$, with respect to function $\psi \in \mathscr{L}([0, b], \mathbb{X})$ and type $0 \leqslant \beta \leqslant 1$. They proved the existence and uniqueness of solution by using the Banach contraction mapping principle and Leray-Schauder alternative theorem. For more informations refer to [2, 3, 12, 22, 24, 29].

Motivated by all of the above, we consider the following equation

$$D_{0^+}^{\alpha,\beta;\psi} u(t) = Au(t) + f(t, u(t)), \quad 0 < \alpha < 1, \quad 0 < \beta < 1; \quad t \geqslant 0$$

where $D_{0^+}^{\alpha,\beta;\psi}(.)$ denotes the ψ-Hilfer fractional derivative of order α and type β.

The main purpose is to study the existence and uniqueness of (ω,c)-asymptotically periodic mild solution. For this, we use the well known Banach fixed point principle. We also establish Ulam-Hyers stability. The rest of our paper is organized as follows: In Section 2 we present some preliminary results that are useful in this paper to prove the results. In Section 3, we first obtain the Volterra integral equivalent equation and propose the mild solution of equation (1). Further, we prove the existence and uniqueness of solution. Finally , in Section 4, we establish ulam-Hyers stability of the solution.

2. Preliminaries

Throughout this paper, we assume that $(\mathbb{X},\|.\|)$ is a complex Banach space and we will denote by $C(\mathbb{R},\mathbb{X})$ the collection of all continuous functions from $\mathbb{R}$ into $\mathbb{X}$, and $BC(\mathbb{R},\mathbb{X})$ the collection of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$, and we set

$$C_0(\mathbb{X}) := \left\{h \in C(\mathbb{R}_+,\mathbb{X}) : \lim_{t\to\infty} h(t) = 0\right\}.$$

In this section, we give some notations, definitions and results on ψ-Hilfer fractional derivative and (ω,c)-asymptotically periodic functions.

DEFINITION 1. The Euler's Gamma function is given by:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt \;\text{ for }\; \alpha > 0.$$

Futhermore, $\Gamma(1) = 1$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for any $\alpha > 0$.

DEFINITION 2. The Laplace transform of a function g is denoted and defined by:

$$\mathscr{L}\{g(t)\}(s) = G(s) = \int_0^\infty g(t)e^{-st}dt \;\text{ for }\; s > 0.$$

DEFINITION 3. [36] The two-parameter Mittag-Leffler function is defined by the series expansions:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \;\text{ for }\; \alpha > 0,\ \beta > 0,\ z \in \mathbb{C},$$

where $\Gamma(t)$ is the gamma function.

LEMMA 1. [36, 42] *Let* $0 < \alpha < 2$, *and* $\beta \in \mathbb{R}$ *be arbitrary. We suppose that* μ *is such that* $\frac{\pi\alpha}{2} < \mu < \min\{\pi,\pi\alpha\}$. *Then there exists a constant* $C = C(\alpha,\beta,\mu) > 0$ *such that*

$$|E_{\alpha,\beta}(z)| \leqslant \frac{C}{1+|z|}, \quad \mu \leqslant |\arg(z)| \leqslant \pi.$$

PROPOSITION 1. [18] *Let $t \in \mathbb{R}$ and $0 < \alpha \leqslant 1$. Then the following holds:*

$$E_{\alpha,1}(t) > 0, \ \ E_{\alpha,\alpha}(t) > 0, \ \ \text{and} \ \ E'_{\alpha,\alpha}(t) > 0. \tag{3}$$

Moreover,

$$\lim_{t\to\infty} E_{\alpha,1}(t) = \infty, \ \ \lim_{t\to-\infty} E_{\alpha,1}(t) = 0. \tag{4}$$

LEMMA 2. [37] *Given $\lambda > 0$, and $0 < \alpha < 2$. Then the following identities hold:*

$$\partial_t E_{\alpha,1}\left(-\lambda t^{\alpha}\right) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda t^{\alpha}\right), \tag{5}$$

and

$$\partial_t \left(t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda t^{\alpha}\right)\right) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^{\alpha}), \ \ \text{for all} \ \ t > 0. \tag{6}$$

DEFINITION 4. The left sided fractional integral of a function f of order α with respect to ψ is

$$I_{0^+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \psi'(t)\left(\psi(x) - \psi(t)\right)^{\alpha-1} f(t)dt.$$

DEFINITION 5. The left sided Hilfer fractional derivative of a function f of order α and type β is

$$D_{0^+}^{\alpha,\beta} f(x) = I_{0^+}^{\gamma-\alpha} \frac{d}{dx}\left(I_{0^+}^{1-\gamma}\right) f(x)$$

where $\gamma = \beta(1-\alpha) + \alpha$.

DEFINITION 6. The left sided with respect to ψ-Hilfer fractional derivative of a function f of order α and type β is

$$D_{0^+}^{\alpha,\beta;\psi} f(x) = I_{0^+}^{(\gamma-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\left(I_{0^+}^{1-\gamma;\psi}\right) f(x).$$

THEOREM 1. [38] *If $f \in C^1(\mathbb{R}, \mathbb{X})$, then*

$$I_{0^+}^{\alpha;\psi} D_{0^+}^{\alpha,\beta;\psi} f(x) = f(x) - \frac{\left(\psi(x) - \psi(0)\right)^{\gamma-1}}{\Gamma(\gamma)} I_{0^+}^{(1-\gamma);\psi} f(0)$$

and

$$D_{0^+}^{\alpha,\beta;\psi} I_{0^+}^{\alpha;\psi} f(x) = f(x).$$

THEOREM 2. [38] *Let $f, \ g \in C^1(\mathbb{R}, \mathbb{X})$, $\alpha > 0$ and $0 < \beta < 1$. Then*

$$D_{0^+}^{\alpha,\beta;\psi} f(x) = D_{0^+}^{\alpha,\beta;\psi} g(x) \Longleftrightarrow f(x) = g(x) + c\left(\psi(x) - \psi(0)\right)^{\gamma-1}, \ \ c \in \mathbb{R}.$$

In what follows, we will recall some definitions and properties of both of (ω,c)-periodic and (ω,c)-asymptotically periodic functions.

DEFINITION 7. [6, 30] A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)-periodic if there exist $c \in \mathbb{C} \setminus \{0\}$ and $w > 0$ such that

$$f(t+\omega) = cf(t), \ \forall t \in \mathbb{R}.$$

ω is called the c-period of f.

We denote by $P_{\omega c}(X)$ the collection of all functions $f \in C(\mathbb{R}, \mathbb{X})$ which are (ω, c)-periodic. When $c = 1$ ($c = -1$ resp.) we recover the ω-periodic (resp. ω-antiperiodic) case.

THEOREM 3. [30] (Theorem 2.4) *Suppose* $f \in C^1(\mathbb{R}, \mathbb{X}) \cap P_{\omega c}(\mathbb{R}, \mathbb{X})$. *Then* $f' \in P_{\omega c}(\mathbb{R}, \mathbb{X})$.

DEFINITION 8. [7] A function $h \in C(\mathbb{R}, \mathbb{X})$ is called c-asymptotic if $c^\wedge(-t)h(t) \in C_0(\mathbb{X})$, that is,

$$\lim_{t \to \infty} c^\wedge(-t)h(t) = 0$$

where $c^\wedge(-t) = c^{-t/\omega}$. The collection of those functions will be denoted by $C_{0,c}(\mathbb{X})$.

DEFINITION 9. [7] A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)-asymptotically periodic if $f = g + h$ where $g \in P_{\omega,c}(\mathbb{R}, \mathbb{X})$ and $h \in C_{0,c}(\mathbb{X})$. The collection of those functions(with the same c-period ω for the first component) will be denoted by $AP_{\omega c}(\mathbb{X})$.

LEMMA 3. [7] *Let* $\alpha \in \mathbb{C}$. *Then*

1. $(f+g) \in AP_{\omega c}(\mathbb{X})$ *and* $\alpha h \in AP_{\omega,c}(\mathbb{X})$ *whenever* $f, g, h \in AP_{\omega c}(\mathbb{X})$.
2. *If* $\tau \geqslant 0$ *is constant, then* $f_\tau(t) := f(t+\tau) \in AP_{\omega,c}(\mathbb{X})$ *whenever* $f \in AP_{\omega c}(\mathbb{X})$.
3. *Let* $g \in P_{\omega,c}(\mathbb{X})$ *and* $h \in C_{0,c}(\mathbb{X})$ *such that* $g, h \in C^1(\mathbb{R}, \mathbb{X})$. *Then the derivative of* $(f = g + h) \in AP_{\omega c}(\mathbb{X})$ *belongs to* $AP_{\omega,c}(\mathbb{X})$.

THEOREM 4. [7] *Let* $f(t,x) := g(t,x) + h(t,x)$ *where* $g(t+\omega, cx) = cg(t,x)$ *and* $h \in C_{0,c}(\mathbb{X}, \mathbb{X})$.

Let us assume the following conditions:

(1) $h_t(z) = c^\wedge(-t)h(c^\wedge(t)z)$ *is uniformly continuous for* z *in any bounded subset of* $\mathbb{X}$ *uniformly for* $t \geqslant d$ *and* $h_t(z) \to 0$ *as* $t \to \infty$ *uniformly in* z.

(2) *There exists* $v \in BC(\mathbb{R}_+, \mathbb{R}_+)$ *such that*

$$\|f(t,u_1) - f(t,u_2)\| \leqslant v(t)\|u_1 - u_2\|, \ \forall\, u_1,\ u_2 \in \mathbb{X},\ t \in \mathbb{R}_+.$$

If $u \in AP_{\omega c}(\mathbb{X})$, *then* $f(.,u(.)) \in AP_{\omega c}(\mathbb{X})$.

THEOREM 5. [7] $AP_{\omega c}([d,\infty) \times \mathbb{X}, \mathbb{X})$ *is a Banach space with the norm*

$$\|f\|_{a\omega c} := \sup_{t \geqslant d} \||c|^\wedge(-t)f(t)\|.$$

For the uniqueness result, we will need the following theorem.

THEOREM 6. [12] (Contraction Mapping Principle) *Let $\mathbb{X}$ be a Banach space, $D \subset \mathbb{X}$ be a closed and $A : D \to D$ a contraction mapping (i.e., $\|Ax - Ay\| \leqslant k\|x - y\|$ for some $k \in (0,1)$) and for $x, y \in D$. Then, A has a unique fixed point.*

In what follows, we will consider $t \in \mathbb{R}_+$. Therefore we will use $\|f\|_{a\omega c}$ as

$$\|f\|_{a\omega c} := \sup_{t \geqslant 0} \| |c|^{\wedge}(-t) f(t) \|.$$

3. Existence and uniqueness of the mild solution

We suppose that ψ is an increasing and positive function, having a continuous derivative ψ' and such that $\psi(0) = 0$, $\lim_{t \longrightarrow +\infty} \psi(t) = +\infty$, and $\psi \in C^1(\mathbb{R}, \mathbb{X}) \cap P_\omega(\mathbb{X})$. We assume also that there exist positive constants $\eta, p \geqslant 0$ such that $\psi(t) \leqslant \eta t^p$.

