

ANALYTICAL, ASYMPTOTIC AND INTEGRAL REPRESENTATIONS FOR A DOUBLE SUM

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Abstract. We give an analytical representation for the double sum

$$\sum_{1 \leq i < j \leq n} \frac{1}{[(n+i)(n+j)]^k}$$

in terms of the polygamma functions, where k is any given positive integer. Based on this result, we present an asymptotic formula as $n \rightarrow \infty$ and an integral representation for this sum.

1. Introduction

Izán Pérez, in a private communication to the first author, formulated the following limit:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq i < j \leq n} \frac{1}{\sqrt{(n+i)(n+j)}} = 6 - 4\sqrt{2}. \quad (1.1)$$

We give here a generalization of this limit by considering the double sum

$$S_r(n) = \sum_{1 \leq i < j \leq n} \frac{1}{[(n+i)(n+j)]^r}, \quad (1.2)$$

where in this section the index $r > 0$ is an arbitrary real quantity, but will subsequently be restricted to be a positive integer.

It is easy to see that

$$n^{2r-2} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{[(n+i)(n+j)]^r} = \left(n^{r-1} \sum_{i=1}^n \frac{1}{(n+i)^r} \right)^2 = \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{\left(1 + \frac{i}{n}\right)^r} \right)^2.$$

By the definition of a definite integral, we therefore have

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{2r-2} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{[(n+i)(n+j)]^r} &= \left(\int_0^1 \frac{1}{(1+x)^r} dx \right)^2 \\ &= \begin{cases} \left(\frac{2^{1-r} - 1}{1-r} \right)^2 & r \neq 1 \\ (\ln 2)^2 & r = 1. \end{cases} \end{aligned} \quad (1.3)$$

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In a similar manner, we have

$$n^{2r-1} \sum_{i=1}^n \frac{1}{(n+i)^{2r}} = \frac{1}{n} \sum_{i=1}^n \frac{1}{\left(1 + \frac{i}{n}\right)^{2r}} \rightarrow \begin{cases} \frac{2^{1-2r} - 1}{1 - 2r} & r \neq 1 \\ \ln 2 & r = 1, \end{cases}$$

as $n \rightarrow \infty$, which implies that

$$\lim_{n \rightarrow \infty} n^{2r-2} \sum_{i=1}^n \frac{1}{(n+i)^{2r}} = 0. \quad (1.4)$$

Let

$$a_{ij} = \frac{1}{[(n+i)(n+j)]^r} \quad (i, j = 1, 2, \dots, n),$$

where $r > 0$. Then it follows that

$$S_r(n) = \sum_{1 \leq i < j \leq n} a_{ij} = \frac{1}{2} \left\{ \sum_{i=1}^n \sum_{j=1}^n a_{ij} - \sum_{j=1}^n a_{jj} \right\}. \quad (1.5)$$

Noting that (1.4) holds, we have

$$\lim_{n \rightarrow \infty} n^{2r-2} \sum_{j=1}^n a_{jj} = 0.$$

Then from (1.3) and (1.5) we finally obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{2r-2} S_r(n) &= \frac{1}{2} \lim_{n \rightarrow \infty} n^{2r-2} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{[(n+i)(n+j)]^r} \\ &= \begin{cases} \frac{1}{2} \left(\frac{2^{1-r} - 1}{1 - r} \right)^2 & r \neq 1, \\ \frac{1}{2} (\ln 2)^2 & r = 1, \end{cases} \end{aligned} \quad (1.6)$$

which generalizes the formula (1.1). The choice $r = 1/2$ in (1.6) yields (1.1).

Now let $r = k$, where k is any given positive integer. In this paper, we establish an analytical representation for the sum $S_k(n)$ defined in (1.2) in terms of the polygamma functions. The polygamma functions $\psi^{(n)}(x)$ for $n = 1, 2, \dots$ are defined by

$$\psi^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}} \ln \Gamma(x) = \frac{d^n}{dx^n} \psi(x),$$

where $\Gamma(x)$ is the gamma function, and $\psi(x)$ (also known as the digamma function) is defined by

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) \quad \text{and} \quad \psi^{(0)}(x) = \psi(x).$$

Based on this exact representation for $S_k(n)$, we present an asymptotic formula as $n \rightarrow \infty$ in Section 2 and an integral representation for this sum in Section 3.

