

ON AN EXTENSION OF EXTENDED BETA AND HYPERGEOMETRIC FUNCTIONS

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Abstract. Motivated mainly by certain interesting recent extensions of the Gamma, Beta and hypergeometric functions, we introduce here new extensions of the Beta function, hypergeometric and confluent hypergeometric functions. We systematically investigate several properties of each of these extended functions, such as their various integral representations, Mellin transforms, derivatives, transformations, summation formulas, generating function and asymptotics. Relevant connections of certain special cases of the main results presented herewith are also pointed out.

1. Introduction

Extensions of a number of well-known special functions have been investigated recently by several authors (see [1, 2, 3, 4, 7, 8, 9, 11, 13, 15]). In 1994, Chaudhry and Zubair [4] introduced the following extensions of the incomplete Gamma functions in the form

$$\gamma(\alpha, x; p) := \int_0^x t^{\alpha-1} \exp\left(-t - \frac{p}{t}\right) dt \quad (\Re(p) > 0; p = 0, \Re(\alpha) > 0) \quad (1)$$

and

$$\Gamma(\alpha, x; p) := \int_x^\infty t^{\alpha-1} \exp\left(-t - \frac{p}{t}\right) dt \quad (\Re(p) \geq 0) \quad (2)$$

with $|\arg x| < \pi$. These functions satisfy the following decomposition formula:

$$\begin{aligned} \gamma(\alpha, x; p) + \Gamma(\alpha, x; p) &\equiv \Gamma_p(x) = \int_0^\infty t^{\alpha-1} \exp\left(-t - \frac{p}{t}\right) dt \\ &= 2p^{\alpha/2} K_\alpha(2\sqrt{p}) \quad (\Re(p) > 0), \end{aligned} \quad (3)$$

where, here and in the following, $K_\nu(x)$ denotes the modified Bessel function. In 1997, Chaudhry *et al.* [1, Eq. (1.7)] presented the following extension of the Beta function given by

$$\begin{aligned} B(x, y; p) &:= \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(-\frac{p}{t(1-t)}\right) dt \\ &(\Re(p) > 0; p = 0, \Re(x) > 0, \Re(y) > 0) \end{aligned} \quad (4)$$

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and showed that this extension has certain connections with the Macdonald, error and Whittaker functions.

More recently, Chaudhry *et al.* [2] used $B(x, y; p)$ to extend the hypergeometric and the confluent hypergeometric functions as follows:

$$F_p(a, b; c; z) := \sum_{n=0}^{\infty} (a)_n \frac{B(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!} \tag{5}$$

$$(p \geq 0, |z| < 1; \Re(c) > \Re(b) > 0)$$

and

$$\Phi_p(b; c; z) := \sum_{n=0}^{\infty} \frac{B(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!} \tag{6}$$

$$(p \geq 0; \Re(c) > \Re(b) > 0).$$

Among several interesting and potentially useful properties of the extended hypergeometric functions defined by (5) and (6), the following integral representations were also given by Chaudhry *et al.* [2, Eq. (3.2)] and [2, Eq. (3.6)]:

$$F_p(a, b; c; z) := \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \exp\left(-\frac{p}{t(1-t)}\right) dt \tag{7}$$

$$(|\arg(1-z)| < \pi; p > 0; p = 0 \text{ and } \Re(c) > \Re(b) > 0)$$

and

$$\Phi_p(b; c; z) := \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \exp\left(zt - \frac{p}{t(1-t)}\right) dt \tag{8}$$

$$(p > 0; p = 0 \text{ and } \Re(c) > \Re(b) > 0),$$

respectively. They investigated these functions and gave their various integral representations, beta distribution, certain properties including differentiation formulas, Mellin transform, transformation formulas, recurrence relations, summation formula, asymptotic formulas and certain interesting connections with some well-known special functions. It is clearly seen that in the case $p = 0$, (3), (4), (5) and (6) reduce to the usual Gamma function, Beta function, Gauss hypergeometric and confluent hypergeometric functions, respectively.