LEMMA 4. *The equation (1) is equivalent to the Volterra integral equation*

$$u(t) = \frac{(\psi(t))^{\gamma-1}}{\Gamma(\gamma)} I_{0^+}^{(1-\gamma);\psi} u(0) + \frac{A}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} f(\tau, u(\tau)) d\tau. \tag{7}$$

Proof. Applying the operator $I_{0^+}^{\alpha;\psi}$ to the both sides of equation (1), we get

$$I_{0^+}^{\alpha;\psi} D_{0^+}^{\alpha,\beta;\psi} u(t) = A I_{0^+}^{\alpha;\psi} u(t) + I_{0^+}^{\alpha;\psi} f(t, u(t)). \tag{8}$$

From Theorem 1 the left hand side of (8) becomes

$$I_{0^+}^{\alpha;\psi} D_{0^+}^{\alpha,\beta;\psi} u(t) = u(t) - \frac{(\psi(t))^{\gamma-1}}{\Gamma(\gamma)} I_{0^+}^{(1-\gamma);\psi} u(0).$$

So, it follows that

$$u(t) = \frac{(\psi(t))^{\gamma-1}}{\Gamma(\gamma)} I_{0^+}^{(1-\gamma);\psi} u(0) + \frac{A}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} f(\tau, u(\tau)) d\tau.$$

Conversely, if u satisfies (7) then,

$$u(t) - \frac{(\psi(t))^{\gamma-1}}{\Gamma(\gamma)} I_{0^+}^{(1-\gamma);\psi} u(0) = A I_{0^+}^{\alpha,\psi} u(t) + I_{0^+}^{\alpha;\psi} f(t, u(t)),$$

we deduce that

$$I_{0^+}^{\alpha;\psi} D_{0^+}^{\alpha,\beta;\psi} u(t) = A I_{0^+}^{\alpha,\psi} u(t) + I_{0^+}^{\alpha;\psi} f(t, u(t)). \tag{9}$$

Applying the operator $D_{0^+}^{\alpha,\beta;\psi}$ to both sides of equation (9) while taking into account Theorem 1 and Theorem 2, we obtain

$$\begin{aligned} D_{0^+}^{\alpha,\beta;\psi}u(t) &= D_{0^+}^{\alpha,\beta;\psi}\left[AI_{0^+}^{\alpha;\psi}u(t)+I_{0^+}^{\alpha;\psi}f(t,u(t))\right] \\ &= AD_{0^+}^{\alpha,\beta;\psi}.I_{0^+}^{\alpha;\psi}u(t)+D_{0^+}^{\alpha,\beta;\psi}.I_{0^+}^{\alpha;\psi}f(t,u(t)) \\ &= Au(t)+f(t,u(t)). \quad \square \end{aligned}$$

LEMMA 5. *Suppose that (7) is fulfilled, then we have the following integral equation:*

$$\begin{aligned} u(x) = &\int_0^x \psi'(\sigma)(\psi(x)-\psi(\sigma))^{\gamma-2}\left(\int_0^{+\infty} T\left(\left(\frac{\psi(\sigma)}{\eta}\right)^{\alpha}\right)\rho_\alpha(\eta)\,d\eta\right)u_0 d\sigma \\ &+\int_0^x \psi'(\sigma)R(\psi(x)-\psi(\sigma))f(\sigma,u(\sigma))\,d\sigma \end{aligned}$$

where

$$R(\tau)=\int_0^{+\infty} T(\theta\tau^{\alpha})\tau^{\alpha-1}\theta^{\frac{-1}{\alpha}}\rho_\alpha\left(\theta^{-1/\alpha}\right)d\theta$$

and

$$u_0 = I_{0^+}^{(1-\gamma);\psi}u(0).$$

Proof. Let us set $x=\psi(t)$ and make the change of variable $\sigma=\psi(\tau)$. Then, $d\sigma=\psi'(\tau)d\tau$ and (7) becomes

$$\begin{aligned} u\left(\psi^{-1}(x)\right) = &\frac{x^{\gamma-1}}{\Gamma(\gamma)}I_{0^+}^{(1-\gamma);\psi}u(0)+\frac{A}{\Gamma(\alpha)}\int_0^x (x-\sigma)^{\alpha-1}u(\psi^{-1}(\sigma))d\sigma \\ &+\frac{1}{\Gamma(\alpha)}\int_0^x (x-\sigma)^{\alpha-1}f(\psi^{-1}(\sigma),u(\psi^{-1}(\sigma)))d\sigma. \end{aligned} \tag{10}$$

If we set $u_0=I_{0^+}^{(1-\gamma);\psi}u(0)$, $z(x)=u\left(\psi^{-1}(x)\right)$ and $g(x)=f(\psi^{-1}(x),u(\psi^{-1}(x)))$, then (10) becomes

$$z(x)=\frac{x^{\gamma-1}}{\Gamma(\gamma)}u_0+\frac{A}{\Gamma(\alpha)}\int_0^x (x-\sigma)^{\alpha-1}z(\sigma)\,d\sigma+\frac{1}{\Gamma(\alpha)}\int_0^x (x-\sigma)^{\alpha-1}g(\sigma)\,d\sigma. \tag{11}$$

Applying the Laplace transform on the both sides of (11), we have

$$Z(s)=s^{-\gamma}u_0+s^{-\alpha}AZ(s)+s^{-\alpha}G(s).$$

Which implies that

$$\left(\frac{s^{\alpha}I-A}{s^{\alpha}}\right)Z(s)=s^{-\gamma}u_0+s^{-\alpha}G(s).$$

Therefore

$$Z(s)=s^{\alpha-\gamma}\left(s^{\alpha}I-A\right)^{-1}u_0+\left(s^{\alpha}I-A\right)^{-1}G(s) \tag{12}$$

where $Z(s) = \mathscr{L}[z(t)](s)$ and $G(s) = \mathscr{L}[g(t)](s)$ are the Laplace transforms of z and g respectively. Because of $\mathscr{L}[T(t)](s) = (s^\alpha I - A)^{-1}$, then it follows from (12) that

$$\begin{aligned} Z(s) &= s^{\alpha-\gamma}\mathscr{L}[T(\sigma)](s^\alpha)u_0 + \mathscr{L}[T(\sigma)](s^\alpha)G(s) \\ &= s^{\alpha-\gamma}u_0\int_0^{+\infty} e^{-s^\alpha\sigma}T(\sigma)\,d\sigma + \int_0^{+\infty} e^{-s^\alpha\sigma}T(\sigma)\,d\sigma \int_0^{+\infty} e^{-s\tau}g(\tau)\,d\tau. \end{aligned} \quad (13)$$

Considering the following change of variable $\sigma = \delta^\alpha$, equation (13) gives

$$\begin{aligned} Z(s) &= \alpha s^{\alpha-\gamma}u_0\int_0^{+\infty} e^{-(s\delta)^\alpha}T(\delta^\alpha)\delta^{\alpha-1}d\delta \\ &\quad + \alpha\int_0^{+\infty} e^{-(s\delta)^\alpha}T(\delta^\alpha)\delta^{\alpha-1}d\delta\int_0^{+\infty} e^{-s\tau}g(\tau)\,d\tau \\ &= \alpha s^{1-\gamma}u_0\int_0^{+\infty} T(\delta^\alpha)(s\delta)^{\alpha-1}e^{-(s\delta)^\alpha}d\delta \\ &\quad + \alpha\int_0^{+\infty}\left(\int_0^{+\infty} T(\delta^\alpha)\delta^{\alpha-1}e^{-(s\delta)^\alpha}e^{-s\tau}g(\tau)\,d\tau\right)d\delta \\ &= \alpha s^{1-\gamma}u_0\int_0^{+\infty} T(\delta^\alpha)\frac{d}{d\delta}\left(\frac{-1}{\alpha s}e^{-(s\delta)^\alpha}\right)d\delta \\ &\quad + \alpha\int_0^{+\infty}\left(\int_0^{+\infty} T(\delta^\alpha)\delta^{\alpha-1}e^{-(s\delta)^\alpha}e^{-s\tau}g(\tau)\,d\tau\right)d\delta. \end{aligned}$$

Because of

$$\int_0^{+\infty} e^{-s\eta}\rho_\alpha(\eta)\,d\eta = e^{-s^\alpha}$$

then

$$\begin{aligned} Z(s) &= -s^{-\gamma}u_0\int_0^{+\infty} T(\delta^\alpha)\frac{d}{d\delta}\left(\int_0^{+\infty} e^{-s\delta\eta}\rho_\alpha(\eta)\,d\eta\right)d\delta \\ &\quad + \alpha\int_0^{+\infty}\left(\int_0^{+\infty} T(\delta^\alpha)\delta^{\alpha-1}\left(\int_0^{+\infty} e^{-s\delta\eta}\rho_\alpha(\eta)\,d\eta\right)e^{-s\tau}g(\tau)\,d\tau\right)d\delta \\ &= s^{1-\gamma}u_0\int_0^{+\infty}\left(\int_0^{+\infty} T(\delta^\alpha)e^{-s\delta\eta}\rho_\alpha(\eta)\,\eta\,d\eta\right)d\delta \\ &\quad + \alpha\int_0^{+\infty}\left(\int_0^{+\infty}\left(\int_0^{+\infty} T(\delta^\alpha)\delta^{\alpha-1}e^{-s\tau}g(\tau)e^{-s\delta\eta}\rho_\alpha(\eta)\,d\eta\right)d\tau\right)d\delta. \end{aligned} \quad (14)$$

Now, we make the change of variable $\theta = \eta\delta \Longrightarrow d\theta = \eta d\delta$. Equation (14) becomes

$$\begin{aligned} Z(s) &= s^{1-\gamma}u_0\int_0^{+\infty}\left(\int_0^{+\infty} T\left(\left(\frac{\theta}{\eta}\right)^\alpha\right)e^{-s\theta}\rho_\alpha(\eta)\,d\eta\right)d\theta \\ &\quad + \alpha\int_0^{+\infty}\left(\int_0^{+\infty}\left(\int_0^{+\infty} T\left(\left(\frac{\theta}{\eta}\right)^\alpha\right)\frac{\theta^{\alpha-1}}{\eta^\alpha}e^{-s\tau}g(\tau)e^{-s\theta}\rho_\alpha(\eta)\,d\eta\right)d\tau\right)d\theta. \end{aligned} \quad (15)$$

