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FURTHER RESULTS ON VALUE DISTRIBUTION OF L–FUNCTIONS

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Abstract. With the aid of weighted sharing we study the uniqueness of meromorphic functions concerning nonlinear differential polynomials that share a nonzero polynomial with the same of L-functions. Our results in the paper will improve, extend a results due to Fang Liu, Xiao-Min Li and Hong-Xun Yi [5].

1. Introduction

L-functions, with the Riemann zeta function as a prototype, are important objects in number theory, and value distribution of L-functions has been studied extensively, which can be found, for example in Steuding [25]. Value distribution of L-functions concerns distribution of the zeros of L-functions L and, more generally, the c-points of L, i.e., the roots of the equation $L(s) = c$, or the values in the preimage $L^{-1} = \{s \in \mathbb{C} : L(s) = c\}$, where and throughout the paper, s denotes the complex variable in the complex plane $\mathbb{C}$ and c denotes a complex value. L-functions can be analytically continued as meromorphic functions in $\mathbb{C}$.

It is well-known that a nonconstant meromorphic function in $\mathbb{C}$ is completely determined by five such preimages (cf. [8, 22, 28, 30], which is a famous theorem due to Nevanlinna and often referred to as Nevanlinna's uniqueness or unicity theorem. Two meromorphic functions f and g in the complex plane are said to share a value $c \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in $\mathbb{C}$. Moreover, f and g are said to share a value c CM (counting multiplicities) if they share the value c and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities. In terms of sharing values, two nonconstant meromorphic functions in $\mathbb{C}$ must be identically equal if they share five values IM, and one must be a Möbius transform of the other if they share four values CM.

Throughout the paper, an L-function always means an L-function L in the Selberg class, which includes the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and essentially those Dirichlet series where one might expect a Riemann hypothesis. Such an L-function is defined to be a Dirichlet series $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, satisfying the following axioms (cf. [24, 25]): (i) Ramanujan hypothesis. $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$ (ii) Analytic continuation. There is a non-negative integer k such that $(s-1)^k L(s)$ is an entire function of finite order. (iii) Functional equation. L satisfies a functional equation of type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1-\overline{s})},$$

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where

$$\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\nu_j \geqslant 0$ and $|\omega| = 1$. (iv) Euler product hypothesis. $L(s) = \prod_p exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$ with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p.

In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane $\mathbb{C}$. We denote by $N_{k)}(r, \frac{1}{(f-a)})$ the counting function for zeros of $f-a$ with multiplicity $\leqslant k$, and by $\overline{N}_{k)}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, \frac{1}{(f-a)})$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted.

Let z_0 be a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p > q \geqslant 1$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p = q \geqslant 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g), N_E^{1)}(r,a;g), \overline{N}_E^{(2}(r,a;g)$.

Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f,g share the value a with weight k.

The definition implies that if f,g share the value a with weight k then z_0 is an a-point of f with multiplicity $m(\leqslant k)$ if and only if it is an a-point of g with multiplicity $m(\leqslant k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f,g share (a,k) to mean that f,g share the value a with weight k. Clearly if f,g share (a,k) then f,g share (a,p) for any integer p, $0 \leqslant p < k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively. We denote $\rho(f)$ for order of $f(z)$.

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

We first recall the following result due to Steuding [25], which actually holds without the Euler product hypothesis:

THEOREM A. ([25], p. 152) *If two L-functions L_1 and L_2 with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $L_1 = L_2$.*

Later on, Li [17] proved the following result to deal with a question posed by Chung-Chun Yang (cf. [17]):

THEOREM B. ([17]) *Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a nonconstant L-function L share a CM and b IM, then $L = f$.*

In 1997, Lahiri [12] posed the following question:

What can be said about the relationship between two meromorphic functions f and g, when two differential polynomials, generated by f and g respectively, share some nonzero finite value? In this direction, Fang [4] *and Yang-Hua* [28] *respectively proved the following results:*

THEOREM C. ([4]) *Let f and g be two nonconstant entire functions, and let n and k be two positive integers such that $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants, satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.*

THEOREM D. ([28]) *Let f and g be two nonconstant meromorphic functions, and let $n \geqslant 11$ be a positive integer. If f^nf' and g^ng' share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants, satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In 2017, Fang LIU, Xiao-Min LI and Hong-Xun YI proved the following results.