In 1997, Chaudhry and Zubair [5] considered the following extensions of the incomplete gamma functions (1) and (2) in the form

$$\gamma_V(\alpha, x; p) := \sqrt{\frac{2p}{\pi}} \int_0^x t^{\alpha-\frac{3}{2}} \exp(-t) K_{V+\frac{1}{2}}\left(\frac{p}{t}\right) dt \tag{9}$$

and

$$\Gamma_V(\alpha, x; p) := \sqrt{\frac{2p}{\pi}} \int_x^{\infty} t^{\alpha-\frac{3}{2}} \exp(-t) K_{V+\frac{1}{2}}\left(\frac{p}{t}\right) dt \tag{10}$$

$$(|\arg x| < \pi, \Re(p) > 0, -\infty < \alpha < \infty).$$

respectively. The functions (9) and (10) satisfy the following decomposition formula:

$$\begin{aligned} & \gamma_v(\alpha, x; p) + \Gamma_v(\alpha, x; p) \\ &= 2^{\alpha-2} \pi^{-1} \sqrt{p} G_{4,0}^{0,4} \left(\frac{p^2}{16} \left| \frac{1}{2} \left(v + \frac{1}{2} \right), -\frac{1}{2} \left(v + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right) \quad (11) \\ & \quad (\Re(p) > 0, -\infty < \alpha < \infty), \end{aligned}$$

where G is the Meijer G -function. From the fact that [14, Eq. (10.39.2)]

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad (12)$$

it is easily seen that (9) and (10) reduce to the generalized incomplete gamma functions (1) and (2) when $v = 0$.

Motivated essentially by the demonstrated potential for applications of these generalized incomplete gamma functions in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [5, 6] and the references cited therein), we introduce here another interesting extension of the extended Beta function $B(x, y; p)$ given by

$$B_v(x, y; p) := \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} K_{v+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt \quad (\Re(p) > 0), \quad (13)$$

where \sqrt{p} takes its principal value. We note that this extension preserves the symmetry in the parameters x and y , and we have

$$B_v(x, y; p) = B_v(y, x; p).$$

REMARK 1. The special case of (13) when $v = 0$ is easily seen to reduce to the extended beta function (4) upon making use of (12).

Further, making use of the extended Beta function $B_v(x, y; p)$ in (13), we consider other extensions of the extended Gauss hypergeometric and the confluent hypergeometric functions. For each of these new extensions we obtain various integral representations, properties and Mellin transforms, together with differentiation, transformation, summation, generating function and asymptotic formulas. Relevant connections of certain special cases of the main results presented herewith are also pointed out.

2. Integral representations of $B_v(x, y; p)$

In this section, we obtain several integral representations of $B_v(x, y; p)$ for integer values of v and consider certain special cases.

THEOREM 1. *The following various integral representations for $B_\nu(x, y; p)$ in (13) hold true:*

$$B_\nu(x, y; p) = 2\sqrt{\frac{2p}{\pi}} \int_0^{\frac{\pi}{2}} \cos^{2(x-1)} \theta \sin^{2(y-1)} \theta K_{\nu+\frac{1}{2}}(p \sec^2 \theta \csc^2 \theta) d\theta, \quad (14)$$

$$B_\nu(x, y; p) = \sqrt{\frac{2p}{\pi}} \int_0^\infty \frac{u^{x-\frac{3}{2}}}{(1+u)^{x+y-1}} K_{\nu+\frac{1}{2}}\left(p\left(2+u+\frac{1}{u}\right)\right) du \quad (15)$$

and

$$B_\nu(x, y; p) = 2^{2-x-y} \sqrt{\frac{2p}{\pi}} \int_{-1}^1 (1+u)^{x-\frac{3}{2}} (1-u)^{y-\frac{3}{2}} K_{\nu+\frac{1}{2}}\left(\frac{4p}{1-u^2}\right) du \quad (16)$$

$$(\Re(p) > 0).$$

Proof. Equations (14), (15) and (16) can be obtained by employing the transformations $t = \cos^2 \theta$, $t = u/(1+u)$ and $t = (1+u)/2$ in (13), respectively. \square

REMARK 2. Clearly, when $\nu = 0$, the results in Theorem 1 reduce to the corresponding integrals in [1].

THEOREM 2. *The following representation for integer values of $\nu = n \in \mathbb{N}$ in (13) holds true:*

$$B_n(x, y; p) = \sum_{m=0}^n \frac{(2p)^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} B(x+m, y+m; p) \quad (17)$$

$$(n \in \mathbb{N} := \{1, 2, 3 \dots\} \text{ and } \Re(p) > 0).$$

Proof. Using the fact (see, for example, [14, Eq. (10.49.12)])

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{m=0}^n \frac{(2z)^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \quad (n \in \mathbb{N}) \quad (18)$$

in (13) and applying the definition (4), we obtain the desired representation (17). \square