With the following change of variable $\tau+\theta=t \Longrightarrow d\theta=dt$, the second term of right hand side of (15) becomes

$$\begin{aligned}
&\alpha\int_0^{+\infty}\left(\int_0^{+\infty}\left(\int_0^{+\infty}T\left(\left(\frac{\theta}{\eta}\right)^{\alpha}\right)\frac{\theta^{\alpha-1}}{\eta^{\alpha}}e^{-s\tau}g(\tau)e^{-s\theta}\rho_\alpha(\eta)\,d\eta\right)d\tau\right)d\theta\\
&=\alpha\int_0^{+\infty}\left(\int_0^{+\infty}\left(\int_0^{t}T\left(\frac{(t-\tau)^{\alpha}}{\eta^{\alpha}}\right)\frac{(t-\tau)^{\alpha-1}}{\eta^{\alpha}}e^{-st}g(\tau)\rho_\alpha(\eta)\,d\tau\right)d\eta\right)dt\\
&=\int_0^{+\infty}e^{-st}\left(\alpha\int_0^{+\infty}\left(\int_0^{t}T\left(\frac{(t-\tau)^{\alpha}}{\eta^{\alpha}}\right)\frac{(t-\tau)^{\alpha-1}}{\eta^{\alpha}}g(\tau)\rho_\alpha(\eta)\,d\tau\right)d\eta\right)dt\\
&=\mathscr{L}[h(t)](s)
\end{aligned}$$

where

$$h(t)=\alpha\int_0^{+\infty}\left(\int_0^{t}T\left(\frac{(t-\tau)^{\alpha}}{\eta^{\alpha}}\right)\frac{(t-\tau)^{\alpha-1}}{\eta^{\alpha}}g(\tau)\rho_\alpha(\eta)\,d\tau\right)d\eta.$$

If we make the change of variable $\theta=\eta^{-\alpha}\Longrightarrow d\eta=\frac{-1}{\alpha}\theta^{\frac{-1}{\alpha}-1}d\theta$, we obtain

$$\begin{aligned}
h(t)&=\alpha\int_0^{+\infty}\left(\int_0^{t}T\left(\theta(t-\tau)^{\alpha}\right)(t-\tau)^{\alpha-1}\theta g(\tau)\rho_\alpha\left(\theta^{-1/\alpha}\right)d\tau\right)\left(\frac{1}{\alpha}\theta^{\frac{-1}{\alpha}-1}d\theta\right)\\
&=\int_0^{+\infty}\left(\int_0^{t}T\left(\theta(t-\tau)^{\alpha}\right)(t-\tau)^{\alpha-1}\theta^{\frac{-1}{\alpha}}g(\tau)\rho_\alpha\left(\theta^{-1/\alpha}\right)d\tau\right)d\theta\\
&=\int_0^{t}\left(\int_0^{+\infty}\left(T\left(\theta(t-\tau)^{\alpha}\right)(t-\tau)^{\alpha-1}\theta^{\frac{-1}{\alpha}}g(\tau)\rho_\alpha\left(\theta^{-1/\alpha}\right)d\theta\right)\right)d\tau\\
&=\int_0^{t}R(t-\tau)g(\tau)\,d\tau\\
&=R*g(t)
\end{aligned}$$

where

$$R(\tau)=\int_0^{+\infty}T\left(\theta\tau^{\alpha}\right)\tau^{\alpha-1}\theta^{\frac{-1}{\alpha}}\rho_\alpha\left(\theta^{-1/\alpha}\right)d\theta.$$

In the other hand we have

$$\begin{aligned}
&s^{1-\gamma}u_0\int_0^{+\infty}\left(\int_0^{+\infty}T\left(\left(\frac{\theta}{\eta}\right)^{\alpha}\right)e^{-s\theta}\rho_\alpha(\eta)\,d\eta\right)d\theta\\
&=s^{1-\gamma}u_0\int_0^{+\infty}e^{-s\theta}\left(\int_0^{+\infty}T\left(\left(\frac{\theta}{\eta}\right)^{\alpha}\right)\rho_\alpha(\eta)\,d\eta\right)d\theta\\
&=\left(\mathscr{L}\left[\theta^{\gamma-2}\right](s)\right)\left(\mathscr{L}\left[\int_0^{+\infty}T\left(\left(\frac{\theta}{\eta}\right)^{\alpha}\right)u_0\rho_\alpha(\eta)\,d\eta\right](s)\right)\\
&=\mathscr{L}[i(t)](s)\,\mathscr{L}[j(t)](s)\\
&=\mathscr{L}[i*j(t)](s)
\end{aligned}$$

where

$$i(t) = t^{\gamma-2} \quad \text{and} \quad j(t) = \int_0^{+\infty} T\left(\left(\frac{t}{\eta}\right)^{\alpha}\right)\rho_\alpha(\eta)\,d\eta.$$

Thus, (15) becomes

$$Z(s) = \mathscr{L}\left[i * j(t)\,u_0 + R * g(t)\right](s),$$

therefore

$$z(t) = i * j(t)\,u_0 + R * g(t). \tag{16}$$

Since $z(t) = u\left(\psi^{-1}(t)\right)$ and $g(t) = f\left(\psi^{-1}(t), u(\psi^{-1}(t))\right)$, then (16) becomes

$$\begin{aligned} u\left(\psi^{-1}(t)\right) &= i * j(t)\,u_0 + R * f\left(\psi^{-1}(t)\right) \\ &= \int_0^t i(t-\tau)\,j(\tau)\,u_0 d\tau + \int_0^t R(t-\tau)\,g(\tau)\,d\tau. \end{aligned}$$

If we set $x = \psi^{-1}(t)$, then

$$u(x) = \int_0^{\psi(x)} i(\psi(x)-\tau)\,j(\tau)\,u_0 d\tau + \int_0^{\psi(x)} R(\psi(x)-\tau)\,g(\tau)\,d\tau.$$

The change of variable $\tau = \psi(\sigma) \Longrightarrow d\tau = \psi'(\sigma)\,d\sigma$ allowed us to writte

$$\begin{aligned} u(x) = &\int_0^x i(\psi(x)-\psi(\sigma))\,j(\psi(\sigma))\,u_0\psi'(\sigma)\,d\sigma \\ &+ \int_0^x R(\psi(x)-\psi(\sigma))\,g(\psi(\sigma))\,\psi'(\sigma)\,d\sigma \end{aligned}$$

$$\begin{aligned} u(x) = &\int_0^x \psi'(\sigma)\,(\psi(x)-\psi(\sigma))^{\gamma-2}\left(\int_0^{+\infty} T\left(\left(\frac{\psi(\sigma)}{\eta}\right)^{\alpha}\right)\rho_\alpha(\eta)\,d\eta\right)u_0 d\sigma \\ &+ \int_0^x \psi'(\sigma)\,R(\psi(x)-\psi(\sigma))\,f(\sigma, u(\sigma))\,d\sigma. \quad \square \end{aligned}$$

DEFINITION 10. A function u is said to be a mild solution to the problem (1) if u verifies

$$\begin{aligned} u(t) = &\int_0^t \psi'(\sigma)\,(\psi(t)-\psi(\sigma))^{\gamma-2}\,Q(\sigma)u_0 d\sigma \\ &+ \int_0^t \psi'(\sigma)\,R(\psi(t)-\psi(\sigma))\,f(\sigma, u(\sigma))\,d\sigma, \end{aligned} \tag{17}$$

where

$$Q(\tau) = \int_0^{+\infty} T\left(\left(\frac{\psi(\tau)}{\eta}\right)^{\alpha}\right)\rho_\alpha(\eta)\,d\eta, \tag{18}$$

$$R(\tau) = \int_0^{+\infty} \alpha\theta\Phi_\alpha(\theta)\,T(\theta\tau^{\alpha})\,\tau^{\alpha-1}d\theta, \tag{19}$$

and

$$\Phi_\alpha(\theta)=\frac{1}{\alpha}\theta^{-\frac{1}{\alpha}-1}\rho_\alpha\left(\theta^{-1/\alpha}\right), \tag{20}$$

with Φ_α is a probability density function, i.e.:

$$\Phi_\alpha(\theta)\geqslant 0 \text{ for all } \theta\geqslant 0,$$
$$\int_0^{+\infty}\Phi_\alpha(\theta)\,d\theta=1.$$

PROPOSITION 2. [18] *Let $0<\alpha\leqslant 1$. Considering the probability density Φ_α defined by (20), the following hold:*

(i) $\int_0^{+\infty}\Phi_\alpha(\theta)e^{-z\theta}d\theta=E_{\alpha,1}(-z)$, $z\in\mathbb{C}$.

(ii) $\int_0^{+\infty}\alpha\theta\Phi_\alpha(\theta)e^{-z\theta}d\theta=E_{\alpha,\alpha}(-z)$, $z\in\mathbb{C}$.

Based on Proposition 2, we obtain the following proposition.

PROPOSITION 3. *The following results hold:*

(i) $\|Q(\tau)\|_{\mathbb{X}}\leqslant ME_{\alpha,1}\left(-\lambda(\psi(\tau))^\alpha\right)$, $\tau\geqslant 0$,

(ii) $\|R(\tau)\|_{\mathbb{X}}\leqslant M\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda\tau^\alpha)$, $\tau\geqslant 0$,

where $Q(\tau)$ and $R(\tau)$ are respectively defined by (18) and (19), M,λ two positive constants satisfying (2).

Proof. Let $\tau\geqslant 0$, we have

$$\begin{aligned}\|Q(\tau)\|_{\mathbb{X}}&=\|\int_0^{+\infty}T\left(\left(\frac{\psi(\tau)}{\eta}\right)^\alpha\right)\rho_\alpha(\eta)\,d\eta\|_{\mathbb{X}}\\&\leqslant\int_0^{+\infty}\|T\left(\left(\frac{\psi(\tau)}{\eta}\right)^\alpha\right)\|_{\mathbb{X}}\rho_\alpha(\eta)\,d\eta.\end{aligned}$$

Using (2), we obtain

$$\begin{aligned}\|Q(\tau)\|_{\mathbb{X}}&\leqslant M\int_0^{+\infty}e^{-\lambda\left(\left(\frac{\psi(\tau)}{\eta}\right)^\alpha\right)}\rho_\alpha(\eta)\,d\eta\\&\leqslant M\int_0^{+\infty}e^{-\lambda(\psi(\tau))^\alpha\eta^{-\alpha}}\rho_\alpha(\eta)\,d\eta.\end{aligned}$$

Considering the change of variable $\theta=\eta^{-\alpha}$, we have $\eta=\theta^{-\frac{1}{\alpha}}$ and $d\eta=-\frac{1}{\alpha}\theta^{-\frac{1}{\alpha}-1}d\theta$.