THEOREM E. ([5]) *Let f be a nonconstant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > 3k+6$. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.*

THEOREM F. ([5]) *Let f be a nonconstant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > 3k+6$. If $(f^n)^{(k)}(z) - z$ and $(L^n)^{(k)}(z) - z$ share 0 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.*

Now it is natural to ask the following question which is the motivation of the paper.

QUESTION. *Can a CM shared value be replaced by $(p(z), l)$ in Theorems E and F?*

In the paper, our main concern is to find the possible answer of the above question. The following are the main results of the paper.

THEOREM 1. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, let L be an nonconstant L-function and let n and k be two positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(p(z), l)$, where $p(z)$ be a nonzero polynomial with deg $(p) = m$ and f and L share $(\infty, 0)$. Suppose one of the following conditions hold:*

a. $l \geqslant 3$ and $n > 3k+4$;

b. $l = 2$ and $n > 3k+6$;

c. $l = 1$ and $n > 3k+7$;

d. $l = 0$ and $n > 7k+11$.

Then $f = tL$ *for a constant* t *satisfying* $t^n = 1$.

THEOREM 2. *Let* f *be a nonconstant entire function in* $\mathbb{C}$, *let* L *be an nonconstant* L*-function, and let* n *and* k *be two positive integers. If* $(f^n)^{(k)}$ *and* $(L^n)^{(k)}$ *share* $(p(z),l)$, *where* $p(z)$ *be a nonzero polynomial with deg* $(p) = m$ *and* f *and* L *share* $(\infty,0)$. *Suppose one of the following conditions hold:*

I. $l \geqslant 2$ *and* $n > 2k+4$ *;*

II. $l = 1$ *and* $n > \frac{5k+9}{2}$ *;*

III. $l = 0$ *and* $n > 5k+7$.

Then $f = tL$ *for a constant* t *satisfying* $t^n = 1$.

2. Some lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{1}$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right). \tag{2}$$

LEMMA 1. [27] *Let* f *be a non-constant meromorphic function and let* $a_n(z)(\not\equiv 0), a_{n-1}(z), \ldots, a_0(z)$ *be meromorphic functions such that* $T(r,a_i(z)) = S(r,f)$ *for* $i = 0,1,2,\ldots,n$. *Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r,f) + S(r,f).$$

LEMMA 2. [29] *Suppose that* f *is a nonconstant meromorphic function in the complex plane and* k *is a positive integer. Then*

$$N(r,0;f^{(k)}) \leqslant N(r,0;f) + k\overline{N}(r,\infty,f) + O(\log(T(r,f)) + \log r),$$

as $r \to \infty$, *outside of a possible exceptional set of finite linear measure.*

LEMMA 3. [33] *Let* f *be a nonconstant meromorphic function, and* p,k *be positive integers. Then*

$$N_p(r,0;f^{(k)}) \leqslant T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{3}$$

$$N_p(r,0;f^{(k)}) \leqslant k\overline{N}(r,\infty,f) + N_{p+k}(r,0;f) + S(r,f). \tag{4}$$

LEMMA 4. [14] *If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}|f\neq 0)\leqslant k\overline{N}(r,\infty,f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geqslant k)+S(r,f).$$

LEMMA 5. *Suppose that f and g be two non-constant meromorphic functions. Let $F=[f^n]^{(k)}$, $G=[g^n]^{(k)}$, where n,k are positive integers. If f,g share ∞ IM and $V\equiv 0$, then $F\equiv G$.*

Proof. Suppose $V\equiv 0$. Then by integration we obtain

$$1-\frac{1}{F}\equiv A\Big(1-\frac{1}{G}\Big).$$

If z_0 is a pole of f then it is a pole of g. Hence from the definition of F and G we have $\frac{1}{F(z_0)}=0$ and $\frac{1}{G(z_0)}=0$. So $A=1$ and hence $F\equiv G$. □

