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ON A SUBCLASS OF ANALYTIC CLOSE–TO–CONVEX FUNCTIONS IN q–ANALOGUE ASSOCIATED WITH JANOWSKI FUNCTIONS

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Abstract. In this article we define a new subclass of analytic multivalent close-to-convex functions in q-calculus associated with Janowski functions. We investigate some geometric properties such as sufficiency criteria, distortion problem, growth theorem, radii of starlikeness and convexity and coefficient estimates for this class.

1. Introduction and definitions

The q-calculus, which is calculus without limits, has attracted the mathematicians due to its numerous physical and mathematical applications. The generalization of derivative and integral in q-calculus which are known as q-analogue of derivative and integral were introduced and studied by Jackson [11, 12]. Aral and Gupta [4, 5, 6] used this concept and defined the q-Baskakov Durrmeyer operator using q-beta function. Similarly, the authors in [3, 7] gave the generalization of some complex operators called q-Picard and q-Gauss-Weierstrass singular integral operators. Srivastava and Bansal [19, pp. 62] used this concept in Geometric function theory and introduced the q-generalization of starlike functions for the first time, see also [18, pp. 347 *et seq.*]. In 2014, the q-analogue of Ruscheweyh operators were studied by Kanas and Răducanu [14], and investigated their properties. Later Mohammed and Darus [2] and Mahmood and Sokół [15] studied this differential operator.

In this article we introduce the q-analogue of a subclass of close-to-convex multivalent functions in association with Janowski functions and study its geometric properties like sufficiency criteria, coefficient bounds, radii problems and distortion theorem.

Let A_p denote the class of all analytic multivalent functions f that are analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \ (z \in \mathbb{D}), \tag{1}$$

where p is a positive integer.

The q-derivative of a function f is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \ (z \neq 0), \tag{2}$$

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where $0 < q < 1$. It can easily be seen that for $n \in \mathbb{N}$ and $z \in \mathbb{D}$

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n,q] \, a_n z^{n-1}, \tag{3}$$

where

$$[n,q] = \frac{1-q^n}{1-q} = 1 + \sum_{l=1}^{n} q^l, \quad [0,q] = 0.$$

For any non-negative integer n the q-number shift factorial is defined by

$$[n,q]! = \begin{cases} 1, \; n = 0, \\ [1,q]\,[2,q]\,[3,q] \cdots [n,q], \; n \in \mathbb{N}. \end{cases}$$

Consequently the q-generalized Pochhammer symbol for $x > 0$ is

$$[x,q]_n = \begin{cases} 1, \; n = 0, \\ [x,q][x+1,q] \ldots [x+n-1,q], \; n \in \mathbb{N}, \end{cases}$$

and for $x > 0$, let the q-gamma function

$$\Gamma_q(x+1) = [x,q]\,\Gamma_q(x); \; \Gamma_q(1) = 1.$$

Motivated by the previous discussion above and having in mind [8, 10, 13, 17, 20, 22], we define a new subclass $\mathcal{K}_{p,q}(\alpha, \delta, A, B)$ of A_p as follows.

DEFINITION 1. Let $-1 \leqslant B < A \leqslant 1$, $0 \leqslant \alpha < 1$ and $0 < q < 1$. Then a function $f \in A_p$ belongs to the class $\mathcal{K}_{p,q}(\alpha, \delta, A, B)$ if

$$\frac{t^p z^{p+1} \partial_q F_\delta(z)}{[p,q] g(z) g(tz)} \prec \frac{p + [pB + (p-\alpha)(A-B)]z}{p(1+Bz)}, \tag{4}$$

where $g(z)$ is in the class of p-valent starlike functions of order $\frac{1}{2}$ denoted by $\mathcal{S}_p^*(1/2)$ and

$$F_\delta(z) = \frac{(1-\delta)[p,q] f(z) + \delta z \partial_q f(z)}{[p,q]},$$

with $\prec$ denotes subordination.

We note that

1. For $A = 1$, $B = -1$, $\delta = 0$ and $q \to 1^-$ we get the class of multivalent close-to-convex functions order α.
2. For $A = 1$, $B = -1$, $\delta = 0$, $\alpha = 0$ and $q \to 1^-$ the class of multivalent close-to-convex functions occurs.
3. For $A = 1$, $B = -1$, $\delta = 0$, $p = 1$ and $q \to 1^-$ the class of close-to-convex functions of order α is covered.