Therefore

$$\begin{aligned}\|Q(\tau)\|_{\mathbb{X}}&\leqslant M\int_0^{+\infty}\frac{1}{\alpha}\rho_\alpha\left(\theta^{-\frac{1}{\alpha}}\right)\theta^{-\frac{1}{\alpha}-1}e^{-\lambda(\psi(\tau))^\alpha\theta}d\theta\\&\leqslant M\int_0^{+\infty}\Phi_\alpha(\theta)e^{-\lambda(\psi(\tau))^\alpha\theta}d\theta.\end{aligned}$$

From the equality (i) in Proposition 2, we have:

$$\|Q(\tau)\|_{\mathbb{X}} \leqslant ME_{\alpha,1}\left(-\lambda\left(\psi(\tau)\right)^{\alpha}\right), \quad \tau \geqslant 0.$$

According to (2) and the equality (ii) in Proposition 2, we have:

$$\begin{aligned}\|R(\tau)\|_{\mathbb{X}} &= \|\int_0^{+\infty} \alpha\theta\Phi_\alpha(\theta)\,T(\theta\tau^\alpha)\,\tau^{\alpha-1}d\theta\|_{\mathbb{X}}\\ &\leqslant \int_0^{+\infty} \alpha\theta\Phi_\alpha(\theta)\,\|T(\theta\tau^\alpha)\|_{\mathbb{X}}\tau^{\alpha-1}d\theta\\ &\leqslant M\tau^{\alpha-1}\int_0^{+\infty} \alpha\theta\Phi_\alpha(\theta)\,e^{-\lambda\tau^\alpha\theta}d\theta\\ &\leqslant M\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda\tau^\alpha), \quad \tau\geqslant 0. \quad \square\end{aligned}$$

PROPOSITION 4. *Let $c\in\mathbb{C}\setminus\{0\}$ and $\omega>0$. Then the function u define by:*

$$u(t)=\int_0^t \psi'(\sigma)\,(\psi(t)-\psi(\sigma))^{\gamma-2}\,Q(\sigma)u_0 d\sigma$$

belongs to $AP_{\omega c}(\mathbb{X})$ if $|c|\geqslant 1$.

Proof. Let $t\geqslant 0, c\in\mathbb{C}\setminus\{0\}$, $\omega>0$ such that $|c|\geqslant 1$ and $u\in\mathbb{X}$, we have:

$$\begin{aligned}\|c^\wedge(-t)u(t)\|_{\mathbb{X}} &= \|c^{-t/\omega}\int_0^t \psi'(\sigma)\,(\psi(t)-\psi(\sigma))^{\gamma-2}\,Q(\sigma)u_0 d\sigma\|_{\mathbb{X}}\\ &\leqslant \int_0^t |c|^{-t/\omega}\psi'(\sigma)\,(\psi(t)-\psi(\sigma))^{\gamma-2}\,\|Q(\sigma)\|_{\mathbb{X}}\|u_0\|_{\mathbb{X}}d\sigma.\end{aligned}$$

From the equality (i) in Proposition 3, we have:

$$\begin{aligned}\|c^\wedge(-t)u(t)\|_{\mathbb{X}} &\leqslant M\|u_0\|_{\mathbb{X}}\int_0^t |c|^{-t/\omega}\psi'(\sigma)\,(\psi(t)-\psi(\sigma))^{\gamma-2}\,E_{\alpha,1}\left(-\lambda\left(\psi(\sigma)\right)^{\alpha}\right)d\sigma\\ &\leqslant M\|u_0\|_{\mathbb{X}}\int_0^t |c|^{-t/\omega}\psi'(\sigma)\,(\eta t^p)^{\gamma-2}\,E_{\alpha,1}\left(-\lambda\left(\psi(\sigma)\right)^{\alpha}\right)d\sigma\\ &\leqslant M\eta^{\gamma-2}\|u_0\|_{\mathbb{X}}t^{p(\gamma-2)}\int_0^t |c|^{-t/\omega}\psi'(\sigma)\,E_{\alpha,1}\left(-\lambda\left(\psi(\sigma)\right)^{\alpha}\right)d\sigma.\end{aligned}$$

Take $\tau=\psi(\sigma)$, we have $d\sigma=\frac{1}{\psi'(\sigma)}d\tau$, since $|c|\geqslant 1$, then $|c|^{-t/\omega}\leqslant 1$. Therefore, using Lemma 1, we have

$$\begin{aligned}\|c^\wedge(-t)u(t)\|_{\mathbb{X}} &\leqslant M\eta^{\gamma-2}\|u_0\|_{\mathbb{X}}t^{p(\gamma-2)}\int_0^{\psi(t)} E_{\alpha,1}(-\lambda\tau^\alpha)\,d\tau\\ &\leqslant MC\eta^{\gamma-2}\|u_0\|_{\mathbb{X}}t^{p(\gamma-2)}\int_0^{\psi(t)} \frac{1}{1+\lambda\tau^\alpha}d\tau.\end{aligned}$$

Setting $v=\tau^{\alpha}$, we get

$$\|c^{\wedge}(-t)u(t)\|_{\mathbb{X}} \leqslant \frac{MC\eta^{\gamma-2}\|u_0\|_{\mathbb{X}}}{\alpha} t^{p(\gamma-2)} \int_0^{\psi(t)^{\alpha}} \frac{v^{\frac{1}{\alpha}-1}}{1+\lambda v} dv.$$

Take $z=\lambda v$, we have

$$\begin{aligned}
\|c^{\wedge}(-t)u(t)\|_{\mathbb{X}} &\leqslant \frac{MC\eta^{\gamma-2}\|u_0\|_{\mathbb{X}}}{\alpha\lambda^{\frac{1}{\alpha}}} t^{p(\gamma-2)} \int_0^{\lambda\psi(t)^{\alpha}} \frac{z^{\frac{1}{\alpha}-1}}{1+z} dz \\
&\leqslant \frac{MC\eta^{\gamma-2}\|u_0\|_{\mathbb{X}}}{\alpha\lambda^{\frac{1}{\alpha}}} t^{p(\gamma-2)} \int_0^{+\infty} \frac{z^{\frac{1}{\alpha}-1}}{1+z} dz \\
&\leqslant \frac{MC\eta^{\gamma-2}\|u_0\|_{\mathbb{X}}}{\alpha\lambda^{\frac{1}{\alpha}}} t^{p(\gamma-2)} B\left(\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right) \\
&\leqslant \frac{MC\eta^{\gamma-2}\|u_0\|_{\mathbb{X}}}{\alpha\lambda^{\frac{1}{\alpha}}} t^{p(\gamma-2)} \Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right) \\
&\leqslant \frac{MC\frac{\pi}{\alpha}\eta^{\gamma-2}\|u_0\|_{\mathbb{X}}}{\alpha\lambda^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right)} t^{p(\gamma-2)}.
\end{aligned}$$

As $\gamma-2=-(1-\alpha)(1-\beta)-1<0$, then

$$\lim_{t\to\infty} t^{p(\gamma-2)}=0.$$

Hence

$$\lim_{t\to\infty} c^{\wedge}(-t)u(t)=0,$$

therefore $u\in C_{0,c}(\mathbb{X})\subseteq AP_{\omega c}(\mathbb{X})$.

Finally $u\in AP_{\omega c}(\mathbb{X})$ if $|c|\geqslant 1$. □

PROPOSITION 5. *Let $f\in P_{\omega c}(\mathbb{X})$. Then the function u defined by:*

$$u(t)=\int_{-\infty}^{t} \psi'(\sigma)R(\psi(t)-\psi(\sigma))f(\sigma)d\sigma$$

belongs to $P_{\omega c}(\mathbb{X})$.

Proof. Let $\omega>0$, we have:

$$u(t+\omega)=\int_{-\infty}^{t+\omega} \psi'(\sigma)R(\psi(t+\omega)-\psi(\sigma))f(\sigma)d\sigma.$$

Making the change of variable $\xi=\sigma-\omega$, we obtain

$$u(t+\omega)=\int_{-\infty}^{t} \psi'(\xi+\omega)R(\psi(t+\omega)-\psi(\xi+\omega))f(\xi+\omega)d\xi.$$

Since $\psi \in C^1(\mathbb{X}) \cap P_\omega(\mathbb{X})$ and using Theorem 3, then

$$\psi(t+\omega) = \psi(t)$$

and

$$\psi'(\xi+\omega) = \psi'(\xi).$$

We have also $f \in P_{\omega c}(\mathbb{X})$, so

$$f(\xi+\omega) = cf(\xi).$$

Thus

$$\begin{aligned} u(t+\omega) &= c\int_{-\infty}^{t} \psi'(\xi) R(\psi(t) - \psi(\xi)) f(\xi)\, d\xi \\ &= cu(t). \end{aligned}$$

Consequently $u \in P_{\omega c}(\mathbb{X})$. □

PROPOSITION 6. *Let $f \in C_{0,c}(\mathbb{X})$, and define the function u by:*

$$u(t) = \int_{-\infty}^{t} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) f(\sigma)\, d\sigma.$$

Then u belongs to $C_{0,c}(\mathbb{X})$ if $|c| \geqslant 1$.

Proof. We have:

$$\begin{aligned} \|c^\wedge(-t)u(t)\|_{\mathbb{X}} &= \|c^{-t/\omega}\int_{-\infty}^{t} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) f(\sigma)\, d\sigma\|_{\mathbb{X}} \\ &= \|\int_{-\infty}^{t} c^{-(t-\sigma)/\omega} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) c^{-\sigma/\omega} f(\sigma)\, d\sigma\|_{\mathbb{X}} \\ &\leqslant \int_{-\infty}^{t} |c|^{-(t-\sigma)/\omega} \psi'(\sigma) \|R(\psi(t) - \psi(\sigma))\|_{\mathbb{X}} \|c^{-\sigma/\omega} f(\sigma)\|_{\mathbb{X}} d\sigma \\ &\leqslant \left(\sup_{\sigma \in (-\infty, t]} \left\{ \|c^{-\sigma/\omega} f(\sigma)\|_{\mathbb{X}} \right\} \right) \\ &\quad \times \int_{-\infty}^{t} |c|^{-(t-\sigma)/\omega} \psi'(\sigma) \|R(\psi(t) - \psi(\sigma))\|_{\mathbb{X}} d\sigma \end{aligned}$$

Using the equality (ii) in Proposition 3, we obtain

$$\begin{aligned} &\|c^\wedge(-t)u(t)\|_{\mathbb{X}} \\ &\leqslant M \left(\sup_{\sigma \in (-\infty, t]} \left\{ \|c^{-\sigma/\omega} f(\sigma)\|_{\mathbb{X}} \right\} \right) \\ &\quad \times \int_{-\infty}^{t} |c|^{-(t-\sigma)/\omega} \psi'(\sigma) (\psi(t) - \psi(\sigma))^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda (\psi(t) - \psi(\sigma))^\alpha\right) d\sigma. \end{aligned}$$

As $|c|\geqslant 1$ and $t-\sigma\geqslant 0$ implies that $|c|^{-(t-\sigma)/\omega}\leqslant 1$.