LEMMA 6. [16] *Let f_1 and f_2 be two non-constant meromorphic functions satisfying $\overline{N}(r,0;f_i)+\overline{N}(r,\infty;f_i)=S(r;f_1,f_2)$ for $i=1,2$. If $f_1^sf_2^t-1$ is not identically zero for arbitrary integers s and $t(|s|+|t|>0)$, then for any positive ε, we have*

$$N_0(r,1;f_1,f_2)\leqslant \varepsilon T(r)+S(r;f_1,f_2),$$

where $N_0(r,1;f_1,f_2)$ denotes the reduced counting function related to the common 1-points of f_1 and f_2 and $T(r)=T(r,f_1)+T(r,f_2)$, $S(r;f_1,f_2)=o(T(r))$ as $r\to\infty$ possibly outside a set of finite linear measure

LEMMA 7. [2] *Let f and g be two non-constant meromorphic functions sharing $(1,k_1)$, where $2\leqslant k_1\leqslant\infty$. Then*

$$\begin{aligned}\overline{N}(r,1;f|=2)+2\overline{N}(r,1;f|=3)+\ldots+(k_1-1)\overline{N}(r,1;f|=k_1)+k_1\overline{N}_L(r,1;f)\\ +(k_1+1)\overline{N}_L(r,1;g)+k_1\overline{N}_E^{(k_1+1}(r,1;g)\leqslant N(r,1;g)-\overline{N}(r,1;g).\end{aligned}\tag{5}$$

LEMMA 8. [1] *Let F and G be two non-constant meromorphic functions sharing $(1,1)$ and $H\not\equiv 0$. Then*

$$\begin{aligned}T(r,F)\leqslant N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F)\\ +\frac{1}{2}\overline{N}(r,\infty;F)+S(r,F)+S(r,G).\end{aligned}$$

LEMMA 9. [1] *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $H\not\equiv 0$. Then*

$$\begin{aligned}T(r,F)\leqslant N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)\\ +\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,F)+S(r,G).\end{aligned}$$

LEMMA 10. [11] *Let f and g be two nonconstant meromorphic functions. If $(f^n)^{(k)} = (g^n)^{(k)}$ and $n > k+1$, then $f = tg$ for a constant t such that $t^n = 1$.*

LEMMA 11. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, let L be an nonconstant L-function, and let n and k be two positive integers with $H \not\equiv 0$. If $F = (f^n)^{(k)}$ and $G = (L^n)^{(k)}$ share $(1,k_1)$, and f and L share $(\infty,0)$, then*

$$(n-1)\overline{N}(r,\infty;f) \leqslant (k+1)\{T(r,f)+T(r,L)\}+\overline{N}_*(r,1;F,G)+O(\log r).$$

Proof. Suppose ∞ is an e.v.P. of f and L then the lemma follows immediately.

Next suppose ∞ is not an e.v.P. of f and L. Since $H \not\equiv 0$ from Lemma 5 we have $V \not\equiv 0$. We suppose that z_0 is a pole of f with multiplicity q and a pole of L with multiplicity r. Clearly z_0 is a pole of F with multiplicity $nq+k$ and a pole of G with multiplicity $nr+k$. Noting that f,L share $(\infty,0)$ from the definition of V it is clear that z_0 is a zero of V with multiplicity atleast $n+k-1$. Now using the Milloux theorem [[8], p. 55] and Lemma 1, we obtain from the definition of V that

$$m(r,V) = O(\log r).$$

Then by using Valiron-Mokhonko lemma (cf. [22]) and (4) we get

$$\begin{aligned}
&(n+k-1)\overline{N}(r,\infty;f)\\
&\leqslant N(r,0;V)\\
&\leqslant T(r,V)+O(1)\\
&\leqslant N(r,\infty;V)+m(r,0;V)+O(1)\\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+O(\log r)\\
&\leqslant N_{k+1}(r,0;f^n)+k\overline{N}(r,\infty;f)+N_{k+1}(r,0;L^n)+k\overline{N}(r,\infty;L)+\overline{N}_*(r,1;F,G)+O(\log r)\\
&\leqslant (k+1)\overline{N}(r,0;f)+(k+1)\overline{N}(r,0;L)+k\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)+O(\log r).
\end{aligned} \tag{6}$$

