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ASYMPTOTICS OF PROLATE SPHEROIDAL WAVE FUNCTIONS

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Abstract. Uniform asymptotic approximations are obtained for the prolate spheroidal wave functions, for both the angular functions $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ $(-1<x<1)$ and radial functions $Ps_n^m\left(x,\gamma^2\right)$ $(1<x<\infty)$. Here $\gamma\to\infty$, and the results are uniformly valid in the stated intervals, m and n are integers, with m bounded and n satisfying $0\leqslant m\leqslant n\leqslant 2\pi^{-1}\gamma(1-\delta)$, where $\delta\in(0,1)$ is fixed. The results are obtained by an application of certain existing asymptotic solutions of differential equations, and involve elementary, Bessel, and parabolic cylinder functions. An asymptotic relationship between the prolate spheroidal equation separation parameter and the other parameters is also obtained, and error bounds are available for all approximations.

1. Introduction

Separation of the wave equation in prolate spheroidal coordinates leads to the prolate spheroidal wave equation (PSWE)

$$\left(1-z^2\right)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+\left(\lambda-\frac{\mu^2}{1-z^2}+\gamma^2\left(1-z^2\right)\right)y=0, \tag{1.1}$$

where λ and μ are separation constants, and γ is proportional to the frequency (see [30] and [40]).

Solutions of (1.1), the prolate spheroidal wave functions (PSWFs), are viewed as depending on the parameters μ and γ from the equation, as well as an implicitly defined parameter ν (which describes the behavior of solutions at infinity). This latter parameter is the so-called characteristic exponent, and for details see [1, §8.1.1].

The parameter λ is usually regarded as an eigenvalue admitting an eigensolution that is bounded at both $z=\pm1$, which is equivalent to $\mu=m$ and $\nu=n$ being integers (see [1] and [19]). Most of the literature focuses on PSWFs with these parameters being integers, since this is the most useful case in practical applications. We shall assume this as well throughout this paper.

We consider the important case of $\gamma\to\infty$ (which corresponds, for example, to high-frequency scattering in acoustics). In this case it is known [1, p. 186] that $\lambda\to-\infty$, and we shall assume this here. With the exception of §5, our results will be uniformly valid for m bounded, n small or large, and specifically

$$0\leqslant m\leqslant n\leqslant 2\pi^{-1}\gamma(1-\delta), \tag{1.2}$$

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where (here and throughout) $\delta \in (0,1)$ is arbitrarily chosen.

Although we will consider the case z complex, our primary concern will be for z real (denoted by x), and in particular the so-called angular ($-1 < x < 1$) and the radial ($1 < x < \infty$) cases.

We shall apply several existing general asymptotic theories, each of which contains explicit error bounds. The large range of validity (1.2), along with error bounds, signifies a considerable improvement upon existing results. We will primarily be concerned with z real, but we shall use complex-valued argument results as needed to obtain our final results.

One of the principal difficulties in the asymptotic and numerical study of PSWFs is the determination of the eigenvalues, particularly for large value(s) of the other parameters. We mention that an extensive the theory of PSWFs with arbitrary complex parameters μ and ν was developed in [19] and [20].

PSWFs were first studied by Niven [24] in heat conduction in spheroidal bodies, and were subsequently investigated by a number of authors (see [3], [12], [17], [33], [34], [36]). For certain values of the parameters PSWFs are eigenfunctions of the finite Fourier transform, and hence these functions play an important role in signal analysis. These band-limited functions are often encountered in physics, engineering, statistics; see [4], [32], [38], In [22] the PSWE is also shown to play a fundamental role in Laplace's tidal equations. PSWFs also have important applications in fluid dynamics [13], geophysics and theoretical cosmological models [8], atomic and molecular physics [15], [28], and biophysics [6].

Despite being studied extensively over the decades there are significant gaps in the literature on the rigorous analysis of their asymptotic behavior, and their computation is non-trivial, particularly for large values of the parameters. The literature contains many asymptotic approximations and expansions (see [2], [7], [14], [21], [23], [31], [35], [40]), but most are heuristic, all parameters fixed except γ, and with little or no error analysis. For computational techniques see [11], [16], [18], [29], [37], [39].

We mention that in [9] rigorous results (with explicit error bounds) for more than one large parameter were derived for PSWFs, using a theory of a coalescing turning point and double pole [5], but not for the parameter range under consideration in this paper. Specifically, in comparison to the current paper in which $\lambda < 0$, in [9] the case $\lambda > 0$ was assumed, which does not have many of the applications described above.

The PSWE (1.1) has regular singularities at $z = \pm 1$, each with exponents $\pm\frac{1}{2}m$. When $\gamma = 0$ the PSWE degenerates into the associated Legendre equation (regular singularities at $z = \pm 1$ and $z = \infty$), which for $-1 < x < 1$ has solutions the Ferrers functions $\mathsf{P}_\nu^\mu(x)$, and for complex z the associated Legendre functions $P_\nu^\mu(z)$.

The significant difference is that if $\gamma \neq 0$ the PSWE has an irregular singularity at infinity. In fact, one can show (from an algebraic form of Floquet's theorem [1, §8.1.1]) that there exists a solution $S_\nu^{\mu(1)}(z,\gamma)$ (in the notation of [19]), with the property

$$S_\nu^{\mu(1)}\left(ze^{p\pi i},\gamma\right) = e^{p\nu\pi i} S_\nu^{\mu(1)}(z,\gamma), \tag{1.3}$$

for any integer p. The LHS of (1.3) denotes the branch of the function after completing p negative half-circuits about $z = \infty$ (equivalently, p positive half-circuits about $z =$

± 1).

This solution can be expressed as an infinite series involving Bessel functions of the first kind. Specifically, for integral μ and ν we have from [1, §8.3]

$$S_n^{m(1)}(z,\gamma) = \left(\frac{\pi}{2\gamma z}\right)^{1/2} \frac{\left(z^2-1\right)^{-m/2} z^m}{A_n^m(\gamma^2)} \sum_{k=-k^+}^{\infty} a_{n,k}^m\left(\gamma^2\right) J_{n+2k+(1/2)}(\gamma z), \tag{1.4}$$

where

$$k^{\pm} = \left\lfloor \tfrac{1}{2}(n \pm m) \right\rfloor. \tag{1.5}$$

We parenthetically note that related solutions $S_n^{m(j)}(z,\gamma)$ $(j=3,4)$ are defined below

In (1.4) the coefficients are defined by the three-term recurrence relation

$$A_{n,k}^m\left(\gamma^2\right) a_{n,k-1}^m\left(\gamma^2\right) + \left\{\lambda_n^m\left(\gamma^2\right) + B_{n,k}^m\left(\gamma^2\right)\right\} a_{n,k}^m\left(\gamma^2\right) + C_{n,k}^m a_{n,k+1}^m\left(\gamma^2\right) = 0, \tag{1.6}$$

where

$$A_{n,k}^m\left(\gamma^2\right) = \frac{(n-m+2k-1)(n-m+2k)}{(2n+4k-3)(2n+4k-1)}\gamma^2, \tag{1.7}$$

$$B_{n,k}^m\left(\gamma^2\right) = \frac{2\left[(n+2k)(n+2k+1)+m^2-1\right]}{(2n+4k-1)(2n+4k+3)}\gamma^2 - (n+2k)(n+2k+1), \tag{1.8}$$

and

$$C_{n,k}^m\left(\gamma^2\right) = \frac{(n+m+2k+1)(n+m+2k+2)}{(2n+4k+3)(2n+4k+5)}\gamma^2. \tag{1.9}$$

The normalising constant $A_n^m\left(\gamma^2\right)$ is defined by

$$A_n^m\left(\gamma^2\right) = \sum_{k=-k^+}^{\infty} (-1)^k a_{n,k}^m\left(\gamma^2\right). \tag{1.10}$$

We also remark that the coefficients $a_{n,k}^m\left(\gamma^2\right)$ vanish for $k \leqslant -1-k^+$, where k^+ is defined by (1.5).