Thus

$$\|c^{\wedge}(-t)u(t)\|_{\mathbb{X}}\leqslant M\left(\sup_{\sigma\in\left(-\infty,t\right]}\left\{\|c^{-\sigma/\omega}f(\sigma)\|_{\mathbb{X}}\right\}\right)\times\int_{-\infty}^{t}\psi'(\sigma)(\psi(t)-\psi(\sigma))^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(\sigma))^{\alpha}\right)d\sigma.$$

Let us consider the integral

$$J=\int_{-\infty}^{t}\psi'(\sigma)(\psi(t)-\psi(\sigma))^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(\sigma))^{\alpha}\right)d\sigma.$$

Making the change of variable $x=\psi(t)-\psi(\sigma)$, we have $d\sigma=-\frac{1}{\psi'(\sigma)}dx$.

Further, ψ is an increasing and positive function, and $\psi(0)=0$, then $\lim_{\sigma\longrightarrow-\infty}\psi(\sigma)=0$.

The above integral becomes

$$J=-\int_{\psi(t)}^{0}x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^{\alpha})\,dx.$$

According to (5), we have

$$x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^{\alpha})=-\frac{1}{\lambda}\partial_x E_{\alpha,1}(-\lambda x^{\alpha}) \tag{21}$$

and so

$$\begin{aligned}J&=\frac{1}{\lambda}\int_{\psi(t)}^{0}\partial_x E_{\alpha,1}(-\lambda x^{\alpha})\,dx\\&=\frac{1}{\lambda}\left[E_{\alpha,1}(-\lambda x^{\alpha})\right]_{\psi(t)}^{0}\\&=\frac{1}{\lambda}\Big(E_{\alpha,1}(0)-E_{\alpha,1}\left(-\lambda(\psi(t))^{\alpha}\right)\Big).\end{aligned}$$

Since

$$\lim_{t\to\infty}\psi(t)=\infty,$$

we deduce from Proposition 1 that

$$\lim_{t\longrightarrow\infty}E_{\alpha,1}\left(-\lambda(\psi(t))^{\alpha}\right)=0.$$

Since $E_{\alpha,1}(0)=1$, we finally get

$$\lim_{t\longrightarrow\infty}J=\frac{1}{\lambda}.$$

Moreover, we know that $f \in C_{0,c}(\mathbb{X})$, then

$$\lim_{t\to\infty} \sup_{\sigma\in\left(-\infty,t\right]} \left\{\|c^{-\sigma/\omega} f(\sigma)\|_{\mathbb{X}}\right\} = 0.$$

Since

$$\|c^{\wedge}(-t)u(t)\|_{\mathbb{X}} \leqslant M\left(\sup_{\sigma\in\left(-\infty,t\right]} \left\{\|c^{-\sigma/\omega} f(\sigma)\|_{\mathbb{X}}\right\}\right) \times J,$$

therefore

$$\lim_{t\to\infty} c^{\wedge}(-t)u(t) = 0.$$

Consequently $u \in C_{0,c}(\mathbb{X})$.

This ends the proof of the proposition. □

PROPOSITION 7. *Let $f \in AP_{\omega c}(\mathbb{X})$. Then the function u defined by:*

$$u(t) = \int_{-\infty}^{t} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) f(\sigma) d\sigma$$

belongs to $AP_{\omega c}(\mathbb{X})$ if $|c| \geqslant 1$.

Proof. Since $f \in AP_{\omega c}(\mathbb{X})$, then there exist $f_1 \in P_{\omega c}(\mathbb{X})$ and $f_2 \in C_{0,c}(\mathbb{X})$ such that $f = f_1 + f_2$.

It follows that

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) (f_1(\sigma) + f_2(\sigma)) d\sigma \\ &= \int_{-\infty}^{t} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) f_1(\sigma) d\sigma + \int_{-\infty}^{t} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) f_2(\sigma) d\sigma \\ &= u_1(t) + u_2(t) \end{aligned}$$

with

$$u_1(t) = \int_{-\infty}^{t} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) f_1(\sigma) d\sigma,$$

and

$$u_2(t) = \int_{-\infty}^{t} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) f_2(\sigma) d\sigma.$$

From Proposition 5 and Proposition 6, we can say that $u_1 \in P_{\omega c}(\mathbb{X})$ and $u_2 \in C_{0,c}(\mathbb{X})$.

Consequently $u \in AP_{\omega c}(\mathbb{X})$. □

PROPOSITION 8. *Let $f \in AP_{\omega c}(\mathbb{X})$ and define u by:*

$$u(t) = \int_{0}^{t} \psi'(\sigma) R(\psi(t) - \psi(\sigma)) f(\sigma) d\sigma.$$

Then u belongs to $AP_{\omega c}(\mathbb{X})$ if $|c| \geqslant 1$.

Proof. Since $f \in AP_{\omega c}(\mathbb{X})$, then there exist $f_1 \in P_{\omega c}(\mathbb{X})$ and $f_2 \in C_{0,c}(\mathbb{X})$ such that $f = f_1 + f_2$.

We have

$$\begin{aligned} u(t) &= \int_0^t \psi'(\sigma) R(\psi(t)-\psi(\sigma))(f_1(\sigma)+f_2(\sigma))\,d\sigma \\ &= \int_0^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_1(\sigma)\,d\sigma + \int_0^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_2(\sigma)\,d\sigma \\ &= \int_{-\infty}^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_1(\sigma)\,d\sigma - \int_{-\infty}^0 \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_1(\sigma)\,d\sigma \\ &\quad + \int_0^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_2(\sigma)\,d\sigma \\ &= \int_{-\infty}^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_1(\sigma)\,d\sigma + \int_0^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_2(\sigma)\,d\sigma \\ &= u_1(t) + u_2(t) \end{aligned}$$

where

$$u_1(t) = \int_{-\infty}^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_1(\sigma)\,d\sigma$$

and

$$u_2(t) = \int_0^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_2(\sigma)\,d\sigma.$$

According to Proposition 5, $u_1 \in P_{\omega c}(\mathbb{X})$.

Now, let us show that $u_2 \in C_{0,c}(\mathbb{X})$.

Let $\varepsilon > 0$. As $f_2 \in C_{0,c}(\mathbb{X})$, then there exists $T > 0$ such that for all $\sigma > T$,

$$\|c^{-\sigma/\omega} f_2(\sigma)\|_{\mathbb{X}} < \varepsilon. \tag{22}$$

In order to show that $u_2 \in C_{0,c}(\mathbb{X})$, we consider: $t > T$, hence

$$u_2(t) = \int_0^T \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_2(\sigma)\,d\sigma + \int_T^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_2(\sigma)\,d\sigma,$$

and

$$\|c^\wedge(-t) u_2(t)\|_{\mathbb{X}} \leqslant \sum_{i=1}^2 I_i(t),$$

where

$$\begin{aligned} I_1(t) &= \|c^\wedge(-t) \int_0^T \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_2(\sigma)\,d\sigma\|_{\mathbb{X}}; \\ I_2(t) &= \|c^\wedge(-t) \int_T^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_2(\sigma)\,d\sigma\|_{\mathbb{X}}. \end{aligned}$$

We have:

$$I_1(t) = \|c^{\wedge}(-t)\int_0^T \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f_2(\sigma)\, d\sigma\|_{\mathbb{X}}$$
$$\leqslant \int_0^T |c|^{-(t-\sigma)/\omega} \psi'(\sigma) \|R(\psi(t)-\psi(\sigma))\|_{\mathbb{X}} \|c^{-\sigma/\omega} f_2(\sigma)\|_{\mathbb{X}} d\sigma.$$

We know that $t > T$ and $0 \leqslant \sigma \leqslant T$, therefore $-(t-\sigma) \leqslant 0$. If $|c| \geqslant 1$, it implies that $|c|^{-(t-\sigma)/\omega} \leqslant 1$.

According to the equality (ii) in Proposition 3, we have also

$$\|R(\psi(t)-\psi(\sigma))\|_{\mathbb{X}} \leqslant M(\psi(t)-\psi(\sigma))^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(\sigma))^{\alpha}\right). \tag{23}$$

Hence

$$I_1(t) \leqslant M\left(\sup_{0\leqslant\sigma\leqslant T}\left\{\|c^{-\sigma/\omega} f_2(\sigma)\|_{\mathbb{X}}\right\}\right)\int_0^T \psi'(\sigma)(\psi(t)-\psi(\sigma))^{\alpha-1}$$
$$\times E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(\sigma))^{\alpha}\right) d\sigma.$$

Making the change of variable $\xi = \psi(t)-\psi(\sigma)$ and using (21), we obtain

$$I_1(t) \leqslant -M\left(\sup_{0\leqslant\sigma\leqslant T}\left\{\|c^{-\sigma/\omega} f_2(\sigma)\|_{\mathbb{X}}\right\}\right)\int_{\psi(t)}^{\psi(t)-\psi(T)} \xi^{\alpha-1} E_{\alpha,\alpha}(-\lambda\xi^{\alpha})\, d\xi$$
$$\leqslant \frac{M}{\lambda}\left(\sup_{0\leqslant\sigma\leqslant T}\left\{\|c^{-\sigma/\omega} f_2(\sigma)\|_{\mathbb{X}}\right\}\right)\int_{\psi(t)}^{\psi(t)-\psi(T)} \partial_{\xi} E_{\alpha,1}(-\lambda\xi^{\alpha})\, d\xi$$
$$= \frac{M}{\lambda}\left(\sup_{0\leqslant\sigma\leqslant T}\left\{\|c^{-\sigma/\omega} f_2(\sigma)\|_{\mathbb{X}}\right\}\right)\left[E_{\alpha,1}(-\lambda\xi^{\alpha})\right]_{\psi(t)}^{\psi(t)-\psi(T)}$$
$$= \frac{M}{\lambda}\left(\sup_{0\leqslant\sigma\leqslant T}\left\{\|c^{-\sigma/\omega} f_2(\sigma)\|_{\mathbb{X}}\right\}\right)$$
$$\times\left(E_{\alpha,1}\left(-\lambda(\psi(t)-\psi(T))^{\alpha}\right) - E_{\alpha,1}\left(-\lambda(\psi(t))^{\alpha}\right)\right).$$