This gives

$$(n-1)\overline{N}(r,\infty;f) \leqslant (k+1)\{T(r,f)+T(r,L)\}+\overline{N}_*(r,1;F,G)+O(\log r). \tag{7}$$

This completes the proof of the Lemma. □

LEMMA 12. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, let L be an nonconstant L-function and $F = \frac{(f^n)^{(k)}}{p(z)}$, $G = \frac{(L^n)^{(k)}}{p(z)}$, where $p(z)$ be a nonzero polynomial with deg $(p) = m$, n and k be two positive integers such that $n > 3k+2$. If f and L share $(\infty,0)$ and $H \equiv 0$ then either $[f^n]^{(k)}[L^n]^{(k)} \equiv p^2$ or $f^n \equiv L^n$.*

Proof. Since $H \equiv 0$, on integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \tag{8}$$

where a,b are constants and $a \neq 0$. From (8) it is clear that F and G share $(1,\infty)$.

We now discuss the following three cases.

Case 1. Let $b \neq 0$ and $a \neq b$.

If $b=-1$, then from (8) we have

$$F \equiv \frac{-a}{G-a-1}.$$

Therefore,

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+O(\log r).$$

So in view of Lemma 3 and the second fundamental theorem we get

$$\begin{aligned}
nT(r,L) &\leqslant T(r,G)+N_{k+1}(r.0;L^n)-\overline{N}(r,0;G)+O(\log r)\\
&\leqslant \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,a+1;G)+N_{k+1}(r.0;L^n)-\overline{N}(r,0;G)+O(\log r)\\
&\leqslant \overline{N}(r,\infty;L)+N_{k+1}(r.0;L^n)+\overline{N}(r,\infty;f)+O(\log r)\\
&\leqslant (k+1)\overline{N}(r,0;L)+T(r,f)+O(\log r)\\
&\leqslant (k+1)T(r,L)+T(r,f)+O(\log r)\\
&\leqslant (k+2)T(r,L)+O(\log r).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leqslant T(r,L)$ for $r \in I$. So for $r \in I$, we get a contradiction from above since $n>3k+2$.

If $b \neq -1$, from (8) we obtain that

$$F-\left(1+\frac{1}{b}\right) \equiv \frac{-a}{b^2[G+\frac{a-b}{b}]}.$$

So,

$$\overline{N}\left(r,\frac{(b-a)}{b};G\right)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+O(\log r).$$

Using Lemma 3 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a=b$.

If $b=-1$, then from (8) we have

$$FG \equiv 1,$$

that is

$$[f^n]^{(k)}\,[L^n]^{(k)} \equiv p^2.$$

If $b \neq -1$, from (8) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore,

$$\overline{N}\left(r,\frac{1}{1+b};G\right)=\overline{N}(r,0;F).$$

So in view of Lemma 3 and the second fundamental theorem we get

$$\begin{aligned}
nT(r,L) &\leqslant T(r,G) + N_{k+1}(r.0;L^n) - \overline{N}(r,0;G) + O(\log r) \\
&\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\Big(r,\frac{1}{1+b};G\Big) \\
&\quad + N_{k+1}(r.0;L^n) - \overline{N}(r,0;G) + O(\log r) \\
&\leqslant \overline{N}(r,\infty;L) + (k+1)\overline{N}(r.0;L) + \overline{N}(r,0;F) + O(\log r) \\
&\leqslant (k+1)\overline{N}(r,0;L) + (k+1)\overline{N}(r.0;f) + k\overline{N}(r,\infty;f) + O(\log r) \\
&\leqslant (k+1)T(r,L) + (2k+1)T(r,f) + O(\log r).
\end{aligned}$$

So for $r \in I$ we have

$$nT(r,L) \leqslant (3k+2)T(r,L) + O(\log r),$$

which is a contradiction since $n > 3k+2$.