From (1.4), (1.10) and the well-known behavior of the Bessel function at infinity [25, Chap. 12, §1.2] we observe that $S_\nu^{\mu(1)}(z,\gamma)$ has the important property

$$S_\nu^{\mu(1)}(z,\gamma) = \frac{\cos\left\{\gamma z - \frac{1}{2}\pi(\nu+1)\right\}}{\gamma z}\left\{1 + O\left(\frac{1}{z}\right)\right\} \quad (z \to \infty), \tag{1.11}$$

for $|\arg(z)| < \pi$. With our assumption that $\nu = n =$ integer, all solutions of ((1.1)) are single-valued in the z-plane having a cut along the interval $[-1,1]$.

Other fundamental solutions at infinity are given by $S_n^{m(j)}(z,\gamma)$ $(j=3,4)$. These are defined by

$$S_n^{m(j)}(z,\gamma) = \left(\frac{\pi}{2\gamma z}\right)^{1/2} \frac{\left(z^2-1\right)^{-m/2} z^m}{A_n^m(\gamma^2)} \sum_{k=-k^+}^{\infty} a_{n,k}^m\left(\gamma^2\right) H_{n+2k+(1/2)}^{(j-2)}(\gamma z), \tag{1.12}$$

where $H_{\nu}^{(1,2)}(z)$ are the Hankel functions of the first and second kinds, respectively.

The solutions $S_n^{m(j)}(z,\gamma)$ have the fundamental properties

$$S_n^{m(3)}(z,\gamma) = i^{-n-1}\frac{e^{i\gamma z}}{\gamma z}\left\{1+O\left(\frac{1}{z}\right)\right\} \quad (z\to\infty), \tag{1.13}$$

and

$$S_n^{m(4)}(z,\gamma) = i^{n+1}\frac{e^{-i\gamma z}}{\gamma z}\left\{1+O\left(\frac{1}{z}\right)\right\} \quad (z\to\infty), \tag{1.14}$$

for $|\arg(z)| < \pi$. In particular $S_n^{m(3)}(z,\gamma)$ is the unique solution that is recessive in the upper half plane, and $S_n^{m(4)}(z,\gamma)$ is the unique solution that is recessive in the lower half plane.

An important connection formula, which comes directly from the corresponding one relating the J Bessel function to Hankel functions, is given by

$$S_n^{m(1)}(z,\gamma) = \tfrac{1}{2}\left\{S_n^{m(3)}(z,\gamma)+S_n^{m(4)}(z,\gamma)\right\}. \tag{1.15}$$

For $n \geqslant m \geqslant 0$ and $-1 < x < 1$ there is a solution $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ defined in terms of Ferrers functions by (see [1, §8.2])

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right) = \sum_{k=-k^-}^{\infty}(-1)^k a_{n,k}^m\left(\gamma^2\right)\mathrm{P}_{n+2k}^m(x). \tag{1.16}$$

This is the unique solution having the property of being recessive at $x=1$, and in particular has the property

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right) = K_n^m\left(\gamma^2\right)(1-x)^{m/2}\{1+O(1-x)\} \quad \left(x\to 1^-\right), \tag{1.17}$$

where $K_n^m\left(\gamma^2\right)$ is a constant given by

$$K_n^m\left(\gamma^2\right) = \frac{(-1)^m}{2^{m/2}m!}\sum_{k=-k^-}^{\infty}(-1)^k\frac{(n+2k+m)!}{(n+2k-m)!}a_{n,k}^m\left(\gamma^2\right). \tag{1.18}$$

From (1.16) and [27, eq. 14.7.17] we remark that $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ also has the fundamental property of also being bounded at $x=-1$; this is consequence of the characteristic exponent $\nu = n$ being an integer. In this case the function satisfies the normalisation condition

$$\int_{-1}^{1}\left\{\mathrm{Ps}_n^m\left(x,\gamma^2\right)\right\}^2 dx = \frac{2(n+m)!}{(2n+1)(n-m)!}. \tag{1.19}$$

As mentioned above, the separation constant $\lambda = \lambda_n^m\left(\gamma^2\right)$ is regarded as an eigenvalue for the case m and n integers, and hence admits the eigensolution $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ that is bounded at $x=\pm 1$.

For our purposes the PSWE (1.1) therefore takes the form

$$\left(1-z^2\right)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \left\{\lambda_n^m\left(\gamma^2\right) - \frac{m^2}{1-z^2} + \gamma^2\left(1-z^2\right)\right\}y = 0. \tag{1.20}$$

For x real and lying in $(1,\infty)$ (the radial case) we have the following solution of (1.20)

$$Ps_n^m\left(x,\gamma^2\right) = \sum_{k=-k^-}^{\infty} (-1)^k a_{n,k}^m\left(\gamma^2\right) P_{n+2k}^m(x), \tag{1.21}$$

which also defines $Ps_n^m\left(z,\gamma^2\right)$ for complex z; in this case $Ps_n^m\left(z,\gamma^2\right)$ is entire if m is even, and if m odd $\left(1-z^2\right)^{1/2} Ps_n^m\left(z,\gamma^2\right)$ is entire.

Now since m and n are integers, it is a straightforward to show from (1.21) that

$$Ps_n^m\left(ze^{\pi i},\gamma^2\right) = (-1)^n Ps_n^m\left(z,\gamma^2\right), \tag{1.22}$$

which is (unique) property of the Floquet solution $S_n^{m(1)}(z,\gamma)$. Hence

$$S_n^{m(1)}(z,\gamma) = (-1)^n (n-m)! V_n^m(\gamma) Ps_n^m\left(z,\gamma^2\right), \tag{1.23}$$

for some constant $V_n^m(\gamma)$, and hence from the known behavior of $S_n^{m(1)}(z,\gamma)$ at infinity

$$Ps_n^m\left(z,\gamma^2\right) = V_n^m(\gamma)\frac{\sin\left\{\gamma z - \frac{1}{2}\pi n\right\}}{\gamma z}\left\{1+O\left(\frac{1}{z}\right)\right\} \quad (z\to\infty). \tag{1.24}$$

An explicit expression, in terms of $a_{n,k}^m\left(\gamma^2\right)$, for the constant $V_n^m(\gamma)$ can be obtained from (1.11), (1.21), (1.23), and letting $z\to\infty$.

From [1, p. 171] we also note the recessive behavior

$$Ps_n^m\left(z,\gamma^2\right) = K_n^m\left(\gamma^2\right)(z-1)^{m/2}\left\{1+O(z-1)\right\} \quad (z\to 1), \tag{1.25}$$

where $K_n^m\left(\gamma^2\right)$ is given by (1.18).

The plan of the paper is as follows. In §2 we obtain Liouville-Green approximations for $S_n^{m(j)}(z,\gamma)$ $(j=3,4)$ where z is complex, and use these to obtain an asymptotic approximation for the radial PSWF $Ps_n^m\left(x,\gamma^2\right)$ which is uniformly valid in the interval $1+\delta \leqslant x < \infty$. In §3 the approximation for $Ps_n^m\left(x,\gamma^2\right)$ is extended to $1<x<\infty$ by applying the theory of differential equations having a simple pole, which involves the Bessel function of the first kind. Also in this section an asymptotic relationship involving $\lambda_n^m\left(\gamma^2\right)$ and the parameters m and n is obtained, by matching the Liouville-Green and Bessel function approximations at infinity.

In §4 the angular PSWF $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ is approximated, with the intervals $1-\delta_0 \leqslant x < 1$ and $0 \leqslant x \leqslant 1-\delta_0$ considered separately (for some positive constant δ_0). In the former interval the asymptotic approximation involves the modified Bessel function of the first kind, and in the latter interval the parabolic cylinder function is used. In §5 the approximation involving the parabolic cylinder function is simplified under the assumption n being bounded. Finally, in §6 we summarise the main results of the paper.