We know that

$$\lim_{t\to\infty} E_{\alpha,1}\left(-\lambda(\psi(t)-\psi(T))^{\alpha}\right) = 0$$

and

$$\lim_{t\to\infty} E_{\alpha,1}\left(-\lambda(\psi(t))^{\alpha}\right) = 0$$

because

$$\lim_{t\to-\infty} E_{\alpha,1}(t) = 0,$$

then

$$\lim_{t\to\infty} E_{\alpha,1}\left(-\lambda(\psi(t)-\psi(T))^{\alpha}\right) - E_{\alpha,1}\left(-\lambda(\psi(t))^{\alpha}\right) = 0.$$

Since $f_2 \in C_{0,c}(\mathbb{X})$, therefore

$$\sup_{0\leqslant\sigma\leqslant T}\left\{\|c^{-\sigma/\omega} f_2(\sigma)\|_{\mathbb{X}}\right\} < \infty.$$

Consequently

$$\lim_{t\to\infty} I_1(t) = 0.$$

We have

$$\begin{aligned}I_2(t) &= \|c^{\wedge}(-t)\int_T^t \psi'(\sigma)R(\psi(t)-\psi(\sigma))f_2(\sigma)d\sigma\|_{\mathbb{X}}\\ &\leqslant \int_T^t |c|^{-(t-\sigma)/\omega}\psi'(\sigma)\|R(\psi(t)-\psi(\sigma))\|_{\mathbb{X}}\|c^{-\sigma/\omega}f_2(\sigma)\|_{\mathbb{X}}d\sigma.\end{aligned}$$

According to (22) and (23), and consider the change of variable $\xi = \psi(t)-\psi(\sigma)$, we obtain

$$\begin{aligned}I_2(t) &\leqslant \varepsilon M\int_T^t |c|^{-(t-\sigma)/\omega}\psi'(\sigma)(\psi(t)-\psi(\sigma))^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(\sigma))^{\alpha}\right)d\sigma\\ &\leqslant -\varepsilon M\int_{\psi(t)-\psi(T)}^{0} |c|^{-(t-\sigma)/\omega}\xi^{\alpha-1}E_{\alpha,\alpha}(-\lambda\xi^{\alpha})d\xi.\end{aligned}$$

We note that $T \leqslant \sigma \leqslant t$ implies that $-(t-\sigma) \leqslant 0$. Then, if $|c| \geqslant 1$, we have $|c|^{-(t-\sigma)/\omega} \leqslant 1$.

According to (21), the above inequality becomes:

$$\begin{aligned}I_2(t) &\leqslant \frac{\varepsilon M}{\lambda}\int_{\psi(t)-\psi(T)}^{0} \partial_{\xi}E_{\alpha,1}(-\lambda\xi^{\alpha})d\xi\\ &= \frac{\varepsilon M}{\lambda}\left[E_{\alpha,1}(-\lambda\xi^{\alpha})\right]_{\psi(t)-\psi(T)}^{0}\\ &= \frac{\varepsilon M}{\lambda}\left(E_{\alpha,1}(0)-E_{\alpha,1}\left(-\lambda(\psi(t)-\psi(T))^{\alpha}\right)\right).\end{aligned}$$

As

$$\lim_{t\to\infty} E_{\alpha,1}\left(-\lambda(\psi(t)-\psi(T))^{\alpha}\right) = 0$$

and

$$E_{\alpha,1}(0) = 1$$

then

$$I_2(t) \leqslant \frac{\varepsilon M}{\lambda}.$$

Hence

$$\lim_{t\to\infty} I_2(t) = 0.$$

Hence, $u_2 \in C_{0,c}(\mathbb{X})$ if $|c| \geqslant 1$. Finally, $u \in AP_{\omega c}(\mathbb{X})$ if $|c| \geqslant 1$.
The proof is complete. □

In the following result, we need the following assumption:

(H1) $h_\tau(z) = c^\wedge(-t)h(c^\wedge(t)z)$ is uniformly continuous for z in any bounded subset of $\mathbb{X}$ uniformly for $t \geqslant d$ and $h_\tau(z) \to 0$ as $t \to \infty$ uniformly in z .

(H2) $f \in AP_{\omega c}(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X})$ and there exists a constant $L_f > 0$ such that:

$$\|f(t,x) - f(t,y)\| \leqslant L_f \|x-y\|, \text{ for all } t \in \mathbb{R}_+, \ x,y \in \mathbb{X}.$$

(H3) The inequality (2) hold.

Now, we present our first result for the problem (1) which relies on the Banach fixed point principle.

THEOREM 7. *Under the previous assumptions, if we assume that* $|c| \geqslant 1$, (H_1)–(H_3) *hold. Then there exists a unique* (ω,c)*-asymptotically periodic mild solution to equation* (1)*, provided that there is a constant*

$$\Omega_\lambda := \frac{L_f M}{\lambda} < 1.$$

Proof. Consider the operator $\Gamma : AP_{\omega c}(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X}) \to AP_{\omega c}(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X})$ such that

$$\begin{aligned}\Gamma u(t) := &\int_0^t \psi'(\sigma)(\psi(t)-\psi(\sigma))^{\gamma-2} Q(\sigma) u_0 d\sigma \\ &+ \int_0^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) f(\sigma, u(\sigma)) d\sigma. \end{aligned} \tag{24}$$

In view of Theorem 4, Proposition 4 and 8, and Lemma 3, Γ is well-defined.

Let $u,v \in AP_{\omega c}(\mathbb{X})$, then

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\|_{a\omega c} \\
&= \| \int_0^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) (f(\sigma,u(\sigma)) - f(\sigma,v(\sigma))) d\sigma \|_{a\omega c} \\
&= \sup_{t\geqslant 0} \left\{ \| |c|^{-t/\omega} \int_0^t \psi'(\sigma) R(\psi(t)-\psi(\sigma)) (f(\sigma,u(\sigma)) - f(\sigma,v(\sigma))) d\sigma \|_{a\omega c} \right\} \\
&\leqslant \sup_{t\geqslant 0} \left\{ \int_0^t |c|^{-t/\omega} \psi'(\sigma) \|R(\psi(t)-\psi(\sigma))\|_{\mathbb{X}} \|f(\sigma,u(\sigma)) - f(\sigma,v(\sigma))\|_{\mathbb{X}} d\sigma \right\} \\
&\leqslant L_f \sup_{t\geqslant 0} \left\{ \int_0^t |c|^{-t/\omega} \psi'(\sigma) \|R(\psi(t)-\psi(\sigma))\|_{\mathbb{X}} \|u(\sigma) - v(\sigma)\|_{\mathbb{X}} d\sigma \right\} \\
&\leqslant L_f \sup_{t\geqslant 0} \left\{ \int_0^t |c|^{-(t-\sigma)/\omega} \psi'(\sigma) \|R(\psi(t)-\psi(\sigma))\|_{\mathbb{X}} \|c^{-\sigma/\omega}(u(\sigma) - v(\sigma))\|_{\mathbb{X}} d\sigma \right\} \\
&\leqslant L_f M \sup_{t\geqslant 0} \left\{ \int_0^t |c|^{-(t-\sigma)/\omega} \psi'(\sigma) (\psi(t)-\psi(\sigma))^{\alpha-1} \right. \\
&\quad \left. \times E_{\alpha,\alpha}\left(-\lambda (\psi(t)-\psi(\sigma))^\alpha\right) d\sigma \right\} \|u-v\|_{a\omega c}.
\end{aligned}$$

As $|c| \geqslant 1$ and $t-\sigma \geqslant 0$, then $|c|^{-(t-\sigma)/\omega} \leqslant 1$. In addition, making the change of variable $\xi = \psi(t) - \psi(\sigma)$, we obtain:

$$\|\Gamma u(t) - \Gamma v(t)\|_{a\omega c} \leqslant L_f M \sup_{t \geqslant 0} \left\{ \int_0^{\psi(t)} \xi^{\alpha-1} E_{\alpha,\alpha}(-\lambda \xi^{\alpha})\, d\sigma \right\} \|u-v\|_{a\omega c}.$$

Using (21), we have

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\|_{a\omega c} &\leqslant -\frac{L_f M}{\lambda} \sup_{t \geqslant 0} \left\{ \int_0^{\psi(t)} \partial_\xi E_{\alpha,1}(-\lambda \xi^{\alpha})\, d\sigma \right\} \|u-v\|_{a\omega c} \\
&\leqslant \frac{L_f M}{\lambda} \sup_{t \geqslant 0} \left[E_{\alpha,1}(-\lambda \xi^{\alpha}) \right]_{\psi(t)}^{0} \|u-v\|_{a\omega c} \\
&\leqslant \frac{L_f M}{\lambda} \sup_{t \geqslant 0} \left\{ E_{\alpha,1}(0) - E_{\alpha,1}\left(-\lambda (\psi(t))^{\alpha}\right) \right\} \|u-v\|_{a\omega c} \\
&= \frac{L_f M}{\lambda} \sup_{t \geqslant 0} \left\{ 1 - E_{\alpha,1}\left(-\lambda (\psi(t))^{\alpha}\right) \right\} \|u-v\|_{a\omega c}.
\end{aligned}$$

It is clear that

$$\sup_{t \geqslant 0} \left\{ 1 - E_{\alpha,1}\left(-\lambda (\psi(t))^{\alpha}\right) \right\} = 1,$$

setting $\Omega_\lambda = \frac{L_f M}{\lambda}$, we get

$$\|\Gamma u(t) - \Gamma v(t)\|_{a\omega c} \leqslant \Omega_\lambda \|u-v\|_{a\omega c}.$$

When $\Omega_\lambda < 1$, we deduce by the Banach contraction principle that Γ has a unique fixed point $u \in AP_{\omega c}(\mathbb{X})$, solution of equation (1).

Finally, equation (1) has a unique (ω,c)-asymptotically periodic mild solution. □

4. Ulam-Hyers stability

In what follows, we discuss the Ulam-Hyers stability of the solution to the equation (1).