Case 3. Let $b = 0$. From (8) we obtain

$$F \equiv \frac{G+a-1}{a}. \tag{9}$$

If $a \neq 1$ then from (9) we obtain

$$\overline{N}(r,1-a;G) = \overline{N}(r,0;F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $a = 1$ and from (9) we obtain

$$F \equiv G.$$

Then by the Lemma 10 we have

$$f^n \equiv L^n. \quad \square$$

3. Proof of Theorem 1

Proof. Suppose that d is the degree of L. Then $d = 2\sum_{i=1}^{k} \lambda_j$, where k and λ_j are respectively the positive integer and the positive real number in the axiom (iii) of the definition of L- function.

Then we have that

$$T(r,L) = \frac{d}{\Pi} r\log r + O(r) \tag{10}$$

(cf. [[25], p. 150]). Clearly, f and L are transcendental meromorphic functions (cf. [[5], p. 43]). Note that an L- function at most has one pole $z = 1$ in the complex plane.

Let $F = \frac{(f^n)^{(k)}}{p(z)}$ and $G = \frac{(L^n)^{(k)}}{p(z)}$. It follows that F and G share $(1,l)$ except the zeros of $p(z)$ and f,g share $(\infty,0)$.

Case 1. Let $H \not\equiv 0$.

Subcase 1.1. $l \geqslant 1$.

From (1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G with different multiplicities, (iv) zeros of $F^{'}(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned}N(r,\infty;H) \leqslant\ & \overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)\\ &+\overline{N}_0\left(r,0;F'\right)+\overline{N}_0\left(r,0;G'\right),\end{aligned} \tag{11}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $a(z_0) \neq 0,\infty$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1) \leqslant N(r,0;H) \leqslant N(r,\infty;H)+O(\log r). \tag{12}$$

While $l \geqslant 3$, using (11) and (12) we get

$$\begin{aligned}\overline{N}(r,1;F) &\leqslant N(r,1;F|=1)+\overline{N}(r,1;F|\geqslant 2)\\ &\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)+\overline{N}_*(r,1;F,G)\\ &\quad+\overline{N}(r,1;F|\geqslant 2)+\overline{N}_0\left(r,0;F'\right)+\overline{N}_0\left(r,0;G'\right)+O(\log r).\end{aligned} \tag{13}$$

Now in view of Lemmas 4 and 7 we get

$$\begin{aligned}&\overline{N}_0\left(r,0;G'\right)+\overline{N}(r,1;F|\geqslant 2)+\overline{N}_*(r,1;F,G)\\ &\leqslant \overline{N}_0\left(r,0;G'\right)+\overline{N}(r,1;F|=2)+\overline{N}(r,1;F|=3)+\ldots+\overline{N}(r,1;F|=l)\\ &\quad+\overline{N}_E^{(l+1}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G)\\ &\leqslant \overline{N}_0\left(r,0;G'\right)-\overline{N}(r,1;F|=3)-\ldots-(l-2)\overline{N}(r,1;F|=l)-(l-1)\overline{N}_L(r,1;F)\\ &\quad-l\overline{N}_L(r,1;G)-(l-1)\overline{N}_E^{(l+1}(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G)\\ &\leqslant \overline{N}_0\left(r,0;G'\right)+N(r,1;G)-\overline{N}(r,1;G)-(l-2)\overline{N}_L(r,1;F)-(l-1)\overline{N}_L(r,1;G)\\ &\leqslant N(r,0;G^{'}|G\neq 0)-(l-2)\overline{N}_L(r,1;F)-(l-1)\overline{N}_L(r,1;G)\\ &\leqslant \overline{N}(r,0;G)+\overline{N}(r,\infty;G)-(l-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G)\\ &\leqslant \overline{N}(r,0;G)+\overline{N}(r,\infty;G)-\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G).\end{aligned} \tag{14}$$