2. Liouville-Green asymptotics: the radial case

Making the transformation $w=\left(z^2-1\right)^{1/2} y$ in (1.20) we remove the first derivative to obtain

$$\frac{d^2w}{dz^2} = \left\{-\gamma^2 + \frac{\lambda_n^m\left(\gamma^2\right)}{z^2-1} + \frac{m^2-1}{\left(z^2-1\right)^2}\right\} w. \tag{2.1}$$

Now, from [1, p. 186] it is known that for large γ, with m and n bounded, that

$$\lambda_n^m\left(\gamma^2\right) = -\gamma^2 + 2\left(n - m + \tfrac{1}{2}\right)\gamma + O(1). \tag{2.2}$$

With this in mind we define a parameter σ by

$$\lambda_n^m\left(\gamma^2\right) = -\gamma^2\left(1 - \sigma^2\right), \tag{2.3}$$

and throughout we shall assume that

$$0 \leqslant \sigma = \sqrt{1 + \gamma^{-2}\lambda_n^m\left(\gamma^2\right)} \leqslant \sigma_0 < 1, \tag{2.4}$$

where σ_0 is an arbitrary positive constant.

Next, from (2.3) we can express (2.1) in the form

$$\frac{d^2 w}{dz^2} = \left[\gamma^2 f(\sigma, z) + g(z)\right] w, \tag{2.5}$$

where

$$f(\sigma, z) = \frac{\sigma^2 - z^2}{z^2 - 1}, \quad g(z) = \frac{m^2 - 1}{\left(z^2 - 1\right)^2}. \tag{2.6}$$

We observe for large γ the differential equation has turning points at $z = \pm\sigma$, and on account of our assumption (2.4) these turning points lie in the interval $(-1, 1)$, they may coalesce with one another at $z = 0$, but are bounded away from the poles $z = \pm 1$.

We shall construct Liouville-Green approximations for $Ps_n^m\left(z, \gamma^2\right)$, using the theory of [25, Chap. 10]. To this end, we introduce a new independent variable

$$\xi = \int_1^z \{-f(\sigma, t)\}^{1/2} dt = \int_1^z \left(\frac{t^2 - \sigma^2}{t^2 - 1}\right)^{1/2} dt. \tag{2.7}$$

Branch cuts are suitably chosen so that $0 \leqslant \xi < \infty$ for $1 \leqslant z < \infty$.

The RHS of (2.7) can be expressed in terms of the elliptic integral of the second kind [27, eq. 19.2.5]

$$E(a; b) = \int_0^a \left(\frac{1 - b^2 t^2}{1 - t^2}\right)^{1/2} dt = b\int_0^a \left(\frac{b^{-2} - t^2}{1 - t^2}\right)^{1/2} dt. \tag{2.8}$$

Here $b = \sigma^{-1} > 1$, and the branches of the square roots are such that integrand is positive for $0 \leqslant t < b^{-1}$ and negative for $1 < t < \infty$, and continuous elsewhere in the complex t-plane having a cut along the interval $\left[b^{-1}, 1\right]$. We thus have

$$\xi = \sigma E\left(z; \sigma^{-1}\right) - \sigma E\left(1; \sigma^{-1}\right). \tag{2.9}$$

Then with the new dependent variable $W = \{-f\}^{1/4} w$ we obtain

$$\frac{d^2 W}{d\xi^2} = \left[-\gamma^2 + \psi(\xi)\right] W, \tag{2.10}$$

where

$$\psi(\xi) = \frac{m^2-1}{(z^2-1)(z^2-\sigma^2)} + \frac{(1-\sigma^2)\left(6z^4-(3+\sigma^2)z^2-2\sigma^2\right)}{4(z^2-1)(z^2-\sigma^2)^3}. \tag{2.11}$$

We observe that $\psi(\xi) = O\left(\xi^{-2}\right)$ as $\xi \to \infty$, but is unbounded at the singularities $z = \pm 1$, and also at the turning points $z = \pm\sigma$.

From the definition of ξ we find that

$$\xi = z - J(\sigma) + O\left(z^{-1}\right) \quad (z \to \infty), \tag{2.12}$$

where

$$J(\sigma) = 1 - \int_1^\infty \left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2} - 1\right] dt. \tag{2.13}$$

Note $J(0) = 0$ and $J(1) = 1$. Now by Cauchy's theorem

$$0 = \mathrm{Re}\int_{-\infty}^{\infty}\left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2} - 1\right] dt = 2\,\mathrm{Re}\int_0^\infty \left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2} - 1\right] dt. \tag{2.14}$$

Hence

$$\int_1^\infty \left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2} - 1\right] dt = -\,\mathrm{Re}\int_0^1 \left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2} - 1\right] dt, \tag{2.15}$$

and consequently from (2.13)

$$J(\sigma) = 1 + \mathrm{Re}\int_0^1 \left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2} - 1\right] dt = \int_0^\sigma \left(\frac{\sigma^2-t^2}{1-t^2}\right)^{1/2} dt; \tag{2.16}$$

i.e.

$$J(\sigma) = \sigma E\left(\sigma;\sigma^{-1}\right), \tag{2.17}$$

for $\sigma > 0$, in which E is the Elliptic integral of the second kind given by (2.8). Thus

$$\xi = z - \sigma E\left(\sigma;\sigma^{-1}\right) + O\left(z^{-1}\right) \quad (z \to \infty). \tag{2.18}$$

We now apply Theorem 3.1 of [26], with u replaced by γ, and with ξ replaced by $i\xi$. Then, by matching solutions that are recessive at $z = \pm i\infty$, we have from (1.13), (1.14) and (2.18)

$$\begin{aligned} S_n^{m(3)}(z,\gamma) = {} & i^{-1-n}\gamma^{-1}\left[\left(z^2-1\right)\left(z^2-\sigma^2\right)\right]^{-1/4} e^{i\gamma J(\sigma)} \\ & \times \left[e^{i\gamma\xi}\sum_{s=0}^{p-1}(-i)^s\frac{A_s(\xi)}{\gamma^s} + \varepsilon_{p,1}(\gamma,\xi)\right], \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} S_n^{m(4)}(z,\gamma) = {} & i^{1+n}\gamma^{-1}\left[\left(z^2-1\right)\left(z^2-\sigma^2\right)\right]^{-1/4} e^{-i\gamma J(\sigma)} \\ & \times \left[e^{-i\gamma\xi}\sum_{s=0}^{p-1} i^s\frac{A_s(\xi)}{\gamma^s} + \varepsilon_{p,2}(\gamma,\xi)\right]. \end{aligned} \tag{2.20}$$

The error terms $\varepsilon_{p,j}(\gamma,\xi)$ $(j=1,2)$ are bounded by Olver's theorem, and are $O(\gamma^{-p})$ in unbounded domains containing the real interval $1+\delta \leqslant z < \infty$ $(\delta > 0)$. Here the coefficients are defined recursively by $A_0(\xi)=1$ and

$$A_{s+1}(\xi) = -\tfrac{1}{2}A_s'(\xi) + \tfrac{1}{2}\int \psi(\xi) A_s(\xi)\, d\xi \quad (s=0,1,2,\cdots). \tag{2.21}$$

Thus, from (1.15), (1.23), (2.17), (2.19) and (2.20), we obtain the desired Liouville-Green expansion for $Ps_n^m(x,\gamma^2)$. In particular, to leading order, we have

$$Ps_n^m(x,\gamma^2) = \frac{(-1)^n \sin\left(\gamma\xi + \gamma\sigma E\left(\sigma;\sigma^{-1}\right) - \frac{1}{2}n\pi\right) + O\left(\gamma^{-1}\right)}{\gamma(n-m)!V_n^m(\gamma)\left[(x^2-1)(x^2-\sigma^2)\right]^{1/4}}, \tag{2.22}$$

as $\gamma \to \infty$, uniformly for $1+\delta \leqslant x < \infty$. In order for this approximation to be practicable, one requires an asymptotic approximation for $\lambda_n^m(\gamma^2)$ as $\gamma \to \infty$, and we shall discuss this in the next section. We also remark that (2.22) breaks down at the simple pole $x=1$, and in the next section we obtain asymptotic approximations that are valid at this pole.