REMARK 3. The Meijer G -function [9, p. 232, Eq. (5.124)], the Whittaker function $W_{\alpha,\beta}(z)$ [14, p. 334], and the confluent function $U(a, b, z)$ [14, p. 325] are expressible in terms of the extended beta function $B(x, y; p)$ for $\Re(p) > 0$ as follows:

$$B(x, y; p) = \sqrt{\pi} 2^{1-x-y} G_{2,3}^{3,0} \left(4p \left| \begin{matrix} \frac{x+y}{2}, \frac{x+y+1}{2} \\ 0, x, y \end{matrix} \right. \right), \quad (19)$$

$$B(x, x; p) = \sqrt{\pi} 2^{-x} p^{\frac{x-1}{2}} e^{-2p} W_{-\frac{x}{2}, \frac{x}{2}}(4p) \quad (20)$$

and

$$B(x, x; p) = \sqrt{\pi} 2^{1-2x} e^{-4p} U\left(\frac{1}{2}, 1-x, 4p\right). \quad (21)$$

Now, applying the relationships (19)–(21) to (17), we can deduce certain interesting representations of the extended beta function in (13) for integer values of ν . These are given in Corollary 1 below without proof.

COROLLARY 1. *Each of the following representations for integer values of $\nu = n \in \mathbb{N}$ in (17) holds true:*

$$B_n(x, y; p) = \sqrt{\pi} 2^{1-x-y} \sum_{m=0}^n \frac{(8p)^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} G_{2,3}^{3,0} \left(4p \left| \begin{matrix} \frac{x+y+2m}{2}, \frac{x+y+2m+1}{2} \\ 0, x+m, y+m \end{matrix} \right. \right), \quad (22)$$

$$B_n(x, x; p) = \sqrt{\pi} 2^{-x} p^{\frac{x-1}{2}} e^{-2p} \sum_{m=0}^n \frac{(4p\sqrt{p})^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} W_{-\frac{x+m}{2}, \frac{x+m}{2}}(4p) \quad (23)$$

and

$$B_n(x, x; p) = \sqrt{\pi} 2^{1-2x} e^{-4p} \sum_{m=0}^n \frac{(8p)^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} U \left(\frac{1}{2}, 1-x-m, 4p \right). \quad (24)$$

3. Some properties of $B_\nu(x, y; p)$

In this section, we obtain certain properties including functional relations and summation formulas for $B_\nu(x, y; p)$ as follows:

THEOREM 3. *The following functional relation for $B_\nu(x, y; p)$ in (13) holds true:*

$$B_\nu(x+1, y; p) + B_\nu(x, y+1; p) = B_\nu(x, y; p). \quad (25)$$

Proof. The left-hand side of (25) becomes

$$B_\nu(x+1, y; p) + B_\nu(x, y+1; p) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} [t+(1-t)] K_{\nu+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt,$$

which, after a simple algebraic manipulation and application of (13), establishes the desired result. \square

THEOREM 4. *The following summation formula for $B_\nu(x, y; p)$ in (13) holds true:*

$$B_\nu(x, 1-y; p) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_\nu(x+n, 1; p) \quad (\Re(p) > 0). \quad (26)$$

Proof. Application of the binomial theorem

$$(1-t)^{-y} = \sum_{n=0}^{\infty} (y)_n \frac{t^n}{n!} \quad (|t| < 1)$$

to the definition (13) of $B_v(x, y; p)$ yields

$$B_v(x, 1 - y; p) = \sqrt{\frac{2p}{\pi}} \int_0^1 \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^{x+n-\frac{3}{2}} (1-t)^{-\frac{1}{2}} K_{v+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt.$$

Interchange the order of integration and summation in the last expression and use of (13) proves the desired identity. \square

THEOREM 5. *The following infinite summation formula for $B_v(x, y; p)$ in (13) holds true:*

$$B_v(x, y; p) = \sum_{n=0}^{\infty} B_v(x+n, y+1; p) \quad (\Re(p) > 0). \tag{27}$$

Proof. Replacing $(1-t)^{y-1}$ in (13) by its series representation

$$(1-t)^{y-1} = (1-t)^y \sum_{n=0}^{\infty} t^n \quad (|t| < 1),$$

we obtain

$$B_v(x, y; p) = \sqrt{\frac{2p}{\pi}} \int_0^1 (1-t)^{y-\frac{1}{2}} \sum_{n=0}^{\infty} t^{x+n-\frac{3}{2}} K_{v+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt.$$

Interchange the order of integration and summation in the last expression and use of (13) proves the desired identity. \square

REMARK 4. In the special case $v = 0$, (25)–(27) reduce to the corresponding results given in [1].