Hence using (13), (14), Lemmas 3 and 11 we get from the second fundamental theorem that

$$\begin{aligned}
&nT(r,f)\\
&\leqslant T(r,F)+N_{k+2}(r,0;f^n)-N_2(r,0;F)+O(\log r)\\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^n)-N_2(r,0;F)-\overline{N}_0\left(r,0;F^{'}\right)\\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^n)+\overline{N}(r,0;F|\geqslant 2)\\
&\quad +\overline{N}(r,0;G|\geqslant 2)+\overline{N}(r,1;F|\geqslant 2)+\overline{N}_*(r,1;F,G)\\
&\quad +\overline{N}_0\left(r,0;G^{'}\right)-N_2(r,0;F)+O(\log r)\\
&\leqslant 2\overline{N}(r,\infty;f)+\overline{N}(r,\infty;L)+N_{k+2}(r,0;f^n)+N_2(r,0;G)-\overline{N}_*(r,1;F,G)\\
&\quad -\overline{N}_L(r,1;G)+O(\log r)\\
&\leqslant 2\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^n)+N_2\left(r,0;(L^n)^{(k)}\right)-\overline{N}_*(r,1;F,G)+O(\log r)\\
&\leqslant 2\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,0;f)+(k+2)\overline{N}(r,0;L)-\overline{N}_*(r,1;F,G)+O(\log r)\\
&\leqslant (k+2)\{T(r,f)+T(r,L)\}+\frac{2(k+1)}{(n-1)}\{T(r,f)+T(r,L)\}+O(\log r)\\
&\leqslant \left[k+2+\frac{2(k+1)}{n-1}\right]\{T(r,f)+T(r,L)\}+O(\log r).
\end{aligned} \tag{15}$$

In a similar way we can obtain

$$nT(r,L)\leqslant \left[k+2+\frac{(k+1)^2}{n-1}\right]\{T(r,f)+T(r,L)\}+O(\log r). \tag{16}$$

Adding (15) and (16), we get

$$\left[n-2k-4-\frac{(k+3)(k+1)}{n-1}\right]\{T(r,f)+T(r,L)\}\leqslant O(\log r).$$

Since the quantity in the third bracket can be written as

$$\left[\frac{(n-1)^2-(2k+3)(n-1)-(k+3)(k+1)}{n-1}\right], \tag{17}$$

by a simple computation one can easily verify that when

$$n-1>3k+3>\frac{2k+3+\sqrt{(2k+3)^2+4(k+3)(k+1)}}{2}$$

i.e., when $n>3k+4$ we get a contradiction from (17).

While $l\geqslant 2$, like (13), (14) and not using Lemma 11 in (15) we can deduce a contradiction when $n>3k+6$. So we omit the detail.

While $l=1$. From (3) we obtain

$$\begin{aligned} N_2(r,0;F) &\leqslant N_2(r,0;[f^n]^{(k)})+S(r,f) \\ &\leqslant T(r,[f^n]^{(k)})-nT(r,f)+N_{k+2}(r,0;f^n)+S(r,f) \\ &\leqslant T(r,F)-nT(r,f)+N_{k+2}(r,0;f^n)+O\{\log r\}+S(r,f). \end{aligned}$$

Which implies

$$nT(r,f)\leqslant T(r,F)+N_{k+2}(r,0;f^n)-N_2(r,0;F)+O\{\log r\}+S(r,f). \tag{18}$$

Using (18) and Lemma 8, we get

$$\begin{aligned} nT(r,f) &\leqslant \frac{5}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}(r,0;F)+N_{k+2}(r,0;f^n)+N_2\left(r,0;(L^n)^{(k)}\right)+O(\log r) \\ &\leqslant \frac{5}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\left\{N_{k+1}(r,0;f^n)+k\overline{N}(r,\infty;f)\right\}+N_{k+2}(r,0;L^n) \\ &\quad +(k+2)\overline{N}(r,0;f)+O(\log r) \\ &\leqslant \frac{5+k}{2}\overline{N}(r,\infty;f)+\frac{3k+5}{2}\overline{N}(r,0;f)+(k+2)\overline{N}(r,0;L)+O(\log r) \\ &\leqslant (2k+5)T(r,f)+(k+2)T(r,L)+O(\log r) \\ &\leqslant (3k+7)T(r)+O(\log r), \end{aligned} \tag{19}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$.