3. Bessel function approximations: the radial case

We now obtain approximations valid at the simple pole of $f(\sigma,z)$ at $z=1$, using the asymptotic theory of [25, Chap. 12]. We consider $z=x$ real and positive. The appropriate Liouville transformation is now given by

$$\eta = \xi^2 = \left[\int_1^x \{-f(\sigma,t)\}^{1/2} dt\right]^2, \tag{3.1}$$

along with

$$\hat{W} = \left\{\frac{\eta\left(x^2-\sigma^2\right)}{x^2-1}\right\}^{1/4} w, \tag{3.2}$$

which yields the new equation

$$\frac{d^2\hat{W}}{d\eta^2} = \left[-\frac{\gamma^2}{4\eta} + \frac{m^2-1}{4\eta^2} + \frac{\hat{\psi}(\eta)}{\eta}\right]\hat{W}. \tag{3.3}$$

Here

$$\hat{\psi}(\eta) = \frac{1-4m^2}{16\eta} + \frac{m^2-1}{4(x^2-1)(x^2-\sigma^2)} + \frac{\left(1-\sigma^2\right)\left(6x^4-\left(3+\sigma^2\right)x^2-2\sigma^2\right)}{16(x^2-1)\left(x^2-\sigma^2\right)^3}. \tag{3.4}$$

This has the same main features of (2.5), namely a simple pole for the dominant term (for large γ) and a double pole in another term. We note that $x=1$ corresponds to $\eta=0$.

The difference here is that non-dominant term $\hat{\psi}(\eta)$ is now analytic at $\eta=0$, i.e. $x=1$. Neglecting $\hat{\psi}(\eta)$ in (3.3) gives an equation solvable in terms of Bessel functions. We then find (by matching recessive solutions at $x=1$) and applying theorem 4.1 of [25, Chap. 12] (with u replaced by γ and ζ replaced by η)

$$
\begin{aligned}
Ps_n^m\left(x,\gamma^2\right) = c_n^m(\gamma)\left\{\frac{\eta}{(x^2-1)(x^2-\sigma^2)}\right\}^{1/4} \\
\times\left[J_m\left(\gamma\eta^{1/2}\right)+O\left(\gamma^{-1}\right)\operatorname{env}J_m\left(\gamma\eta^{1/2}\right)\right],
\end{aligned}
\tag{3.5}
$$

as $\gamma\to\infty$, uniformly for $1<x<\infty$. Olver's theorem provides an asymptotic expansion in inverse powers of γ, but we present just the leading term here. The so-called envelope env of the J Bessel function is defined by [27, §2.8(iv)].

The constant of proportionality $c_n^m(\gamma)$ can be found by comparing both sides of (3.5) as $x\to1$ ($\eta\to0$). Using

$$
\eta=2\left(1-\sigma^2\right)(x-1)+O\left\{(x-1)^2\right\}\quad(x\to1),
\tag{3.6}
$$

along with (1.25), (2.3) and the behavior of the J Bessel function at the origin (e.g. [25, Chap. 12, §1]) we find that

$$
c_n^m(\gamma)=\left(-\frac{2}{\lambda_n^m(\gamma^2)}\right)^{m/2}m!K_n^m\left(\gamma^2\right).
\tag{3.7}
$$

An asymptotic approximation for this constant is given by (4.46) below.

Next, from the well-known behavior of the J Bessel function at infinity (e.g. see [25, Chap. 12, §1]), we find from (3.5) that

$$
\begin{aligned}
Ps_n^m\left(x,\gamma^2\right) \sim \text{constant}\times\left\{\left(x^2-1\right)\left(x^2-\sigma^2\right)\right\}^{-1/4} \\
\times\left\{\cos\left(\gamma\xi-\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi\right)+O\left(\xi^{-1}\right)\right\}\quad\left(\eta=\xi^2\to\infty\right).
\end{aligned}
\tag{3.8}
$$

However, from (2.22) we observe an alternative expression of the behavior of this function. On comparing both, we deduce that

$$
\gamma\sigma E\left(\sigma;\sigma^{-1}\right)=\left(2N+\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{4}\right)\pi+O\left(\gamma^{-1}\right),
\tag{3.9}
$$

for some integer N, which we show is zero. Now, from (2.2) and (2.3) we have for *fixed* m and n that $\sigma=O\left(\gamma^{-1/2}\right)$ as $\gamma\to\infty$, and more precisely,

$$
\sigma^2=2\left(n-m+\tfrac{1}{2}\right)\gamma^{-1}+O\left(\gamma^{-2}\right).
\tag{3.10}
$$

Thus in this case, using (2.16) and (2.17) in the LHS of (3.9), we have

$$
\tfrac{1}{4}\pi\gamma\sigma^2+O\left(\gamma^{-1}\sigma^4\right)=\left(2N+\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{4}\right)\pi+O\left(\gamma^{-1}\right).
\tag{3.11}
$$

Inserting (3.10) into (3.11) we deduce that $N=0$, at least for fixed m and n; a continuity argument removes this restriction.

It is possible to extend (2.22) and (3.8) to asymptotic expansions, and consequently from (2.16) and (2.17) we arrive at

$$\gamma \int_0^\sigma \left(\frac{\sigma^2 - t^2}{1 - t^2} \right)^{1/2} dt \sim \frac{1}{2}\left(n - m + \frac{1}{2} \right) \pi + \sum_{s=0}^{\infty} \frac{\kappa_s}{\gamma^{2s+1}}, \tag{3.12}$$

for constants κ_s which can be determined in terms of the coefficients appearing in (2.19) and (2.20). From (2.4) we can invert this expansion to provide a means of computing the eigenvalue $\lambda = \lambda_n^m(\gamma^2)$ asymptotically in terms of m and n as $\gamma \to \infty$.

Now, the elliptic integral on the LHS of (3.12) monotonically increases from 0 to 1 as σ increases from 0 to 1. We therefore see that the condition (2.4) (along with our assumption that m is bounded) is equivalent to (1.2).

4. Bessel and parabolic cylinder function approximations: the angular case

Recall $\mathrm{Ps}_n^m(x, \gamma^2)$ is the unique solution with the property of being recessive at $x = \pm 1$. It is also uniquely determined by the property

$$\mathrm{Ps}_n^m(-x, \gamma^2) = (-1)^{m+n}\, \mathrm{Ps}_n^m(x, \gamma^2). \tag{4.1}$$

Thus, it suffices to approximate $\mathrm{Ps}_n^m(x, \gamma^2)$ in the interval $0 \leqslant x < 1$. We consider the subintervals $0 \leqslant x \leqslant 1 - \delta_0$ and $1 - \delta_0 \leqslant x < 1$ separately, where $\delta_0 \in (0, 1 - \sigma_0)$ is arbitrary; recall that σ_0 is defined by (2.4). The significance of this choice is that the turning point $x = \sigma$ is bounded away from the interval $[1 - \delta_0, 1]$.

For $1 - \delta_0 \leqslant x < 1$ we apply theorem 3.1 of [25, Chap. 12]. It can then be shown by utilising the recessive behavior at $x = 1$ ($\eta = 0$) that

$$\mathrm{Ps}_n^m(x, \gamma^2) = c_n^m(\gamma) \left\{ \frac{|\eta|}{(1 - x^2)(x^2 - \sigma^2)} \right\}^{1/4} I_m\left(\gamma |\eta|^{1/2} \right) \left[1 + O\left(\frac{1}{\gamma} \right) \right], \tag{4.2}$$

where $I_m(x)$ is the modified Bessel function, and $c_n^m(\gamma)$ is given by (3.7). Expansions and error bounds are obtainable from Olver's theorem.

The interval $0 \leqslant x \leqslant 1 - \delta_0$ is less straightforward. From (2.5) and (2.6) we observe that equation has the turning point $x = \sigma$ in this interval, and this coalesces with the other turning point $x = -\sigma$ when $\sigma \to 0$. The appropriate asymptotic theory for this situation is provided by [26], and from eq. (2.3) of this reference the appropriate transformation is given by

$$\frac{d\zeta}{dx} = \left(\frac{\sigma^2 - x^2}{(1 - x^2)(\alpha^2 - \zeta^2)} \right)^{1/2}. \tag{4.3}$$

Upon integration, this yields the implicit relationship

$$\int_{-\alpha}^{\zeta} \left(\alpha^2 - \tau^2 \right)^{1/2} d\tau = \int_{-\sigma}^{x} \{-f(\sigma, t)\}^{1/2} dt = \int_{-\sigma}^{x} \left(\frac{\sigma^2 - t^2}{1 - t^2} \right)^{1/2} dt. \tag{4.4}$$

The lower limits are selected to ensure that the turning point $x=-\sigma$ is mapped to a new turning point at $\zeta=-\alpha$ (see (4.10) below). From [26, eq. (2.5)] we find that α is given by

$$\alpha^2=\frac{2}{\pi}\int_{-\sigma}^{\sigma}\left(\frac{\sigma^2-t^2}{1-t^2}\right)^{1/2}dt=\frac{4}{\pi}J(\sigma), \tag{4.5}$$

which ensures that the original turning point $x=\sigma$ is mapped to the turning point at $\zeta=\alpha$ in the transformed equation.