2. An asymptotic formula as $n \rightarrow \infty$

We first show how the double sum $S_k(n)$ can be evaluated in terms of polygamma functions. We have the following theorem.

THEOREM 2.1. *For positive integer k , the sum $S_k(n)$ has the following analytical representation:*

$$S_k(n) = \frac{1}{2} \left\{ \left(\frac{\Psi^{(k-1)}(2n+1) - \Psi^{(k-1)}(n+1)}{(k-1)!} \right)^2 + \frac{\Psi^{(2k-1)}(2n+1) - \Psi^{(2k-1)}(n+1)}{(2k-1)!} \right\}. \quad (2.1)$$

Proof. The polygamma function has the expression (see [2, Eq. 1.2(54)] and [3, Eq. 1.3(54)])

$$\Psi^{(n)}(x+m) = \Psi^{(n)}(x) + (-1)^n n! \sum_{j=1}^m \frac{1}{(x+j-1)^{n+1}}, \quad m, n = 0, 1, 2, \dots \quad (2.2)$$

The choice $(x, m, n) = (n+1, n, k-1)$ in (2.2) yields

$$\sum_{j=1}^n \frac{1}{(n+j)^k} = (-1)^{k-1} \frac{\Psi^{(k-1)}(2n+1) - \Psi^{(k-1)}(n+1)}{(k-1)!}.$$

We then obtain

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{[(n+i)(n+j)]^k} &= \left(\sum_{j=1}^n \frac{1}{(n+j)^k} \right)^2 \\ &= \left(\frac{\Psi^{(k-1)}(2n+1) - \Psi^{(k-1)}(n+1)}{(k-1)!} \right)^2. \end{aligned} \quad (2.3)$$

The choice $(x, m, n) = (n+1, n, 2k-1)$ in (2.2) yields

$$\sum_{j=1}^n \frac{1}{(n+j)^{2k}} = -\frac{\Psi^{(2k-1)}(2n+1) - \Psi^{(2k-1)}(n+1)}{(2k-1)!}. \quad (2.4)$$

We then obtain from (2.3) and (2.4)

$$\begin{aligned} \sum_{1 \leq i < j \leq n} \frac{1}{[(n+i)(n+j)]^k} &= \frac{1}{2} \left\{ \sum_{i=1}^n \sum_{j=1}^n \frac{1}{[(n+i)(n+j)]^k} - \sum_{j=1}^n \frac{1}{(n+j)^{2k}} \right\} \\ &= \frac{1}{2} \left\{ \left(\frac{\Psi^{(k-1)}(2n+1) - \Psi^{(k-1)}(n+1)}{(k-1)!} \right)^2 + \frac{\Psi^{(2k-1)}(2n+1) - \Psi^{(2k-1)}(n+1)}{(2k-1)!} \right\}. \end{aligned}$$

This completes the proof.

We now give the asymptotic expansion of $S_k(n)$.

THEOREM 2.2. For $k = 1, 2, \dots$, the sum $S_k(n)$ has the asymptotic expansion

$$S_k(n) = \frac{1}{2n^{2k-2}} \left\{ C_0(k) + \frac{C_1(k)}{n} + \frac{C_2(k)}{n^2} + O(n^{-3}) \right\} \quad (2.5)$$

as $n \rightarrow \infty$, where the coefficients $C_s(k)$ ($s \leq 2$) when $k \geq 2$ are given by

$$C_0(k) = \left(\frac{1-2^{1-k}}{k-1} \right)^2, \quad C_1(k) = -\frac{(1-2^{-k})(1-2^{1-k})}{k-1} - \frac{1-2^{1-2k}}{2k-1},$$

$$C_2(k) = \left(\frac{1-2^{-k}}{2} \right)^2 + \frac{1-2^{-2k}}{2} + \frac{k(1-2^{-1-k})(1-2^{1-k})}{6(k-1)}. \quad (2.6)$$