4. Mellin transforms of $B_v(x, y; p)$

The Mellin transform of a suitably integrable function $f(t)$ with index s is defined, as usual, by (see [12])

$$\mathcal{M}\{f(\tau) : \tau \rightarrow s\} := \int_0^{\infty} \tau^{s-1} f(\tau) d\tau, \tag{28}$$

whenever the improper integral in (28) exists.

THEOREM 6. *The following Mellin transformation formula for $B_v(x, y; p)$ in (13) holds true:*

$$\mathcal{M}\{B_v(x, y; p) : p \rightarrow s\} = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-v}{2}\right) \Gamma\left(\frac{s+v+1}{2}\right) B(x+s, y+s) \tag{29}$$

$$(\Re(x+s) > 0, \Re(y+s) > 0, \Re(s-v) > 0, \Re(s+v) > -1),$$

where $B(x, y)$ denotes the classical Beta function, provided that each member of (29) exists.

Proof. Using the Definition (28) of the Mellin transform, we find from (13) that

$$\begin{aligned} & \mathcal{M} \{B_V(x, y; p) : p \rightarrow s\} \\ &= \int_0^\infty p^{s-1} \left\{ \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} K_{V+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt \right\} dp \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty p^{s-\frac{1}{2}} \left\{ \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} K_{V+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt \right\} dp. \end{aligned} \quad (30)$$

Upon interchanging the order of integration in (30), which can easily be justified by absolute convergence of the integrals involved under the constraints stated in (29), we find

$$\begin{aligned} & \mathcal{M} \{B_V(x, y; p) : p \rightarrow s\} \\ &= \sqrt{\frac{2}{\pi}} \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} \left\{ \int_0^\infty p^{s-\frac{1}{2}} K_{V+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dp \right\} dt \\ &= \int_0^1 t^{x+s-1} (1-t)^{y+s-1} \left\{ \sqrt{\frac{2}{\pi}} \int_0^\infty w^{s-\frac{1}{2}} K_{V+\frac{1}{2}}(w) dw \right\} dt, \end{aligned} \quad (31)$$

where we have set $w = pt/(1-t)$ in the inner p -integral.

The inner integral may be evaluated with the help of [14, Eq. (10.43.19)]

$$\int_0^\infty t^{s-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(t) dt = 2^{s-\frac{3}{2}} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s+\alpha+1}{2}\right) \quad (32)$$

for $\Re(s-\alpha) > 0$, $\Re(s+\alpha) > -1$, and the integral with respect to t can then be evaluated in terms of the classical Beta function (given by (4) with $p=0$). This evidently completes our derivation of the Mellin transform formula (29). \square

5. The beta distribution of $B_V(x, y; p)$

As a statistical application of $B_V(x, y; p)$, we define the extended beta distribution by employing $B_V(x, y; p)$, where the parameters a and b satisfy $-\infty < a < \infty$, $-\infty < b < \infty$ and $p > 0$ as:

$$f(t) = \begin{cases} \frac{1}{B_V(a, b; p)} \sqrt{\frac{2p}{\pi}} t^{a-\frac{3}{2}} (1-t)^{b-\frac{3}{2}} K_{V+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) & (0 < t < 1), \\ 0, & \text{otherwise.} \end{cases} \quad (33)$$

If r is any real number, then we have the r th moment of X

$$E(X^r) = \frac{B_V(a+r, b; p)}{B_V(a, b; p)} \quad (34)$$

$$(p > 0, -\infty < a < \infty, -\infty < b < \infty).$$

The particular case of (34) when $r = 1$,

$$\mu = E(X) = \frac{B_v(a+1, b; p)}{B_v(a, b; p)}, \tag{35}$$

represents the mean of the distribution, and

$$\sigma^2 = E(X^2) - \{E(X)\}^2 = \frac{B_v(a, b; p)B_v(a+2, b; p) - B_v^2(a+1, b; p)}{B_v^2(a, b; p)} \tag{36}$$

is the variance of the distribution.

The moment generating function of the distribution is

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n) = \frac{1}{B_v(a, b; p)} \sum_{n=0}^{\infty} B_v(a+n, b; p) \frac{t^n}{n!}. \tag{37}$$

The cumulative distribution of (33) can be expressed as

$$F(x) = \frac{B_{v,x}(a, b; p)}{B_v(a, b; p)}, \tag{38}$$

where

$$B_{v,x}(a, b; p) = \sqrt{\frac{2p}{\pi}} \int_0^x t^{a-\frac{3}{2}} (1-t)^{b-\frac{3}{2}} K_{v+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt \tag{39}$$

$(p > 0, -\infty < a < \infty, -\infty < b < \infty)$

is the extended incomplete beta function.