By symmetry $x=0$ is mapped to $\zeta=0$, and so the lower limits in the integrals of (4.4) can be replaced by 0. Thus we have

$$\tfrac{1}{2}\alpha^2\arcsin\left(\frac{\zeta}{\alpha}\right)+\tfrac{1}{2}\zeta\left(\alpha^2-\zeta^2\right)^{1/2}=\sigma E\left(x;\sigma^{-1}\right), \tag{4.6}$$

for $0\leqslant x\leqslant\sigma$ $(0\leqslant\zeta\leqslant\alpha)$.

For $\sigma\leqslant x\leqslant 1-\delta_0$ we have

$$\int_{\alpha}^{\zeta}\left(\tau^2-\alpha^2\right)^{1/2}d\tau=\int_{\sigma}^{x}\{f(\sigma,t)\}^{1/2}dt=\int_{\sigma}^{x}\left(\frac{t^2-\sigma^2}{1-t^2}\right)^{1/2}dt. \tag{4.7}$$

Thus in this case

$$-\tfrac{1}{2}\alpha^2\operatorname{arccosh}\left(\frac{\zeta}{\alpha}\right)+\tfrac{1}{2}\zeta\left(\zeta^2-\alpha^2\right)^{1/2}=\left|\operatorname{Im}\left\{\sigma E\left(x;\sigma^{-1}\right)\right\}\right|. \tag{4.8}$$

With

$$W=\left\{\frac{\sigma^2-x^2}{(\alpha^2-\zeta^2)(1-x^2)}\right\}^{1/4}w, \tag{4.9}$$

we transform (2.5) to the form

$$\frac{d^2W}{d\zeta^2}=\left\{\gamma^2\left(\zeta^2-\alpha^2\right)+\psi(\gamma,\alpha,\zeta)\right\}W, \tag{4.10}$$

where

$$\begin{aligned}\psi(\gamma,\alpha,\zeta)=&\frac{\left(1-m^2\right)\left(\alpha^2-\zeta^2\right)}{(1-x^2)(\sigma^2-x^2)}+\frac{2\alpha^2+3\zeta^2}{4\left(\alpha^2-\zeta^2\right)^2}\\&-\frac{\left(1-\sigma^2\right)\left(\alpha^2-\zeta^2\right)\left\{6x^4-\left(\sigma^2+3\right)x^2-2\sigma^2\right\}}{4\left(1-x^2\right)\left(\sigma^2-x^2\right)^3}.\end{aligned} \tag{4.11}$$

To sharpen the subsequent error bounds it is possible to perturb the parameter by defining a new parameter ω by $\alpha^2=\omega^2+\psi(\gamma,\alpha,0)\gamma^{-2}$, but we shall not pursue this here.

From theorem I of [26], with u replaced by γ, we obtain two independent solutions of (4.10) given by

$$w_1(\gamma,\alpha,\zeta)=U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)+\varepsilon_1(\gamma,\alpha,\zeta), \tag{4.12}$$

and

$$w_2(\gamma,\alpha,\zeta) = \bar{U}\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) + \varepsilon_2(\gamma,\alpha,\zeta). \tag{4.13}$$

Here $U(a,x)$ and $\bar{U}(a,x)$ are the parabolic cylinder functions defined in [26, §5] and [27, §12.2], and are linearly independent for $a<0$. The approximants $U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right)$ and $\bar{U}\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right)$ satisfy the differential equation (4.10) with $\psi(\gamma,\alpha,\zeta)\equiv 0$.

The error terms are bounded by [26, §6], and in particular these show that

$$\varepsilon_1(\gamma,\alpha,\zeta) = O\left(\gamma^{-2/3}\ln(\gamma)\right)\operatorname{env}U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right), \tag{4.14}$$

and

$$\varepsilon_2(\gamma,\alpha,\zeta) = O\left(\gamma^{-2/3}\ln(\gamma)\right)\operatorname{env}\bar{U}\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right), \tag{4.15}$$

uniformly for $0 \leqslant x \leqslant 1-\delta_0$. Here the envelope function env is defined for the parabolic cylinder functions by [27, eq. 14.15.23].

The parabolic cylinder function U has the unique recessive property

$$U\left(-\tfrac{1}{2}a,x\right) \sim x^{(a-1)/2}e^{-x^2/4} \quad (x\to\infty), \tag{4.16}$$

whereas $\bar{U}$ is dominant, with the behavior

$$\bar{U}\left(-\tfrac{1}{2}a,x\right) \sim (2/\pi)^{1/2}\Gamma\left(\tfrac{1}{2}a+\tfrac{1}{2}\right)x^{-(a+1)/2}e^{x^2/4} \quad (x\to\infty); \tag{4.17}$$

see [26, §5]. In addition, from [26, Eqs. (5.12) and (5.13)] and the definitions (4.12) and (4.13), we note that $w_1(\gamma,\alpha,\zeta)$ and $w_2(\gamma,\alpha,\zeta)$ are oscillatory in the ζ interval $[0,\alpha]$, with comparable amplitudes and complementary phases of the argument.

Now, for negative x and ζ, we will also need the solution given in [26]

$$w_4(\gamma,\alpha,\zeta) = \bar{U}\left(-\tfrac{1}{2}\gamma\alpha^2, -\zeta\sqrt{2\gamma}\right) + \varepsilon_4(\gamma,\alpha,\zeta). \tag{4.18}$$

We remark that

$$\varepsilon_j(\gamma,\alpha,0) = \partial\varepsilon_j(\gamma,\alpha,0)/\partial\zeta = 0 \quad (j=2,4), \tag{4.19}$$

and hence

$$w_2(\gamma,\alpha,0) = w_4(\gamma,\alpha,0) = \bar{U}\left(-\tfrac{1}{2}\gamma\alpha^2, 0\right), \tag{4.20}$$

as well as

$$\partial w_2(\gamma,\alpha,0)/\partial\zeta = -\partial w_4(\gamma,\alpha,0)/\partial\zeta = \sqrt{2\gamma}\,\bar{U}'\left(-\tfrac{1}{2}\gamma\alpha^2, 0\right). \tag{4.21}$$

The error bounds for $\varepsilon_4(\gamma,\alpha,\zeta)$ only apply for non-positive ζ. In order to extend the solution to positive values of ζ we use [26, Eqs. (6.23) and (6.24)] to obtain the connection formula

$$\begin{aligned} w_4(\gamma,\alpha,\zeta) = &-\left\{\sin\left(\tfrac{1}{2}\pi\gamma\alpha^2\right) + O\left(\gamma^{-2/3}\right)\right\} w_2(\gamma,\alpha,\zeta) \\ &+ \left\{\cos\left(\tfrac{1}{2}\pi\gamma\alpha^2\right) + O\left(\gamma^{-2/3}\right)\right\} w_1(\gamma,\alpha,\zeta). \end{aligned} \tag{4.22}$$

Now from (2.17), (3.9) and (4.5)

$$\tfrac{1}{2}\pi\gamma\alpha^2 = \left(n-m+\tfrac{1}{2}\right)\pi + O\left(\gamma^{-1}\right). \tag{4.23}$$

Bearing in mind that $w_1(\gamma,\alpha,\zeta)$ is exponentially small compared to $w_2(\gamma,\alpha,\zeta)$ in $0 \leqslant x \leqslant 1-\delta_0$ (except near its zeros) we deduce from (4.22) and (4.23) that

$$\begin{aligned} &w_2(\gamma,\alpha,\zeta) - (-1)^{m+n} w_4(\gamma,\alpha,\zeta) \\ &= 2w_2(\gamma,\alpha,\zeta) + O\left(\gamma^{-2/3}\right)\left\{w_1(\gamma,\alpha,\zeta) + w_2(\gamma,\alpha,\zeta)\right\}. \end{aligned} \tag{4.24}$$

We next express

$$\begin{aligned} \mathrm{Ps}_n^m\left(x,\gamma^2\right) = \left\{\frac{\alpha^2-\zeta^2}{(\sigma^2-x^2)(1-x^2)}\right\}^{1/4} &\left[d_n^m(\gamma)\, w_1(\gamma,\alpha,\zeta)\right. \\ &\left. + e_n^m(\gamma)\left\{w_2(\gamma,\alpha,\zeta) - (-1)^{m+n} w_4(\gamma,\alpha,\zeta)\right\}\right], \end{aligned} \tag{4.25}$$

and we shall determine the constant $d_n^m(\gamma)$ (as well as bounding $e_n^m(\gamma)$) by comparing both sides of this relationship at appropriate values of x.