When $k = 1$, we have

$$C_0(1) = (\ln 2)^2, \quad C_1(1) = -\frac{1}{2}(1 + \ln 2), \quad C_2(1) = \frac{1}{16}(7 + 2 \ln 2). \quad (2.7)$$

Proof. To demonstrate (2.5), we require the asymptotic formula [1, Eq. (6.4.11)]

$$(-1)^{k+1} \psi^{(k)}(x) \sim \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{s=1}^{\infty} B_{2s} \frac{(2s+k-1)!}{(2s)! x^{2s+k}} \quad (2.8)$$

for $x \rightarrow \infty$, where B_n are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi).$$

The expansion (2.8) shows that

$$\frac{\psi^{(k-1)}(n+1)}{(k-1)!} = \frac{(-1)^k}{n^{k-1}} \left\{ \frac{1}{k-1} - \frac{1}{2n} + \frac{k}{12n^2} + O(n^{-3}) \right\} \quad (n \rightarrow \infty)$$

upon using the fact that $B_2 = \frac{1}{6}$. Thus we find

$$\begin{aligned} \Psi_1(n, k) &:= \frac{\psi^{(k-1)}(2n+1) - \psi^{(k-1)}(n+1)}{(k-1)!} \\ &= \frac{(-1)^{k-1}}{n^{k-1}} \left\{ A_0(k) + \frac{A_1(k)}{n} + \frac{A_2(k)}{n^2} + O(n^{-3}) \right\} \end{aligned}$$

and

$$\begin{aligned} \Psi_2(n, k) &:= \frac{\psi^{(2k-1)}(2n+1) - \psi^{(2k-1)}(n+1)}{(2k-1)!} \\ &= \frac{-1}{n^{2k-1}} \left\{ A_0(2k) + \frac{A_1(2k)}{n} + O(n^{-2}) \right\} \end{aligned}$$

as $n \rightarrow \infty$, where

$$A_0(k) = \frac{1 - 2^{1-k}}{k-1}, \quad A_1(k) = -\frac{1}{2}(1 - 2^{-k}), \quad A_2(k) = \frac{1}{12}(1 - 2^{-1-k}).$$

It then follows that

$$\Psi_1^2(n, k) + \Psi_2(n, k) = \frac{1}{n^{2k-2}} \left\{ C_0(k) + \frac{C_1(k)}{n} + \frac{C_2(k)}{n^2} + O(n^{-3}) \right\} \quad (n \rightarrow \infty),$$

where

$$\begin{aligned} C_0(k) &= A_0(k)^2 = \left(\frac{1 - 2^{1-k}}{k-1} \right)^2, \\ C_1(k) &= 2A_0(k)A_1(k) - A_0(2k) = -\frac{(1 - 2^{-k})(1 - 2^{1-k})}{k-1} - \frac{1 - 2^{1-2k}}{2k-1}, \\ C_2(k) &= A_1(k)^2 + 2A_0(k)A_2(k) - A_1(2k) \\ &= \left(\frac{1 - 2^{-k}}{2} \right)^2 + \frac{1 - 2^{-2k}}{2} + \frac{k(1 - 2^{-1-k})(1 - 2^{1-k})}{6(k-1)}. \end{aligned}$$

Substitution of this expansion into (2.1) then yields (2.5).

In the case $k = 1$, we proceed in the same manner making use of the expansion [1, Eq. (6.4.11)]

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{s=1}^{\infty} \frac{B_{2s}}{2sx^{2s}} \quad (x \rightarrow \infty),$$

to find

$$\Psi_1(n, 1) = \ln 2 - \frac{1}{4n} + \frac{1}{16n^2} + O(n^{-3}), \quad \Psi_2(n, 1) = -\frac{1}{2n} + \frac{3}{8n^2} + O(n^{-3}).$$

Hence we obtain

$$S_1(n) = \frac{(\ln 2)^2}{2} - \frac{1 + \ln 2}{4n} + \frac{7 + 2 \ln 2}{32n^2} + O(n^{-3}) \quad (n \rightarrow \infty).$$

This completes the proof.

REMARK 2.1. *Noting that*

$$\lim_{k \rightarrow 1} \frac{1 - 2^{1-k}}{k-1} = \ln 2,$$

we see that the limiting values of the coefficients $C_s(k)$ in (2.6) as $k \rightarrow 1$ agree with those given in (2.7).