DEFINITION 11. The equation (1) is Ulam-Hyers stable if there exists a real number C_ψ such that, for each $\varepsilon > 0$ and for each solution $\widetilde{u} \in AP_{\omega c}(\mathbb{X})$ of inequality:

$$\left| D_{0^+}^{\alpha,\beta;\psi} \widetilde{u}(t) - A\widetilde{u}(t) - f(t, \widetilde{u}(t)) \right| \leqslant \varepsilon, \tag{25}$$

there exists some $u \in AP_{\omega c}(\mathbb{X})$ satisfying

$$\begin{cases} D_{0^+}^{\alpha,\beta;\psi} u(t) = Au(t) + f(t,u(t)), \quad 0<\alpha<1, \quad 0<\beta<1; \quad t \geqslant 0 \\ I_{0^+}^{(1-\gamma);\psi} u(0) = u_0, \quad \gamma = \beta(1-\alpha)+\alpha \end{cases} \tag{26}$$

with

$$\left| \widetilde{u}(t) - u(t) \right| \leqslant C_\psi \varepsilon, \quad t \geqslant 0.$$

DEFINITION 12. The equation (1) is generalized Ulam-Hyers stable if there exists $\varphi \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\varphi(0)=0$ such that, for each $\varepsilon>0$ and for each solution $\widetilde{u}\in AP_{\omega c}(\mathbb{X})$ of inequality:

$$\left|D_{0^+}^{\alpha,\beta;\psi}\widetilde{u}(t)-A\widetilde{u}(t)-f(t,\widetilde{u}(t))\right|\leqslant\varepsilon,$$

there exists some $u\in AP_{\omega c}(\mathbb{X})$ satisfying

$$\begin{cases} D_{0^+}^{\alpha,\beta;\psi}u(t)=Au(t)+f(t,u(t)), \quad 0<\alpha<1, \quad 0<\beta<1; \quad t\geqslant 0\\ I_{0^+}^{(1-\gamma);\psi}u(0)=u_0, \quad \gamma=\beta(1-\alpha)+\alpha\end{cases}$$

with

$$\left|\widetilde{u}(t)-u(t)\right|\leqslant\varphi(\varepsilon), \quad t\geqslant 0.$$

THEOREM 8. *Assume that the hypotheses $(H_2)-(H_3)$ and Theorem 7 hold. Suppose also that $C_\alpha:=\sup_{t\geqslant 0}\left\{\|c^{-t/\omega}(\psi(t))^\alpha\|_{\mathbb{X}}\right\}<\infty$. Then the equation (1) is Ulam-Hyers stable.*

Proof. Let $\varepsilon>0$ and let $\widetilde{u}\in AP_{\omega c}(\mathbb{X})$ be a solution of the inequaty (25). In view of Theorem 7, let $u\in AP_{\omega c}(\mathbb{X})$ be the unique solution of equation (1), we have

$$u(t)=F_u+\int_0^t\psi'(\sigma)R(\psi(t)-\psi(\sigma))f(\sigma,u(\sigma))\,d\sigma$$

with

$$F_u=\int_0^t\psi'(\sigma)(\psi(t)-\psi(\sigma))^{\gamma-2}Q(\sigma)u_0d\sigma.$$

On the one hand, if $I_{0^+}^{(1-\gamma)}u(0)=I_{0^+}^{(1-\gamma)}\widetilde{u}(0)$, then $u_0=\widetilde{u}_0$ it follows that $F_u=F_{\widetilde{u}}$. Hence

$$u(t)=F_{\widetilde{u}}+\int_0^t\psi'(\sigma)R(\psi(t)-\psi(\sigma))f(\sigma,u(\sigma))\,d\sigma.$$

On the other hand, using (25), we have

$$\left|I_{0^+}^{\alpha;\psi}\left(D_{0^+}^{\alpha;\beta;\psi}\widetilde{u}(t)-A\widetilde{u}(t)-f(t,\widetilde{u}(t))\right)\right|\leqslant\varepsilon I_{0^+}^{\alpha;\psi}(1).$$

So

$$\begin{aligned}&\left|\widetilde{u}(t)-F_{\widetilde{u}}-\int_0^t\psi'(\sigma)R(\psi(t)-\psi(\sigma))f(\sigma,\widetilde{u}(\sigma))\,d\sigma\right|\\ &\leqslant\frac{\varepsilon}{\Gamma(\alpha)}\int_0^t\psi'(\sigma)(\psi(t)-\psi(\sigma))^{\alpha-1}d\sigma\\ &\leqslant\frac{\varepsilon}{\alpha\Gamma(\alpha)}(\psi(t))^\alpha\\ &\leqslant\frac{\varepsilon}{\Gamma(\alpha+1)}(\psi(t))^\alpha.\end{aligned}$$

From the above, we have

$$\begin{aligned}
\left|\widetilde{u}(t)-u(t)\right| &= \left|\widetilde{u}(t)-F_{\widetilde{u}}-\int_0^t \psi'(\sigma)R(\psi(t)-\psi(\sigma))f(\sigma,u(\sigma))\,d\sigma\right| \\
&= \Big|\widetilde{u}(t)-F_{\widetilde{u}}-\int_0^t \psi'(\sigma)R(\psi(t)-\psi(\sigma))f(\sigma,\widetilde{u}(\sigma))\,d\sigma \\
&\quad +\int_0^t \psi'(\sigma)R(\psi(t)-\psi(\sigma))\left(f(\sigma,\widetilde{u}(\sigma))-f(\sigma,u(\sigma))\right)d\sigma\Big| \\
&\leqslant \left|\widetilde{u}(t)-F_{\widetilde{u}}-\int_0^t \psi'(\sigma)R(\psi(t)-\psi(\sigma))f(\sigma,\widetilde{u}(\sigma))\,d\sigma\right| \\
&\quad +\left|\int_0^t \psi'(\sigma)R(\psi(t)-\psi(\sigma))\left(f(\sigma,\widetilde{u}(\sigma))-f(\sigma,u(\sigma))\right)d\sigma\right| \\
\left|\widetilde{u}(t)-u(t)\right| &\leqslant \frac{\varepsilon}{\Gamma(\alpha+1)}(\psi(t))^{\alpha} \\
&\quad +\left|\int_0^t \psi'(\sigma)R(\psi(t)-\psi(\sigma))\left(f(\sigma,\widetilde{u}(\sigma))-f(\sigma,u(\sigma))\right)d\sigma\right|
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \|\widetilde{u}-u\|_{a\omega c} \\
&\leqslant \sup_{t\geqslant 0}\left\{\left\|\frac{\varepsilon}{\Gamma(\alpha+1)}c^{-t/\omega}(\psi(t))^{\alpha}\right\|_{\mathbb{X}}\right\} \\
&\quad +\sup_{t\geqslant 0}\left\{\left\|c^{-t/\omega}\int_0^t \psi'(\sigma)R(\psi(t)-\psi(\sigma))\left(f(\sigma,\widetilde{u}(\sigma))-f(\sigma,u(\sigma))\right)d\sigma\right\|_{\mathbb{X}}\right\} \\
&\leqslant \frac{\varepsilon}{\Gamma(\alpha+1)}\sup_{t\geqslant 0}\left\{\left\|c^{-t/\omega}(\psi(t))^{\alpha}\right\|_{\mathbb{X}}\right\} \\
&\quad +L_f\sup_{t\geqslant 0}\left\{\int_0^t |c|^{-t/\omega}\psi'(\sigma)\left\|R(\psi(t)-\psi(\sigma))\right\|_{\mathbb{X}}\|\widetilde{u}(\sigma)-u(\sigma)\|_{\mathbb{X}}d\sigma\right\} \\
&\leqslant \frac{C_\alpha}{\Gamma(\alpha+1)}\varepsilon+L_fM\sup_{t\geqslant 0}\left\{\int_0^t |c|^{-(t-\sigma)/\omega}\psi'(\sigma)(\psi(t)-\psi(\sigma))^{\alpha-1}\right. \\
&\quad \left.\times E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(\sigma))^{\alpha}\right)\left\|c^{-\sigma/\omega}(\widetilde{u}(\sigma)-u(\sigma))\right\|_{\mathbb{X}}d\sigma\right\} \\
&\leqslant \frac{C_\alpha}{\Gamma(\alpha+1)}\varepsilon+L_fM\sup_{t\geqslant 0}\left\{\int_0^t \psi'(\sigma)(\psi(t)-\psi(\sigma))^{\alpha-1}\right. \\
&\quad \left.\times E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(\sigma))^{\alpha}\right)d\sigma\right\}\|\widetilde{u}-u\|_{a\omega c}.
\end{aligned}$$

We know that

$$\begin{aligned}
&L_fM\sup_{t\geqslant 0}\left\{\int_0^t \psi'(\sigma)(\psi(t)-\psi(\sigma))^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(\sigma))^{\alpha}\right)d\sigma\right\} \\
&=\frac{L_fM}{\lambda}\sup_{t\geqslant 0}\left\{1-E_{\alpha,1}\left(-\lambda(\psi(t))^{\alpha}\right)\right\}
\end{aligned}$$

and, as

$$\sup_{t\geqslant 0}\left\{1-E_{\alpha,1}\left(-\lambda\left(\psi\left(t\right)\right)^{\alpha}\right)\right\}=1$$

then

$$\begin{aligned}\|\widetilde{u}-u\|_{a\omega c}&\leqslant\frac{C_\alpha}{\Gamma(\alpha+1)}\varepsilon+\frac{L_fM}{\lambda}\|\widetilde{u}-u\|_{a\omega c}\\ \Rightarrow\left(1-\frac{L_fM}{\lambda}\right)\|\widetilde{u}-u\|_{a\omega c}&\leqslant\frac{C_\alpha}{\Gamma(\alpha+1)}\varepsilon\\ \Rightarrow\|\widetilde{u}-u\|_{a\omega c}&\leqslant\frac{C_\alpha}{\left(1-\frac{L_fM}{\lambda}\right)\Gamma(\alpha+1)}\varepsilon.\end{aligned}$$

Take

$$C_\psi=\frac{C_\alpha}{\left(1-\frac{L_fM}{\lambda}\right)\Gamma(\alpha+1)}$$

it follows that

$$\|\widetilde{u}-u\|_{a\omega c}\leqslant C_\psi\varepsilon.$$

Finally, equation (1) solution is Ulam-Hyers stable. □

THEOREM 9. *Let the conditions of Theorem 8 hold. If there exists* $\varphi\in C(\mathbb{R}_+,\mathbb{R}_+)$*,* $\varphi(0)=0$*, then the equation* (1) *has generalized Ulam-Hyers stability.*

Proof. In a similar arguments of Theorem 8, with putting $\varphi\left(\varepsilon\right)=C_\psi\varepsilon$ and $\varphi(0)=0$, we obtain

$$\|\widetilde{u}-u\|_{a\omega c}\leqslant\varphi\left(\varepsilon\right).$$

Therefore, equation (1) solution is Ulam-Hyers generalized stable. □

Conclusions

In this paper, we proved the existence and uniqueness of (ω,c)-asymptotically periodic mild solutions to some ψ-Hilfer fractional evolution equations of the form:

$$D_t^{\alpha,\beta,\psi}u(t)=Au(t)+f(t,u(t)),\quad 0<\alpha<1,\quad 0<\beta<1;\quad t\geqslant 0$$

using Banach fixed point theorem. Futhermore, we discuss the stability analysis of the proposed problem.