To this end, firstly we assume that $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ is even, so that $m+n$ is also even. Then, setting $x=\zeta=0$ in (4.25), and invoking (4.20), immediately yields

$$d_n^m(\gamma) = \left(\frac{\sigma}{\alpha}\right)^{1/2} \frac{\mathrm{Ps}_n^m\left(0,\gamma^2\right)}{w_1(\gamma,\alpha,0)}. \tag{4.26}$$

An asymptotic approximation for this constant, which does not involve $\mathrm{Ps}_n^m\left(0,\gamma^2\right)$, is given by (4.43) below.

Next, if we differentiate both sides of (4.25) and again set $x=\zeta=0$ we find from the property $\mathrm{Ps}_n^{m\,\prime}\left(0,\gamma^2\right)=0$ that $e_n^m(\gamma) = O\left(\gamma^{-1}\right)$, which is not sharp enough. Instead we match the parabolic cylinder and Bessel function approximations, and their derivatives, at the fixed point $x = 1-\frac{1}{2}\delta_0$ (at which both the parabolic cylinder function and modified Bessel function approximations are valid). Using (4.2) (4.24) and (4.25), we therefore arrive at

$$e_n^m(\gamma) \sim -d_n^m(\gamma)\frac{\mathscr{W}\left\{\left(\zeta^2-\alpha^2\right)^{1/4} U\left(-\frac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right), |\eta|^{1/4} I_m\left(\gamma|\eta|^{1/2}\right)\right\}}{\mathscr{W}\left\{\left(\zeta^2-\alpha^2\right)^{1/4} \bar{U}\left(-\frac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right), |\eta|^{1/4} I_m\left(\gamma|\eta|^{1/2}\right)\right\}}, \tag{4.27}$$

and

$$c_n^m(\gamma) \sim d_n^m(\gamma)\frac{\left(\zeta^2-\alpha^2\right)^{1/2}\mathscr{W}\left\{\bar{U}\left(-\frac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right), U\left(-\frac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right)\right\}}{\mathscr{W}\left\{\left(\zeta^2-\alpha^2\right)^{1/4} \bar{U}\left(-\frac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right), |\eta|^{1/4} I_m\left(\gamma|\eta|^{1/2}\right)\right\}}. \tag{4.28}$$

In both of these the Wronskians $\mathscr{W}$ are with respect to x, and evaluated at $x = 1-\frac{1}{2}\delta_0$ (with η and ζ corresponding to this value).

Next, from (4.7) and [27, Eqs. 12.10.3 – 12.10.6], we have the asymptotic approximations for large γ, fixed $\zeta \in (\alpha, \infty)$ and fixed $\alpha > 0$

$$U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) \sim \left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/4} \frac{\exp\left\{-\gamma\int_\sigma^x \{f(\sigma,t)\}^{1/2} dt\right\}}{\{2\gamma(\zeta^2-\alpha^2)\}^{1/4}}, \tag{4.29}$$

$$U'\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) \sim -\frac{1}{2}\left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/4} \{2\gamma(\zeta^2-\alpha^2)\}^{1/4} \exp\left\{-\gamma\int_\sigma^x \{f(\sigma,t)\}^{1/2} dt\right\}, \tag{4.30}$$

$$\bar{U}\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) \sim 2\left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/4} \frac{\exp\left\{\gamma\int_\sigma^x \{f(\sigma,t)\}^{1/2} dt\right\}}{\{2\gamma(\zeta^2-\alpha^2)\}^{1/4}}, \tag{4.31}$$

and

$$\bar{U'}\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) \sim \left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/4} \{2\gamma(\zeta^2-\alpha^2)\}^{1/4} \exp\left\{\gamma\int_\sigma^x \{f(\sigma,t)\}^{1/2} dt\right\}. \tag{4.32}$$

These, along with

$$|\eta|^{1/4} I_m\left(\gamma|\eta|^{1/2}\right) \sim (2\pi\gamma)^{-1/2} \exp\left\{\gamma\int_x^1 \{f(\sigma,t)\}^{1/2} dt\right\}, \tag{4.33}$$

$$\frac{d\left\{|\eta|^{1/4} I_m\left(\gamma|\eta|^{1/2}\right)\right\}}{dx} \sim -\left(\frac{\gamma}{2\pi}\right)^{1/2} \left(\frac{x^2-\sigma^2}{1-x^2}\right)^{1/2} \exp\left\{\gamma\int_x^1 \{f(\sigma,t)\}^{1/2} dt\right\}, \tag{4.34}$$

and

$$\frac{d\zeta}{dx} = \left\{\frac{x^2-\sigma^2}{(1-x^2)(\zeta^2-\alpha^2)}\right\}^{1/2}, \tag{4.35}$$

can be used to simplify (4.27) and (4.28). In particular, we find that

$$e_n^m(\gamma)\left\{w_2(\gamma,\alpha,\zeta) - (-1)^{m+n} w_4(\gamma,\alpha,\zeta)\right\} = o(1) A \mathrm{env} U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right), \tag{4.36}$$

where the $o(1)$ term is exponentially small as $\gamma \to \infty$ for $x \in [0, 1-\delta_0]$. In addition, we obtain the useful result

$$c_n^m(\gamma) \sim d_n^m(\gamma) \left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/4} \left(\frac{\pi^2}{2\gamma}\right)^{1/4} \exp\left\{-\gamma\int_\sigma^1 \{f(\sigma,t)\}^{1/2} dt\right\}. \tag{4.37}$$

From (4.14) and (4.25)–(4.36), for $m+n$ even, m bounded and n satisfying (1.2), we arrive at our desired result

$$\begin{aligned}\mathrm{Ps}_n^m(x,\gamma^2) &= \frac{\mathrm{Ps}(0,\gamma^2)}{U\left(-\frac{1}{2}\gamma\alpha^2, 0\right)} \left\{\frac{\sigma^2(\alpha^2-\zeta^2)}{\alpha^2(\sigma^2-x^2)(1-x^2)}\right\}^{1/4} \\ &\quad \times \left\{U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) + O\left(\gamma^{-2/3}\ln(\gamma)\right) \mathrm{env} U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right)\right\},\end{aligned} \tag{4.38}$$

as $\gamma \to \infty$, uniformly for $0 \leqslant x \leqslant 1-\delta_0$.

From [26, §5] we note that

$$U\left(-\tfrac{1}{2}\gamma\alpha^2, 0\right) = \pi^{-1/2} 2^{\left(\gamma\alpha^2-1\right)/4} \Gamma\left(\frac{1}{4}\gamma\alpha^2 + \frac{1}{4}\right) \sin\left(\frac{1}{4}\gamma\alpha^2\pi + \frac{1}{4}\pi\right), \tag{4.39}$$

as well as

$$U'\left(-\tfrac{1}{2}\gamma\alpha^2, 0\right) = -\pi^{-1/2} 2^{\left(\gamma\alpha^2+1\right)/4} \Gamma\left(\frac{1}{4}\gamma\alpha^2 + \frac{3}{4}\right) \sin\left(\frac{1}{4}\gamma\alpha^2\pi + \frac{3}{4}\pi\right). \tag{4.40}$$

Thus, on referring to (4.23), we observe that the RHS of (4.39) is bounded away from zero for large γ when $m+n$ is even, and likewise for the RHS of (4.40) when $m+n$ is odd (see (4.42) below).