6. Extension of extended hypergeometric functions

In this section, we extend the extended Gauss hypergeometric and confluent hypergeometric functions in (5) and (6) by making use of $B_v(x, y; p)$ in (13) as follows:

$$F_{p,v}(a, b; c; z) := \sum_{n=0}^{\infty} (a)_n \frac{B_v(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!} \tag{40}$$

$(p \geq 0; |z| < 1, \Re(c) > \Re(b) > 0)$

and

$$\Phi_{p,v}(b; c; z) := \sum_{n=0}^{\infty} \frac{B_v(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!} \tag{41}$$

$(p \geq 0; \Re(c) > \Re(b) > 0).$

REMARK 5. The special case $v = 0$ in (40) and (41) leads to the corresponding extensions given in (5) and (6) by Chaudhry *et al.* [2].

6.1. Integral representations of the extended hypergeometric functions

We obtain integral representations of the extended Gauss hypergeometric and confluent hypergeometric functions as follows:

THEOREM 7. *The following integral representations for the extended hypergeometric functions in (40) and (41) hold true:*

$$F_{p,\nu}(a, b; c; z) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(b, c-b)} \int_0^1 t^{b-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} (1-zt)^{-a} K_{\nu+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt$$

$$(|\arg(1-z)| < \pi; p > 0; \nu = 0 \text{ and } p = 0, \Re(c) > \Re(b) > 0)$$
(42)

and

$$\Phi_{p,\nu}(b; c; z) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(b, c-b)} \int_0^1 t^{b-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} \exp(zt) K_{\nu+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt$$

$$(p > 0; \nu = 0 \text{ and } p = 0, \Re(c) > \Re(b) > 0).$$
(43)

Proof. Substituting the definition of $B_\nu(x, y; p)$ in (13) into (40), we have

$$F_{p,\nu}(a, b; c; z) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(b, c-b)} \int_0^1 t^{b-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} K_{\nu+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) \sum_{n=0}^{\infty} (a)_n \frac{(zt)^n}{n!} dt.$$
(44)

Employing the binomial expansion

$$(1-zt)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{(zt)^n}{n!} \quad (|zt| < 1)$$

in (44), we obtain the integral in (42).

A similar argument can be used to establish the integral representation of the extended confluent hypergeometric function in (43). \square

REMARK 6. The special case $\nu = 0$ of the integrals in (42) and (43) leads to the corresponding integral representations given in Chaudhry *et al.* [2].

THEOREM 8. *The following representations of the extended hypergeometric functions for integer values $\nu = n$ in (42) and (43) hold true:*

$$F_{p,n}(a, b; c; z) = \sum_{m=0}^n \frac{(2p)^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \frac{B(b+m, c-b+m)}{B(b, c-b)} F_p(a, b+m; c+2m; z)$$
(45)

and

$$\Phi_{p,n}(b; c; z) = \sum_{m=0}^n \frac{(2p)^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \frac{B(b+m, c-b+m)}{B(b, c-b)} \Phi_p(b+m; c+2m; z),$$
(46)

where F_p and Φ_p denote the extended functions defined in (5) and (6).

Proof. Using the fact (18) in the integral representation (42) and applying the definition (7), we obtain the desired representation.

A similar argument can be used to establish the representation of the extended confluent hypergeometric function in (46). \square

6.2. Differentiation formulas

Differentiation formulas for the extended Gauss hypergeometric and confluent hypergeometric functions can be found by differentiating (40) and (41) with respect to z as follows:

THEOREM 9. *The following differentiation formulas for the extended hypergeometric functions in (40) and (41) hold true for $p > 0$ and non-negative integer n :*

$$\frac{d^n}{dz^n} \{F_{p,v}(a, b; c; z)\} = \frac{(a)_n (b)_n}{(c)_n} F_{p,v}(a + n, b + n; c + n; z) \tag{47}$$

and

$$\frac{d^n}{dz^n} \{\Phi_{p,v}(b; c; z)\} = \frac{(b)_n}{(c)_n} \Phi_{p,v}(b + n; c + n; z). \tag{48}$$