In a similar way we can obtain

$$nT(r,L)\leqslant (3k+7)T(r)+O(\log r). \tag{20}$$

Combining (19) and (20) we see that

$$nT(r)\leqslant (3k+7)T(r)+O(\log r), \tag{21}$$

i.e.,

$$(n-3k-7)T(r)\leqslant O(\log r). \tag{22}$$

Since $n>3k+7$, (22) leads to a contradiction.

Subcase 1.2. $l=0$. Using (18) and Lemma 9, we get

$$\begin{aligned} nT(r,f) &\leqslant 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;L)+2\overline{N}(r,0;F)+N_{k+2}(r,0;f^n)+N_2\left(r,0;(L^n)^{(k)}\right) \\ &\quad +\overline{N}\left(r,0;(L^n)^{(k)}\right)+O(\log r) \\ &\leqslant 4\overline{N}(r,\infty;f)+2N_{k+1}(r,0;f^n)+2k\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,0;f) \\ &\quad +N_{k+2}(r,0;L^n)+k\overline{N}(r,\infty;L)+N_{k+1}(r,0;L^n)+k\overline{N}(r,\infty;L)+O(\log r) \\ &\leqslant (2k+4)\overline{N}(r,\infty;f)+(3k+4)\overline{N}(r,0;f)+(2k+3)\overline{N}(r,0;L)+O(\log r) \\ &\leqslant (5k+8)T(r,f)+(2k+3)T(r,L)+O(\log r) \\ &\leqslant (7k+11)T(r)+O(\log r), \end{aligned} \tag{23}$$

where $T(r) = \max\{T(r,f), T(r,g)\}$.

In a similar way we can obtain

$$nT(r,L) \leqslant (7k+11)T(r) + O(\log r). \tag{24}$$

Combining (23) and (24) we see that

$$nT(r) \leqslant (7k+11)T(r) + O(\log r), \tag{25}$$

i.e.,

$$(n-7k-11)T(r) \leqslant O(\log r). \tag{26}$$

Since $n > 7k+11$, (26) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then by Lemma 12, we obtain either

$$(f^n)^{(k)}(L^n)^{(k)} \equiv p^2$$

or

$$f^n \equiv L^n.$$

We consider the following two cases:

Case 1. Suppose that $(f^n)^{(k)}(L^n)^{(k)} \equiv p^2$.

Then,

$$F_1 G_1 \equiv 1 \tag{27}$$

where

$$F_1 = \frac{(f^n)^{(k)}}{p(z)}, \quad G_1 = \frac{(L^n)^{(k)}}{p(z)}. \tag{28}$$

First of all, we prove that 0 is a Picard exceptional value of f and L. Indeed, suppose that $z_0 \notin (z : p(z) = 0)$ is a zero of f with multiplicity m. Then, by the view of (27) we can find that $z_0 = 1$ is a pole of L with multiplicity, say p_1, such that $mn - k = np_1 + k$, and so $(m - p_1)n = 2k$ and so we have $n \leqslant 2k$, which contradicts the assumption $n > 3k+4$. Similarly, we can prove that 0 is a Picard exceptional value of L. On the other hand, by (10) and (27), Valiron-Mokhonko lemma (cf. [22]), a result from Whittaker [[26], p. 82] and the definition of the order of a meromorphic function we have

$$\rho(f) = \rho(f^n) = \rho\left((f^n)^{(k)}\right) = \rho\left((L^n)^{(k)}\right) = \rho(L^n) = \rho(L) = 1. \tag{29}$$

Noting that L has at most one pole $z = 1$ in the complex plane, we have by (27), (29), Lemma 2 that

$$(n+k)\overline{N}(r,\infty;f) \leqslant N\left(r,0;(L^n)^{(k)}\right) \leqslant N(r,0;L^n) + k\overline{N}(r,\infty;L^n) + O(\log r) = O(\log r). \tag{30}$$