For the case $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ odd, equivalently $m+n$ odd, we differentiate both sides of (4.25) with respect to ζ, and then set $x=\zeta=0$. As a result, using (4.3) and (4.21), along with the fact that $\mathrm{Ps}_n^m\left(0,\gamma^2\right)=0$, we obtain

$$d_n^m(\gamma) = \left(\frac{\alpha}{\sigma}\right)^{1/2} \frac{\mathrm{Ps}_n^{m\,\prime}\left(0,\gamma^2\right)}{\partial w_1(\gamma,\alpha,0)/\partial\zeta}. \tag{4.41}$$

Thus, again from (4.36), we conclude for $m+n$ odd, m bounded and n satisfying (1.2), that

$$\begin{aligned}\mathrm{Ps}_n^m\left(x,\gamma^2\right) = {} & \frac{\mathrm{Ps}_n^{m\,\prime}\left(0,\gamma^2\right)}{U'\left(-\frac{1}{2}\gamma\alpha^2,0\right)} \left\{\frac{\alpha^2\left(\alpha^2-\zeta^2\right)}{4\gamma^2\sigma^2\left(\sigma^2-x^2\right)\left(1-x^2\right)}\right\}^{1/4} \\ & \times\left\{U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right) + O\left(\gamma^{-2/3}\ln(\gamma)\right)\mathrm{env}\,U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)\right\},\end{aligned} \tag{4.42}$$

as $\gamma \to \infty$, uniformly for $0 \leqslant x \leqslant 1-\delta$. In this $U'\left(-\frac{1}{2}\gamma\alpha^2,0\right)$ is given by (4.40).

We now show that the proportionality constants in (4.38) and (4.42) can be replaced by one that does not involve $\mathrm{Ps}_n^m\left(0,\gamma^2\right)$ or $\mathrm{Ps}_n^{m\,\prime}\left(0,\gamma^2\right)$. Specifically, from (1.19), (4.2), (4.25), (4.36) and (4.37) we have (for both the even and odd cases) that

$$d_n^m(\gamma) \sim \left\{\frac{(n+m)!}{(2n+1)(n-m)!\,p_n^m(\gamma)}\right\}^{1/2}, \tag{4.43}$$

as $\gamma \to \infty$, again with m bounded and n satisfying (1.2). Here

$$\begin{aligned}p_n^m(\gamma) = \Bigg[& \int_0^{1-\delta_0} \left\{\frac{\alpha^2-\zeta^2}{\left(\sigma^2-x^2\right)\left(1-x^2\right)}\right\}^{1/2} U^2\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)dx \\ & + q_n^m(\gamma)\int_{1-\delta_0}^1 \left\{\frac{|\eta|}{\left(1-x^2\right)\left(x^2-\sigma^2\right)}\right\}^{1/2} I_m^2\left(\gamma|\eta|^{1/2}\right)dx\Bigg],\end{aligned} \tag{4.44}$$

in which

$$q_n^m(\gamma) = \left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/2} \left(\frac{\pi^2}{2\gamma}\right)^{1/2} \exp\left\{-2\gamma\int_\sigma^1 \{f(\sigma,t)\}^{1/2}\,dt\right\}. \tag{4.45}$$

Note also, from (4.37), that under the same conditions

$$c_n^m(\gamma) \sim \left\{ \frac{(n+m)!\,q_n^m(\gamma)}{(2n+1)(n-m)!\,p_n^m(\gamma)} \right\}^{1/2}. \tag{4.46}$$

5. Fixed m and n: the angular case

For fixed m *and* n we can simplify the results of the previous section, by applying the theory of [10]. To this end we observe that (2.1) can be expressed in the form

$$\frac{d^2w}{dx^2} = \left[\frac{\gamma^2 x^2}{1-x^2} - \frac{a\gamma}{1-x^2} + \frac{m^2-1}{(1-x^2)^2} \right] w, \tag{5.1}$$

where

$$a = \lambda\gamma^{-1} + \gamma = 2\left(n-m+\tfrac{1}{2}\right) + O\left(\gamma^{-1}\right), \tag{5.2}$$

the $O\left(\gamma^{-1}\right)$ term being valid for fixed m and n and $\gamma \to \infty$. In particular, a is bounded.

Equation (5.1) is characterised as having a pair of almost coalescent turning points near $x=0$. The appropriate Liouville transformation in this case is given by

$$\frac{1}{2}\rho^2 = \int_0^x \frac{t}{(1-t^2)^{1/2}}dt = 1 - \left(1-x^2\right)^{1/2}. \tag{5.3}$$

Note $x=0$ corresponds to $\rho=0$, and $x=1$ corresponds to $\rho=\sqrt{2}$. Then with

$$W = \frac{x^{1/2}}{\rho^{1/2}(1-x^2)^{1/4}} w, \tag{5.4}$$

we obtain

$$\frac{d^2W}{d\rho^2} = \left[\gamma^2\rho^2 - \gamma a + \gamma\zeta\phi(\rho) + \chi(\rho)\right] W, \tag{5.5}$$

where

$$\phi(\rho) = -\frac{a\rho}{4-\zeta^2}, \tag{5.6}$$

and

$$\chi(\rho) = \frac{\rho^2\left(4m^2-1\right)}{(2-\rho^2)^2} + \frac{7\rho^2-40}{4(4-\rho^2)^2} + \frac{4m^2}{(4-\rho^2)}. \tag{5.7}$$

We remark that $\chi(\rho) = O(1)$ as $\gamma \to \infty$, and this function is analytic at $\rho=0$ $(x=0)$, but is not analytic at $\rho=\sqrt{2}$ $(x=1)$.

Our approximants are again the parabolic cylinder functions $U\left(-\frac{1}{2}a, \rho\sqrt{2\gamma}\right)$ and $\bar{U}\left(-\frac{1}{2}a, \rho\sqrt{2\gamma}\right)$ (c.f. (4.12) and (4.13)). In this form they are solutions of

$$\frac{d^2W}{d\rho^2} = \left[\gamma^2\rho^2 - \gamma a\right] W. \tag{5.8}$$

On comparing this equation with (5.5) we note the extra "large" term $\gamma\zeta\phi(\rho)$. On account of this discrepancy we perturb the independent variable, thus taking as approximants

$$U_1 = \left\{1+\gamma^{-1}\Phi'(\rho)\right\}^{-1/2} U\left(-\tfrac{1}{2}a, \hat{\rho}\sqrt{2\gamma}\right), \tag{5.9}$$

and

$$U_2 = \left\{1+\gamma^{-1}\Phi'(\rho)\right\}^{-1/2} \bar{U}\left(-\tfrac{1}{2}a, \hat{\rho}\sqrt{2\gamma}\right), \tag{5.10}$$

where

$$\hat{\rho} = \rho + \gamma^{-1}\Phi(\rho), \tag{5.11}$$

in which

$$\Phi(\rho) = \frac{1}{2\rho}\int_0^\rho \phi(v)\,dv = \frac{a\ln\left(1-\frac{1}{4}\rho^2\right)}{4\rho}. \tag{5.12}$$

In [10] it is shown that U_j satisfy the differential equation

$$\frac{d^2U}{d\rho^2} = \left\{\gamma^2\rho^2 - \gamma a + \gamma\rho\phi(\rho) + g(\gamma,\rho)\right\}U, \tag{5.13}$$

where $g(\gamma,\rho) = O(1)$ as $\gamma\to\infty$, uniformly for $\rho\in\left[0,\sqrt{2}-\delta\right]$. Thus (5.13) is the appropriate comparison equation to (5.5).

Following [10] we then define

$$\hat{w}_j(\gamma,\rho) = U_j(\gamma,\rho) + \hat{\varepsilon}_j(\gamma,\rho) \quad (j=1,2), \tag{5.14}$$

as exact solutions of (5.5). Explicit error bounds are furnished in [10], and from these it follows that

$$\hat{\varepsilon}_1(\gamma,\rho) = O\left(\gamma^{-1}\ln(\gamma)\right)\mathrm{env}U\left(-\tfrac{1}{2}a, \hat{\rho}\sqrt{2\gamma}\right), \tag{5.15}$$

uniformly for $0\leqslant x\leqslant 1-\delta_0$, and similarly for $\hat{\varepsilon}_2(\gamma,\rho)$.