Proof. Differentiating (40) with respect to z , we have

$$\frac{d}{dz} \{F_{p,v}(a, b; c; z)\} = \sum_{n=1}^{\infty} (a)_n \frac{B_v(b + n, c - b; p)}{B(b, c - b)} \frac{z^{n-1}}{(n - 1)!}$$

which, upon replacing n by $n + 1$ and using the facts that

$$B(b, c - b) = \frac{c}{b} B(b + 1, c - b) \quad \text{and} \quad (\lambda)_{n+1} = \lambda (\lambda + 1)_n,$$

yields

$$\frac{d}{dz} \{F_{p,v}(a, b; c; z)\} = \frac{ab}{c} F_{p,v}(a + 1, b + 1; c + 1; z).$$

The restriction $\Re(c) > \Re(b) > 0$ on the above result may be removed by analytic continuation. Repeated application of this process gives the general form (47). A similar argument can be employed to establish (48) for the extended confluent hypergeometric function. \square

REMARK 7. The special case $v = 0$ of (47) and (48) leads to the corresponding results given in Chaudhry *et al.* [2].

6.3. Transformation and summation formulas

In this section, we obtain transformation and summation formulas for the extended Gauss hypergeometric and confluent hypergeometric functions as follows:

THEOREM 10. *The following transformation formulas for the extended hypergeometric functions hold true when $p > 0$:*

$$F_{p,\nu}(a, b; c; z) = (1-z)^{-a} F_{p,\nu}\left(a, c-b; c; -\frac{z}{1-z}\right) \quad (|\arg(1-z)| < \pi) \quad (49)$$

and

$$\Phi_{p,\nu}(b; c; z) = e^z \Phi_{p,\nu}(c-b; c; -z). \quad (50)$$

Proof. By writing

$$[1-z(1-t)]^{-a} = (1-z)^{-a} \left(1 + \frac{z}{1-z}t\right)^{-a}$$

and replacing t by $1-t$ in (42), we obtain

$$F_{p,\nu}(a, b; c; z) = \frac{(1-z)^{-a}}{B(b, c-b)} \int_0^1 t^{c-b-\frac{3}{2}} (1-t)^{b-\frac{3}{2}} \left(1 + \frac{z}{1-z}t\right)^{-a} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt,$$

which establishes (49).

Again, replacing t by $1-t$ in (43), we find

$$\Phi_{p,\nu}(b; c; z) = \sqrt{\frac{2p}{\pi}} \frac{e^z}{B(b, c-b)} \int_0^1 t^{c-b-\frac{3}{2}} (1-t)^{b-\frac{3}{2}} \exp(-zt) K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt, \quad (51)$$

which establishes (50). \square

REMARK 8. The special case $\nu = 0$ in (49) and (50) leads to the corresponding extensions of Gauss' summation formula and Kummer's first transformation formula, respectively, given in Chaudhry *et al.* [2].

THEOREM 11. *The following summation formula for the extended Gauss hypergeometric function holds true:*

$$F_{p,\nu}(a, b; c; 1) = \frac{B_\nu(b, c-a-b; p)}{B(b, c-b)} \quad (52)$$

$$(p > 0; \nu = 0 \text{ and } p = 0, \Re(c-a-b) > 0).$$

Proof. Setting $z = 1$ in (42) and using the definition (13), we readily obtain the summation formula (52). \square

REMARK 9. The formula (52) in the case $v = 0$ and $p = 0$ reduces to the well-known Gauss summation formula

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \frac{B(b, c-a-b)}{B(b, c-b)} \quad (\Re(c-a-b) > 0).$$

6.4. A generating function for $F_{p,v}(a, b; c; z)$

We have the following theorem:

THEOREM 12. *The following generating function for $F_{p,v}(a, b; c; z)$ holds true*

$$\sum_{n=0}^{\infty} (a)_n F_{p,v}(a+n, b; c; z) \frac{t^n}{n!} = (1-t)^{-a} F_{p,v}\left(a, b; c; \frac{z}{1-t}\right). \quad (53)$$

$(p \geq 0, |t| < 1).$

Proof. Let the left-hand side of (53) be denoted by S . Then, from the definition (40) we have

$$\begin{aligned} S &= \sum_{n=0}^{\infty} (a)_n \left\{ \sum_{k=0}^{\infty} (a+n)_k \frac{B_v(b+k, c-b; p)}{B(b, c-b)} \frac{z^k}{k!} \right\} \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} (a)_k \frac{B_v(b+k, c-b; p)}{B(b, c-b)} \left\{ \sum_{n=0}^{\infty} (a+k)_n \frac{t^n}{n!} \right\} \frac{z^k}{k!} \end{aligned}$$

upon reversal of the order of summation and use of the identity $(a)_n(a+n)_k = (a)_k(a+k)_n$.

