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MEHLER–HEINE TYPE FORMULAS FOR CHARLIER AND MEIXNER POLYNOMIALS II. HIGHER ORDER TERMS

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Abstract. We derive Mehler–Heine type asymptotic expansions for Charlier and Meixner polynomials. These formulas provide good approximations for the polynomials in the neighborhood of $x = 0$, and determine the asymptotic limit of their zeros as the degree n goes to infinity.

1. Introduction

Suppose that $P_n(x)$ is a sequence of orthogonal polynomials and let $x_{k,n}$ denote the zeros of $P_n(x)$

$$P_n(x_{k,n}) = 0, \quad x_{1,n} < x_{2,n} < \cdots < x_{n,n}.$$

Two standard approximations describing the asymptotic behavior of the polynomials $P_n(x)$ as the degree n tends to infinity are Mehler–Heine type formulas (in a region around the smallest zero) and Plancherel-Rotach type formulas (in a region around the largest zero)

$$\underbrace{x_{1,n} < x_{2,n}}_{\text{Mehler-Heine}} < \cdots < \underbrace{x_{n-1,n} < x_{n,n}}_{\text{Plancherel-Rotach}}.$$

Mehler–Heine type formulas were introduced by Heinrich Heine in 1861 [3] and Gustav Mehler [5] in 1868 to analyze the asymptotic behavior of Legendre polynomials. See Watson's book [8, 5.71] for some historical remarks.

In [2], we studied Mehler–Heine type formulas for the Charlier and Meixner polynomials and obtained the following result (there are some minor differences in the formulas because we use monic polynomials in this article):

PROPOSITION 1. *Let*

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}$$

denote the Generalized Hypergeometric Function [7, Chapter 16] *and* $(u)_k$ *the* Pochhammer symbol *(or rising factorial)* [7, 5.2.4],

$$(u)_k = u(u+1)\cdots(u+k-1). \tag{1}$$

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1) If $C_n(x;a)$ denotes the monic Charlier polynomial *defined by* [4, 9.14.1]

$$C_n(x,a) = (-a)^n \; {}_2F_0\left({-n, -x \atop -} ; -\frac{1}{a} \right), \tag{2}$$

then, we have

$$\lim_{n\to\infty} \frac{(-1)^n}{\Gamma(n-x)} C_n(x,a) = \frac{e^a}{\Gamma(-x)}, \quad x \in \mathbb{C}, \tag{3}$$

where $\Gamma(z)$ is the Gamma function [7, Chapter 5].

2) If $M_n(x;\beta,c)$ denotes the monic Meixner polynomial *defined by* [4, 9.10.1]

$$M_n(x;\beta,c) = (\beta)_n \left(\frac{c}{c-1} \right)^n {}_2F_1\left({-n, -x \atop \beta} ; 1 - \frac{1}{c} \right), \quad \beta > 0, \tag{4}$$

then, for $c \in \mathbb{C} \setminus [1,\infty)$ we have

$$\lim_{n\to\infty} \frac{(-1)^n (1-c)^{n+x}}{\Gamma(n-x)} M_n(x;\beta,c) = \frac{(1-c)^{-\beta}}{\Gamma(-x)}, \quad x \in \mathbb{C}, \tag{5}$$

where all functions assume their principal values.

We presented these results at the Special Session on Special Functions and Their Applications, part of the Fall Eastern Sectional Meeting held at Dalhousie University, Halifax, Canada on October 18–19, 2014. Professor Robert Milson was in the audience and inquired about possible error terms of order n^{-1} in the formulas. The purpose of this paper is to answer his question, and extend the previous limits (3) and (5) to full asymptotic expansions.

2. Main results

The monic Charlier polynomials satisfy the orthogonality relation [4, 9.14.2]

$$\sum_{x=0}^{\infty} C_n(x;a) C_m(x;a) \frac{a^x}{x!} = n! a^n e^a \delta_{n,m}, \quad a > 0.$$

PROPOSITION 2. *We have*

$$C_n(x;a) = (-1)^n (-x)_n \; e^a \; {}_1F_1\left({x+1 \atop x-n+1} ; -a \right). \tag{6}$$

Proof. Using the identity [7, 13.6.20]

$$z^n \; {}_2F_0\left({-n, -x \atop -} ; -\frac{1}{z} \right) = (-x)_n \; {}_1F_1\left({-n \atop x+1-n} ; z \right),$$

in (2), we get

$$C_n(x;a) = (-1)^n (-x)_n \; {}_1F_1\left({-n \atop x+1-n} ; a \right). \tag{7}$$