Let us assume that $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ (and hence $m+n$) is even. Similarly to (4.25) we write

$$\begin{aligned}\mathrm{Ps}_n^m\left(x,\gamma^2\right) &= \rho^{1/2}x^{-1/2}\left(1-x^2\right)^{-1/4} \\ &\quad\times\left[\hat{d}_n^m(\gamma)\,\hat{w}_1(\gamma,\rho) + \hat{e}_n^m(\gamma)\left\{\hat{w}_2(\gamma,\rho) - \hat{w}_4(\gamma,\rho)\right\}\right],\end{aligned} \tag{5.16}$$

where $\hat{w}_4(\gamma,\rho)$ is the solution (involving $\bar{U}$) given by eq. (110) of [10]. By matching at $x=\rho=0$ we find

$$\hat{d}_n^m(\gamma) = \frac{Ps_n^m\left(0,\gamma^2\right)}{\hat{w}_1(\gamma,0)}. \tag{5.17}$$

Analogously to the proof of (4.36) it can be shown that

$$\hat{e}_n^m(\gamma)\left\{\hat{w}_2(\gamma,\rho) - \hat{w}_4(\gamma,\rho)\right\} = o(1)\,\hat{A}\,\mathrm{env}U\left(-\tfrac{1}{2}a, \hat{\rho}\sqrt{2\gamma}\right), \tag{5.18}$$

where $o(1)$ is exponentially small for $0 \leqslant x \leqslant 1-\delta_0$ as $\gamma \to \infty$. Consequently, we arrive at our desired result

$$\begin{aligned} \mathrm{Ps}_n^m\left(x,\gamma^2\right) = {} & \frac{\mathrm{Ps}_n^m\left(0,\gamma^2\right)}{U\left(-\frac{1}{2}a,0\right)}\left(\frac{\rho}{x}\right)^{1/2}\left(1-x^2\right)^{-1/4} \\ & \times\left[U\left(-\tfrac{1}{2}a,\hat{\rho}\sqrt{2\gamma}\right)+O\left(\gamma^{-1}\ln(\gamma)\right)\operatorname{env}U\left(-\tfrac{1}{2}a,\hat{\rho}\sqrt{2\gamma}\right)\right], \end{aligned} \tag{5.19}$$

as $\gamma \to \infty$, uniformly for $0 \leqslant x \leqslant 1-\delta_0$.

For the case $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ being odd we likewise obtain, under the same conditions,

$$\begin{aligned} \mathrm{Ps}_n^m\left(x,\gamma^2\right) = {} & \frac{\mathrm{Ps}_n^{m\,\prime}\left(0,\gamma^2\right)}{U'\left(-\frac{1}{2}a,0\right)}\left(\frac{\rho}{2\gamma x}\right)^{1/2}\left(1-x^2\right)^{-1/4} \\ & \times\left[U\left(-\tfrac{1}{2}a,\hat{\rho}\sqrt{2\gamma}\right)+O\left(\gamma^{-1}\ln(\gamma)\right)\operatorname{env}U\left(-\tfrac{1}{2}a,\hat{\rho}\sqrt{2\gamma}\right)\right]. \end{aligned} \tag{5.20}$$

6. Summary

For reference we collect the principal results of the paper. All results are uniformly valid for $\gamma \to \infty$, m and n integers, m bounded, and n satisfying $0 \leqslant m \leqslant n \leqslant 2\pi^{-1}\gamma(1-\delta)$ where $\delta \in (0,1)$ is fixed.

We define $\sigma = \sqrt{1+\gamma^{-2}\lambda_n^m(\gamma^2)}$ and assume $0 \leqslant \sigma \leqslant \sigma_0 < 1$ for an arbitrary fixed positive σ_0. We further define variables $\xi = \xi(x)$ and $\zeta = \zeta(x)$ by

$$\xi = \int_1^x \left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2} dt, \tag{6.1}$$

and

$$\int_\alpha^\zeta \left|\tau^2-\alpha^2\right|^{1/2} d\tau = \int_\sigma^x \left(\frac{\left|t^2-\sigma^2\right|}{1-t^2}\right)^{1/2} dt, \tag{6.2}$$

where

$$\alpha = 2\left\{\frac{1}{\pi}\int_0^\sigma \left(\frac{\sigma^2-t^2}{1-t^2}\right)^{1/2} dt\right\}^{1/2}. \tag{6.3}$$

Then, using the definition above for σ, a uniform asymptotic relationship between $\lambda_n^m\left(\gamma^2\right)$ and the parameters m, n and γ is given implicitly by the relation

$$\gamma\int_0^\sigma \left(\frac{\sigma^2-t^2}{1-t^2}\right)^{1/2} dt = \frac{1}{2}\left(n-m+\frac{1}{2}\right)\pi + O\left(\frac{1}{\gamma}\right). \tag{6.4}$$

The following approximation holds for the radial PSWF

$$\begin{aligned} Ps_n^m\left(x,\gamma^2\right) = {} & \left\{\frac{(n+m)!\,q_n^m(\gamma)}{(2n+1)(n-m)!\,p_n^m(\gamma)}\right\}^{1/2}\left\{\left(x^2-1\right)\left(x^2-\sigma^2\right)\right\}^{-1/4} \\ & \times \xi^{1/2}\left[J_m(\gamma\xi)+O\left(\gamma^{-1}\right)\operatorname{env}J_m(\gamma\xi)\right], \end{aligned} \tag{6.5}$$

this being uniformly valid for $1 < x < \infty$. Here J_m is the Bessel function of the first kind, $\mathrm{env}J_m$ is defined by [27, §2.8(iv)], and the constants $p_n^m(\gamma)$ and $q_n^m(\gamma)$ are given by

$$p_n^m(\gamma) = \left[\int_0^{1-\delta_0} \left\{\frac{\alpha^2-\zeta^2}{(\sigma^2-x^2)(1-x^2)}\right\}^{1/2} U^2\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) dx \right. \\ \left. + q_n^m(\gamma) \int_{1-\delta_0}^1 \left\{\frac{1}{(1-x^2)(x^2-\sigma^2)}\right\}^{1/2} |\xi| I_m^2(\gamma|\xi|)\, dx\right], \tag{6.6}$$

and

$$q_n^m(\gamma) = \left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/2} \left(\frac{\pi^2}{2\gamma}\right)^{1/2} \exp\left\{-2\gamma\int_\sigma^1 \left(\frac{t^2-\sigma^2}{1-t^2}\right)^{1/2} dt\right\}. \tag{6.7}$$

In (6.6) $\delta_0 \in (0, 1-\sigma_0)$ is arbitrarily chosen, I_m is the modified Bessel function of the first kind, and U is the parabolic cylinder function (see [26, §5]).

In terms of the modified Bessel function, we have for the angular PSWF

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right) = \left\{\frac{(n+m)!\,q_n^m(\gamma)}{(2n+1)(n-m)!\,p_n^m(\gamma)}\right\}^{1/2} \left\{\left(1-x^2\right)\left(x^2-\sigma^2\right)\right\}^{-1/4} \\ \times |\xi|^{1/2} I_m(\gamma|\xi|)\left\{1+O\left(\gamma^{-1}\right)\right\}, \tag{6.8}$$

uniformly for $1-\delta_0 \leqslant x < 1$.

Finally, in terms of the parabolic cylinder function, the asymptotic approximation

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right) = \left\{\frac{(n+m)!}{(2n+1)(n-m)!\,p_n^m(\gamma)}\right\}^{1/2} \left\{\frac{\alpha^2-\zeta^2}{(\sigma^2-x^2)(1-x^2)}\right\}^{1/4} \\ \times \left\{U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) + O\left(\gamma^{-2/3}\ln(\gamma)\right) \mathrm{env}U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right)\right\}, \tag{6.9}$$

holds uniformly for $0 \leqslant x \leqslant 1-\delta_0$, where $\mathrm{env}U$ is defined by [27, eq. 14.15.23].

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