3. An integral representation of $S_k(n)$

In this section we obtain an integral representation for the sum $S_k(n)$. We have

THEOREM 3.1. *For $k = 2, 3, \dots$ we have the integral representation for $S_k(n)$ given by*

$$S_k(n) = \frac{1}{2} \left\{ I^2(n, k) + I(n, 2k) \right\}, \quad (3.1)$$

where

$$I(n, k) = \frac{1}{\Gamma(k)} \int_0^\infty \left\{ \frac{2^{-k}}{e^{t/2} - 1} - \frac{1}{e^t - 1} \right\} t^{k-1} e^{-nt} dt.$$

Proof. Using the representation [1, Eq. (6.4.1)] for positive integer n

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} dt \quad (x > 0),$$

we find

$$\psi^{(n)}(x+1) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{e^t - 1} dt \quad (3.2)$$

and

$$\psi^{(n)}(2x+1) = \frac{(-1)^{n+1}}{2^{n+1}} \int_0^\infty \frac{t^n e^{-xt}}{e^{t/2} - 1} dt. \quad (3.3)$$

We then obtain

$$\begin{aligned} \Psi_1^2(n, k) + \Psi_2(n, k) &= \left(\frac{1}{\Gamma(k)} \int_0^\infty \left\{ \frac{2^{-k}}{e^{t/2} - 1} - \frac{1}{e^t - 1} \right\} t^{k-1} e^{-nt} dt \right)^2 \\ &\quad + \frac{1}{\Gamma(2k)} \int_0^\infty \left\{ \frac{2^{-2k}}{e^{t/2} - 1} - \frac{1}{e^t - 1} \right\} t^{2k-1} e^{-nt} dt. \end{aligned} \quad (3.4)$$

Finally, substituting the expression (3.4) into (2.1) yields (2.5). This completes the proof.

The case $k = 1$ is covered by the following theorem.

THEOREM 3.2. *When $k = 1$, we have the representation*

$$S_1(n) = \frac{1}{2} \left[\left\{ \ln 2 - \frac{1}{2} \int_0^\infty \frac{e^{-nt}}{e^{t/2} + 1} dt \right\}^2 + \int_0^\infty \left\{ \frac{1}{4(e^{t/2} - 1)} - \frac{1}{e^t - 1} \right\} t e^{-nt} dt \right]. \quad (3.5)$$

Proof. It follows from (3.2) and (3.3) that

$$\psi'(2n+1) - \psi'(n+1) = \int_0^\infty \left[\frac{1}{4(e^{t/2} - 1)} - \frac{1}{e^t - 1} \right] t e^{-nt} dt. \tag{3.6}$$

Using the recurrence formula $\psi(x+1) = \psi(x) + 1/x$, the duplication formula [1, Eq. (6.3.8)]:

$$\psi(2x) = \frac{1}{2}\psi(x) + \frac{1}{2}\psi\left(x + \frac{1}{2}\right) + \ln 2$$

and the representation [1, Eq. (6.3.21)]:

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right) dt,$$

we find

$$\begin{aligned} \psi(2n+1) - \psi(n+1) &= \ln 2 + \frac{1}{2} \left\{ \psi\left(n + \frac{1}{2}\right) - \psi(n+1) \right\} \\ &= \ln 2 - \frac{1}{2} \int_0^\infty \frac{e^{-nt}}{e^{t/2} + 1} dt. \end{aligned} \tag{3.7}$$

We then obtain from (2.1), (3.6) and (3.7)

$$\begin{aligned} S_1(n) &= \frac{1}{2} \left\{ \left(\psi(2n+1) - \psi(n+1) \right)^2 + \psi'(2n+1) - \psi'(n+1) \right\} \\ &= \frac{1}{2} \left[\left\{ \ln 2 - \frac{1}{2} \int_0^\infty \frac{e^{-nt}}{e^{t/2} + 1} dt \right\}^2 + \int_0^\infty \left\{ \frac{1}{4(e^{t/2} - 1)} - \frac{1}{e^t - 1} \right\} t e^{-nt} dt \right]. \end{aligned}$$

This completes the proof.

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