Therefore,

$$\overline{N}(r,\infty;f) + \overline{N}(r,\infty;L) \leqslant O(\log r). \tag{31}$$

Now we set

$$f_1 = \frac{(f^n)^{(k)}}{(L^n)^{(k)}}, \quad f_2 = \frac{(f^n)^{(k)} - 1}{(L^n)^{(k)} - 1}. \tag{32}$$

By (32) and the assumption that f and L are transcendental meromorphic functions. We have $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$. Suppose that one of f_1 and f_2 is a nonzero constant. Then, by (32) we see that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share ∞ CM. Combining this with (27) we deduce that ∞ is a Picard exceptional value of f and L. Next we suppose that f_1 and f_2 are nonconstant meromorphic functions.

Then, by (27) and (32) we have

$$F_1 = \frac{f_1(1-f_2)}{f_1 - f_2}, \quad G_1 = \frac{1-f_2}{f_1 - f_2}. \tag{33}$$

By (33) we can find that there exists a subset $I \subset (0, +\infty)$ with infinite linear measure such that $S(r) = o(T(r))$ and

$$\begin{aligned} T(r, F_1) &\leqslant 2(T(r, f_1) + T(r, f_2)) + S(r) \\ &\leqslant 8T(r, F_1) + S(r). \end{aligned} \tag{34}$$

These give $S(r, F_1) = S(r; f_1, f_2)$. Also we note that

$$\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r; f_1, f_2),$$

for $i = 1, 2$.

We note that $\overline{N}(r, -1; F_1) \neq S(r, F_1)$, since otherwise by the second fundamental theorem, F_1 will be a constant.

Also we see that

$$\overline{N}(r, -1; F_1) \leqslant N_0(r, 1; f_1, f_2).$$

Thus we have

$$T(r, f_1) + T(r, f_2) \leqslant 4N_0(r, 1; f_1, f_2) + S(r, F_1).$$

Then, by Lemma 6 there exists two mutually prime integers s and $t(|s| + |t| > 0)$ such that

$$f_1^s f_2^t \equiv 1,$$

i.e.,

$$\left[\frac{F_1}{G_1}\right]^s \left[\frac{F_1 - 1}{G_1 - 1}\right]^t \equiv 1. \tag{35}$$

If either s or t is zero then we arrive at contradiction and so $st \neq 0$.

By (35) we consider the following two subcases:

Subcase 1.1. Suppose that $st < 0$, say $s > 0$ and $t < 0$, say $t = -t_1$, where t_1 is some Positive integer. Then, (35) can be written as

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1 - 1}{G_1 - 1}\right]^{t_1}. \tag{36}$$

Let z_1 be a pole of F_1 of multiplicity p_1. Then from (36) we see that z_1 must be a zero of G_1 of multiplicity p_1. Now from (36) we get $2s = t_1$, which is impossible. Hence F_1 has no pole. Similarly we can prove that G_1 also has no poles.

Subcase 1.2. Suppose that $st < 0$ or $st > 0$. Then by (35) we can see that F_1 and G_1 share ∞ CM. This together with (27) and (28) implies that ∞ is a Picard Exceptional value of f and L. Combining this with the obtained result that 0 is a Picard Exceptional value of f and L, we have

$$L(z) = e^{A_2 z + B_2}, \tag{37}$$

where $A_2 \neq 0$ and B_2 are constants. By (37) and Hayman [[8], p. 7] we have

$$T(r,L) = T(r, e^{A_2 z + B_2}) = \frac{|A_2| r}{\pi}(1 + o(1)). \tag{38}$$

Which contradicts (10).

Case 2. Suppose that $f^n = L^n$. Then, we have $f = tL$, where t is a constant satisfying $t^n = 1$.

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

Proof. Noting that $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;L) = 0$, and proceeding in the like manner as the proof of Theorem 1 we obtain the proof of the Theorem 2. □

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