Since by the binomial theorem

$$(1-t)^{-a-k} = \sum_{n=0}^{\infty} (a+k)_n \frac{t^n}{n!} \quad (|t| < 1),$$

identification of the series over k from (40) as $F_{p,v}(a, b; c; z/(1-t))$ then leads to the assertion in (53). \square

6.5. Asymptotic behaviour for large z

We present the asymptotic behaviour of $\Phi_{p,v}(b; c; z)$ for $|z| \rightarrow \infty$ in $\Re(z) > 0$ when it is supposed that $p > 0$. The major contribution to the integrand in (51) for large $|z|$ arises from the neighbourhood of $t = 0$, where the Bessel function may be approximated by its leading asymptotic behaviour

$$K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) \sim \sqrt{\frac{\pi}{2p}} t^{\frac{1}{2}} (1-t)^{\frac{1}{2}} \exp\left(-\frac{p}{t} - \frac{p}{1-t}\right) \quad (t \rightarrow 0).$$

Then, from (51), we obtain with $t = z\tau$

$$\begin{aligned} \Phi_{p,v}(b; c; z) &\simeq \frac{z^{b-c} e^{z-p}}{B(b, c-b)} \int_0^z \tau^{c-b-1} \exp\left(-\tau - \frac{p}{\tau}\right) d\tau \\ &= \frac{z^{b-c} e^{z-p}}{B(c, c-b)} \gamma(c-b, z; pz) \quad (|z| \rightarrow \infty, \Re(z) > 0, p > 0), \end{aligned} \tag{54}$$

where $\gamma(\alpha, x; p)$ is the extended incomplete gamma function defined in (1). We note that the leading term is independent of the parameter v and that the factor e^{-p} was omitted in the case $v = 0$ in [2, Eq. (8.7)]. In the limit $p \rightarrow 0$, we recover the well-known asymptotic behaviour of the ordinary confluent hypergeometric function

$$\Phi(b; c; z) \sim z^{b-c} e^z \frac{\gamma(c-b, z)}{B(b, c-b)} \sim \frac{\Gamma(c)}{\Gamma(b)} z^{b-c} e^z \quad (|z| \rightarrow \infty, \Re(z) > 0).$$

For the function $F_{p,v}(a, b; c; z)$, we apply a slight modification of the analysis described in [16, §16.3] to the integral (42), where for simplicity in presentation we shall suppose the parameters a, b, c and v to be real. With $\xi = -z$, we write

$$F_{p,v}(a, b; c; z) = \sqrt{\frac{2p}{\pi}} \frac{(-z)^{-a}}{B(b, c-b)} \int_0^1 t^{b-a-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} \left(1 + \frac{1}{\xi t}\right)^{-a} K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt,$$

where we employ the series expansion for positive integer n

$$\left(1 + \frac{1}{\xi t}\right)^{-a} = \sum_{k=0}^{n-1} \frac{(-1)^k (a)_k}{(\xi t)^k k!} + R_n(t, \xi)$$

with

$$R_n(t, \xi) = \frac{(-1)^n (a)_n}{(n-1)!} \left(1 + \frac{1}{\xi t}\right)^{-a} \int_0^{1/(\xi t)} u^{n-1} (1+u)^{a-1} du.$$

The finite series yields the contribution to $F_{p,v}(a, b; c; z)$ given by

$$\begin{aligned} &\sqrt{\frac{2p}{\pi}} \frac{(-z)^{-a}}{B(b, c-b)} \sum_{k=0}^{n-1} \frac{(a)_k}{k!} z^{-k} \int_0^1 t^{b-a-k-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt \\ &= \frac{(-z)^{-a}}{B(b, c-b)} \sum_{k=0}^{n-1} \frac{(a)_k}{k!} z^{-k} B_v(b-a-k, c-b; p). \end{aligned}$$