REFERENCES

[1] L. ABADIAS, E. ALVAREZ AND R. GRAU, *(ω,c)-Periodic Mild Solutions to Non-Autonomous Abstract Differential Equations*, Mathematics, vol. 9, no. 5 (2021), pp. 474–489.

[2] S. ABBAS, M. BENCHORA AND A. PETRUSEL, *Ulam stability for Hilfer type fractional differential inclusions via the weakly Picard operators theory*, Fractional Calculus and Applied Analysis, vol. 20, no. 2 (2017), pp. 384–398.

[3] M. S. ABDO, S. K. PANCHAL AND H. S. HUSSIEN, *Fractional Integro-Differential Equations with Nonlocal Conditions and ψ-Hilfer Fractional Derivative*, Mathematical Modelling and Analysis, vol. 24, issue. 4, pp. 564–584, 2019.

[4] M. AGAOGLOU, M. FECKAN AND A. P. PANAGIOTIDOU, *Existence and uniqueness of (ω,c)-periodic solutions of semilinear evolution equations*, International Journal of Dynamical Systems and Differential Equations, vol. 10, no. 2 (2020), pp. 149–166.

[5] R. P. AGARWAL, C. CUEVAS AND H. SOTO, *Pseudo – almost periodic solutions of a class of semilinear fractional differential equations*, Journal of Applied Mathematics and Computing, vol. 37, no. 1–2 (2011), pp. 625–634.

[6] E. ALVAREZ, A. GOMEZ AND M. PINTO, *(ω,c)-periodic functions and mild solutions to abstract fractional integro-differential equation*, Electronic Journal of Qualitative Theory of Differential Equations, no. 16 (2018), pp. 1–8.

[7] E. ALVAREZ, M. PINTO AND S. CASTILLO, *(ω,c)-asymptotically periodic functions, first-order Cauchy problem, and Lasota-Wazewska model with unbounded oscillating production of red cells*, Mathematicals Methods in the Applied Sciences, vol. 43, no. 1 (2020), pp. 305–319.

[8] P. BEDI, A. KUMAR, T. ABDELJAWAD AND A. KHAN, *S-Asymptotically ω-Periodic Mild Solutions and Stability Analysis of Hilfer Fractional Evolution Equations*, Evolution equations and control theory, doi:10.3934/eect.2020089.

[9] D. BRINDLE, *S-Asymptotically ω-Periodic Sequential Solutions to Difference Equations*, Nonlinear Studies, vol. 26, no. 3, pp. 1–12, 2019.

[10] D. BRINDLE AND G. M. N'GUÉRÉKATA, *S-Asymptotically ω-periodic mild solutions to fractional differential equations*, Electronic Journal of Differential Equations, vol. 2020, no. 30 (2020), pp. 1–12.

[11] D. BRINDLE AND G. M. N'GUÉRÉKATA, *Existence results of S-Asymptotically ω-Periodic Mild Solutions to Some Integro-Differential Equations*, PanAmerican Mathematical Journal, vol. 29 (2019), no. 2, pp. 63–74.

[12] P. BORISUT, P. KUMAN, I. AHMED AND W. JIRAKITPUWAPAT, *Existence and Uniqueness for ψ-Hilfer Fractional Differential Equation with Nonlocal Multi-Point Condition*, Math. Meth. Appl. Sci., vol. 44 (2021), pp. 2506–2520.

[13] Y. CHANG, R. ZHANG AND G. M. N'GUÉRÉKATA, *Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations*, Computers and Mathematics with Applications, vol. 64, no. 10 (2012), pp. 3160–3170.

[14] E. CUESTA, *Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations*, Discrete and Continuous Dynamical Systems, Supplement 2007, pp. 277–285.

[15] W. DESCH, R. GRIMMER AND W. SCHAPPACHER, *Some considerations for linear integro-differential equations*, Journal of Mathematical Analysis and Applications, no. 104 (1984), pp. 219–234.

[16] W. DIMBOUR, G. MOPHOU AND G. M. N'GUÉRÉKATA, *S-Asymptotically periodic solutions for partial differential equations with finite delay*, Electronic Journal of Differential Equations, vol. 2011, no. 117 (2011), pp. 1–12.

[17] R. G. FOKO TIOMELA AND G. M. N'GUÉRÉKATA, *(ω,c)-Asymptotically periodic solutions to some fractional integro-differential equations*, Journal of Fractional Calculus and Applications, vol. 13, no. 2(2022), pp. 100–115.

[18] R. G. FOKO TIOMELA AND G. M. N'GUÉRÉKATA, *Optimal (ω,c)-Asymptotically periodic mild solutions to some fractional evolutions equations*, Fractional Differential Calculus, vol. 13, no. 1 (2023), pp. 1–20.

[19] A. GRANAS AND J. DUGUNDJI, *Fixed point theory*, New York, Springer, 2003.

[20] R. C. GRIMMER, *Resolvent operators for integral equations in a Banach Space*, Transactions of the American Mathematical Society, vol. 273, no. 1 (1982), pp. 333–349.

[21] H. R. HENRIQUEZ, M. PIERRI AND P. TABOAS, *On S-asymptotically ω-periodic functions on Banach spaces and applications*, Journal of Mathematical Analysis and Applications, vol. 343, no. 2 (2008), pp. 1119–1130.

[22] F. JARAD, S. HARIKRISHNAN, K. SHAH AND K. KANAGARAJAN, *Existence and Stability Results to a Class of Fractional Random Implicit Differential Equations Involving a Generalized Hilfer Fractional Derivative*, Discrete and Continuous Dynamical Systems Series, vol. 13, no. 3(2020), pp. 723–739.

[23] M. T. KHALLADI, M. KOSTIC, M. PINTO, A. RAHMANI AND D. VELINOV, *C-almost periodic type functions and applications*, Nonautonomous Dynamical Systems, vol. 7, no. 1 (2020), pp. 176–193.

[24] J. P. KHARADE AND K. D. KUCCHE, *On the Impulsive ψ-Hilfer Fractional Differential Equations with Delay*, https://doi.org/10.48550/arXiv.1908.07793.

[25] M. KÉRÉ AND G. M. N'GUÉRÉKATA, *An existence result of (ω,c)-almost periodic mild solutions to some fractional differential equations*, PanAmerican Mathematical Journal, vol. 31, no. 3 (2021), pp. 11–20.

[26] J. LARROUY AND G. M. N'GUÉRÉKATA, *(ω,c)-periodic and asymptotically (ω,c)-periodic mild solutions to fractional Cauchy problems*, Applicable Analysis, vol. 102, no. 3 (2023), pp. 958–976.

[27] M. LI, J. R. WANG AND M. FECKAN, *(ω,c)-Periodic solutions for impulsive differential systems*, Communication in Mathematical Analysis, vol. 74, no. 2 (2018), pp. 35–46.

[28] C. LIZAMA AND R. PONCE, *Bounded solutions to a class of semilinear integro-differential equations in Banach spaces*, Nonlinear Analysis: Theory, Methods and Applications, vol. 74, no. 10 (2011), pp. 3397–3406.

[29] A. E. MFADEL, S. MELLIANI AND M. ELOMARI, *Existence of Solutions for Nonlinear ψ-Caputo-type Fractional Hybrid Differential Equations with Periodic Boundary Conditions*, Asia Pacific Journal of Mathematics, vol. 2022, pp. 9–15.

[30] G. MOPHOU AND G. M. N'GUÉRÉKATA, *An existence result of (ω,c)-periodic mild solutions to some fractional differential equations*, Nonlinear Studies, vol. 27, no. 1 (2020), pp. 167–175.

[31] G. MOPHOU, G. M. N'GUÉRÉKATA AND V. VALMORIN, *Asymptotic behavior of mild solutions of some fractional functional integro-differential equations*, African Diaspora Journal of Mathematics, vol. 16, no. 1 (2013), pp. 70–81.

[32] G. M. N'GUÉRÉKATA, *Almost periodic and almost automorphic functions in abstract spaces*, 2nd edition, Springer, New York (2021).

[33] F. NOROUZI AND G. M. N'GUÉRÉKATA, *Existence results to a ψ-Hilfer neutral fractional evolution equation with infinite delay*, Nonautom. Dyn. Syst., vol. 8 (2021), no. 1, pp. 101–125.

[34] P. V. OUENA, M. KÉRÉ AND G. M. N'GUÉRÉKATA, *Asymptotically (ω,c)-periodic mild solutions to integro-differential equations*, Nonlinear Studies, vol. 32, no. 2 (2024), pp. 617–630.

[35] P. V. OUENA AND M. KÉRÉ, *(ω,c)-Asymptotically periodic mild solutions to semilinear two terms fractional differential equations*, Journal of Nonlinear Evolution Equations and Applications, vol. 2024, no. 5, pp. 67–80 (June 2024).

[36] I. PODLUBNY, *Fractional Differential Equations, An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications*, in: Mathematics in Science and Engineering, vol. 198, Academic Press, Inc., San Diego, CA, 1999, p. xxiv+340.

[37] N. H. TUAN, A. DEBBOUCHE AND T. B. NGOC, *Existence and Regularity of Final Value Problems for Time Fractional Wave Equations*, Computers and Mathematics with Applications, vol. 78 (2019), pp. 1396–1414.

[38] J. VANTERLER DA C. SOUSA AND E. C. DE OLIVEIRA, *On the ψ-Hilfer fractional derivative*, Communication in Nonlinear Science and Numerical Simulation, vol. 60, pp. 72–91, 2018.

[39] J. VANTERLER DA C. SOUSA, K. D. KUCCHE AND E. C. DE OLIVEIRA, *On the Ulam-Hyers stabilities of the solutions of ψ-Hilfer fractional differential equation with abstract Volterra operator*, Math. Appl. Sci., 2019, pp. 1–12.

[40] J. R. WANG, L. REN AND Y. ZHOU, *(ω,c)-Periodic solutions for time varying impulsive differential equations*, Advances in Difference Equations, vol. 2019, no. 259 (2019), pp. 1–9.

[41] J. WANG, Y. ZHOU AND M. FECKAN, *Nonlinear impulsive problems for fractional differential equations and Ulam stability*, Computers and Mathematics with Applications, vol. 64 (2012), pp. 3389–3405.

[42] T. WEI AND Y. ZHANG, *The backward problem for a time-fractional diffusion-wave equation in a bounded domain*, Computers and Mathematics with Applications, vol. 75 (2018), pp. 3632–3648.

[43] J. K. ZAO, Y. K. CHANG AND G. M. N'GUÉRÉKATA, *Asymptotic Behavior of Mild Solutions to Similinear Fractional Differential Equations*, Journal of Optimization Theory and Applications, vol. 156, no. 1 (2013), pp. 106–114.

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