Applying Kummer's transformation [7, 13.2.39]

$$ {}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right) = e^z \, {}_1F_1\left(\begin{matrix} b-a \\ b \end{matrix}; -z\right), $$

we obtain our result. □

COROLLARY 3. *For $x, a = O(1)$, we have*

$$ C_n(x;a) = (-1)^n (-x)_n \; e^a \left[1 + (x+1)an^{-1} + O\left(n^{-2}\right)\right], \quad n \to \infty. \tag{8} $$

Proof. From (6), we have as $n \to \infty$

$$ C_n(x;a) \sim (-1)^n (-x)_n \; e^a \; \left[1 + \frac{x+1}{n-x-1}a + \frac{(x+1)(x+2)}{(n-x-1)(n-x-2)}\frac{a^2}{2}\right], $$

and therefore

$$ C_n(x;a) = (-1)^n (-x)_n \; e^a \; \left[1 + \frac{x+1}{n}a + \frac{(x+1)^2}{n^2}a + \frac{(x+1)(x+2)}{n^2}\frac{a^2}{2} + O\left(n^{-3}\right)\right]. $$

□

REMARK 1. If we use the formula [7, 5.2.5]

$$ (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{9} $$

in (8), rearrange terms and take limits, we recover our previous result (3).

The monic Meixner polynomials satisfy the orthogonality relation [4, 9.10.2]

$$ \sum_{x=0}^{\infty} M_n(x;\beta,c) M_m(x;\beta,c) (\beta)_x \frac{c^x}{x!} = n! c^n (\beta)_n (1-c)^{-\beta-2n} \delta_{n,m}, $$

valid for $\beta > 0$ and $0 < c < 1$.

PROPOSITION 4. *For $c \in \mathbb{C} \setminus [1,\infty)$, we have*

$$ M_n(x;\beta,c) = (-1)^n (-x)_n (1-c)^{-n-x-\beta} \; {}_2F_1\left(\begin{matrix} x+1, x+\beta \\ x+1-n \end{matrix}; \frac{c}{c-1}\right), \tag{10} $$

where we choose the principal branch of $(1-c)^{-n-x-\beta}$.

Proof. Using the identity [7, 15.8.6]

$$ {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; z\right) = \frac{(b)_n}{(c)_n}(1-z)^n \; {}_2F_1\left(\begin{matrix} -n, c-b \\ 1-b-n \end{matrix}; \frac{1}{1-z}\right), $$

in (4) we get

$$M_n(x;\beta,c) = (-1)^n (-x)_n (1-c)^{-n} \ {}_2F_1\left(\begin{matrix} -n, x+\beta \\ x+1-n \end{matrix} ; c \right). \tag{11}$$

Applying the rational transformation [7, 15.8.1]

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} ; z \right) = (1-z)^{-b} \ {}_2F_1\left(\begin{matrix} c-a,b \\ c \end{matrix} ; \frac{z}{z-1} \right), \quad z \in \mathbb{C} \setminus [1,\infty),$$

where $(1-z)^{-b}$ assumes its principal value, the result follows. □

COROLLARY 5. *For* $x = O(1)$ *and* $0 < c < 1$, *we have as* $n \to \infty$

$$M_n(x;\beta,c) = \frac{(-1)^n (-x)_n}{(1-c)^{n+x+\beta}} \left[1 + \frac{(x+1)(x+\beta)c}{1-c} n^{-1} + O\left(n^{-2}\right) \right]. \tag{12}$$

Proof. From (10), we have as $n \to \infty$

$$(1-c)^{n+x+\beta} \frac{M_n(x;\beta,c)}{(-1)^n (-x)_n} \sim 1 + \frac{(x+1)(x+\beta)}{n-x-1} \frac{c}{1-c} + \frac{(x+1)_2 (x+\beta)_2}{(n-x-1)(n-x-2)} \frac{1}{2} \left(\frac{c}{1-c} \right)^2,$$

and therefore

$$(1-c)^{n+x+\beta} \frac{M_n(x;\beta,c)}{(-1)^n (-x)_n} \sim 1 + \frac{(x+1)(x+\beta)}{n} \frac{c}{1-c} + \frac{(x+1)^2 (x+\beta)}{n^2} \frac{c}{1-c} + \frac{1}{2} \frac{(x+1)_2 (x+\beta)_2}{n^2} \left(\frac{c}{1-c} \right)^2. \quad □$$

REMARK 2. If we use (9) in (12), rearrange terms and take limits, we recover our previous result (5).

3. Concluding remarks

We derived asymptotic expansions for the Charlier and Meixner orthogonal polynomials. Our formulas extend the results that we previously obtained in [2] using Tannery's theorem [1]. Although surprisingly simple, these (convergent!) expansions provide excellent approximations for the Charlier and Meixner polynomials in the neighborhood of $x = 0$. They are also very useful in the theory of Sobolev orthogonal polynomials [6].

In a forthcoming sequel, we plan to apply our method to other families of orthogonal polynomials.

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