Now

$$\begin{aligned} |R_n(t, \xi)| &\leq \frac{|(a)_n|}{(n-1)!} \left| \left(1 + \frac{1}{\xi t}\right)^{-a} \int_0^{1/(t|z|)} u^{n-1} (1+u)^{|a|} du \right. \\ &\leq \frac{|(a)_n|}{(n-1)!} \frac{(1+t^{-1})^{|a|}}{(|z|t)^n (\sin \varepsilon)^{|a|}} \int_0^1 w^{n-1} \left(1 + \frac{w}{\xi t}\right)^{|a|} dw \\ &\leq \frac{|(a)_n|}{n!} \frac{(1+t^{-1})^{2|a|}}{(|z|t)^n (\sin \varepsilon)^{|a|}}, \end{aligned}$$

where, with $|z| > 1$, $|\arg \xi| \leq \pi - \varepsilon$, we have used the bounds

$$1 \leq |1 + (\xi t)^{-1}| \leq 1 + t^{-1} \quad (\Re(\xi) \geq 0), \quad |1 + (\xi t)^{-1}| \geq \sin \varepsilon \quad (\Re(\xi) \leq 0).$$

It then follows that, when $|z| > 1$,

$$\begin{aligned} & \left| \int_0^1 t^{b-a-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} R_n(t, \xi) K_{\nu+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt \right| \\ & \leq \frac{|(a)_n|}{n! |z|^n (\sin \varepsilon)^{|a|}} \int_0^1 t^{b-a-n-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} (1+t^{-1})^{2|a|} K_{\nu+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt = O(z^{-n}), \end{aligned}$$

since the integral is convergent when $p > 0$ and is independent of z . The constant implied in the O -symbol becomes infinite as $\varepsilon \rightarrow 0$.

Hence, we obtain the following theorem:

THEOREM 13. *For $n \in \mathbb{N}$ and $p > 0$, we have the asymptotic expansion*

$$F_{p,\nu}(a, b; c; z) = \frac{(-z)^{-a}}{B(b, c-b)} \sum_{k=0}^{n-1} \frac{(a)_k}{k!} z^{-k} B_\nu(b-a-k, c-b; p) + O(z^{-n}) \quad (55)$$

for $|z| \rightarrow \infty$ in $|\arg(-z)| < \pi$.

7. Mellin transforms representations

In this section, we obtain Mellin transform representations for the extended Gauss hypergeometric and confluent hypergeometric functions as follows:

THEOREM 14. *The following Mellin transformation formulas for the extended hypergeometric functions in (40) and (41) hold true:*

$$\begin{aligned} & \mathcal{M} \{ F_{p,\nu}(a, b; c; z) : p \rightarrow s \} \\ & = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) \frac{B(b+s, c+s-b)}{B(b, c-b)} F(a, b+s; c+2s; z) \quad (56) \end{aligned}$$

and

$$\begin{aligned} & \mathcal{M} \{ \Phi_{p,\nu}(b; c; z) : p \rightarrow s \} \\ & = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) \frac{B(b+s, c+s-b)}{B(b, c-b)} \Phi(b+s; c+2s; z) \quad (57) \end{aligned}$$

$$(\Re(s-\nu) > 0, \Re(s+\nu) > -1),$$

provided that each member of (56) and (57) exists.

Proof. Using the definition (28) of the Mellin transform, we find from (42) that

$$\begin{aligned} & \mathcal{M} \left\{ F_{p,v}(a, b; c; z) : p \rightarrow s \right\} \\ &= \int_0^\infty p^{s-1} \left\{ \sqrt{\frac{2p}{\pi}} \frac{1}{B(b, c-b)} \int_0^1 t^{b-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} (1-zt)^{-a} K_{v+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt \right\} dp \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{B(b, c-b)} \int_0^\infty p^{s-\frac{1}{2}} \left\{ \int_0^1 t^{b-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} (1-zt)^{-a} K_{v+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt \right\} dp. \end{aligned}$$

Then, upon interchanging the order of integration by absolute convergence of the integrals involved, we obtain

$$\begin{aligned} & \mathcal{M} \left\{ F_{p,v}(a, b; c; z) : p \rightarrow s \right\} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{B(b, c-b)} \int_0^1 t^{b-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} (1-zt)^{-a} \left\{ \int_0^\infty p^{s-\frac{1}{2}} K_{v+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dp \right\} dt \\ &= \int_0^1 t^{b+s-1} (1-t)^{c-b+s-1} (1-zt)^{-a} \left\{ \sqrt{\frac{2}{\pi}} \int_0^\infty w^{s-\frac{1}{2}} K_{v+\frac{1}{2}}(w) dw \right\} dt, \end{aligned}$$

where we have set $w = pt/(1-t)$ in the inner p -integral. Then, upon use of (32) to evaluate the inner integral followed by (7) (with $p = 0$), we finally obtain (56) subject to the conditions stated. This completes our derivation of the Mellin transform Formula asserted by Theorem 12.

A similar argument can be employed to establish (57) for the extended confluent hypergeometric function. \square

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