Equivalently, a function $f(z) \in A_p$ is in the class $\mathscr{K}_{p,q}(\alpha,\delta,A,B)$ if and only if

$$\left| \frac{\frac{t^p z^{p+1} \partial_q F_\delta(z)}{[p,q] g(z) g(tz)} - 1}{B + (1-\frac{\alpha}{p})(A-B) - B \frac{t^p z^{p+1} \partial_q F_\delta(z)}{[p,q] g(z) g(tz)}} \right| < 1. \tag{5}$$

For our main results we will need the following Lemma.

LEMMA 1. [21] *Let* $h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \prec k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$ *in* $\mathbb{D}$. *If* $k(z)$ *is univalent in* $\mathbb{D}$ *and* $k(\mathbb{D})$ *is convex, then*

$$|d_n| \leqslant |k_1|, \text{ for all } n \in N.$$

2. The main results and their consequences

THEOREM 1. *Let* $g_i(z) \in \mathscr{S}_p^*(\alpha_i)$, $i = 1,2$. *Then*

$$\frac{g_1(t_1 z) g_2(t_2 z)}{t_1^p t_2^p z^p} \in \mathscr{S}_p^*(\gamma),$$

where $\gamma = \alpha_1 + \alpha_2 - 1$ *and* $0 < |t_i| \leqslant 1$.

Proof. Let $g_i(z) \in \mathscr{S}_p^*(\alpha_i)$. Then by definition

$$\operatorname{Re} \frac{t_i z g_i'(t_i z)}{p g_i(t_i z)} > \alpha_i,$$

for all $i = 1,2$. Now writing

$$G(z) = \frac{g_1(t_1 z) g_2(t_2 z)}{t_1^p t_2^p z^p},$$

we obtain that

$$\frac{zG'(z)}{G(z)} = \frac{t_1 z g_1'(t_1 z)}{g_1(t_1 z)} + \frac{t_2 z g_2'(t_2 z)}{g_2(t_2 z)} - p$$
$$\Rightarrow \frac{zG'(z)}{pG(z)} = \frac{t_1 z g_1'(t_1 z)}{p g_1(t_1 z)} + \frac{t_2 z g_2'(t_2 z)}{p g_2(t_2 z)} - 1.$$

It follows that

$$\operatorname{Re} \frac{zG'(z)}{pG(z)} = \operatorname{Re} \frac{t_1 z g_1'(t_1 z)}{p g_1(t_1 z)} + \operatorname{Re} \frac{t_2 z g_2'(t_2 z)}{p g_2(t_2 z)} - 1$$
$$> \alpha_1 + \alpha_2 - 1 = \gamma,$$

which implies that $G(z) \in \mathscr{S}_p^*(\gamma)$, which completes the proof. $\square$

Now for $t_1 = 1$, $t_2 = t$ and $g_1(z) = g_2(z) = g(z)$ we get the following corollary.

COROLLARY 1. *If* $g(z) \in \mathscr{S}_p^*(1/2)$ *then* $G(z) = \frac{g(z)g(tz)}{t^p z^p} \in \mathscr{S}_p^*(0) := \mathscr{S}_p^*$.

THEOREM 2. *Let* $f \in A_p$ *be of the form* (1). *Then the function* $f \in \mathscr{K}_{p,q}(\alpha,\delta,A,B)$ *if and only if the following inequality holds*

$$\sum_{n=1}^{\infty}\Big(p(1+B)[n+p,q]\left((1-\delta)[p,q]+\delta[p+n,q]\right)\left|a_{n+p}\right|$$
$$+\left(p(1+A)-\alpha(A-B)\right)(p+n)[p,q]^2\Big) \leqslant (p-\alpha)(A-B)[p,q]^2. \tag{6}$$

Proof. Suppose that the inequality (6) holds. Then to show that $f \in \mathscr{K}_{p,q}(\alpha,\delta,A,B)$ we only need to prove (5). For this consider

$$H := \left|\frac{\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}-1}{B+(1-\frac{\alpha}{p})(A-B)-B\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}}\right| = \left|\frac{z\partial_q F_\delta(z)-[p,q]G(z)}{\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]G(z)-Bz\partial_q F_\delta(z)}\right|. \tag{7}$$

Now with the help of (2), (3), (1) and

$$G(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}, \ (z \in \mathbb{D}), \tag{8}$$

where p is a positive integer and with

$$\Lambda_n = \frac{(1-\delta)[p,q]+\delta[p+n,q]}{[p,q]}$$

we get that the expression in (7) equals

$$\begin{aligned}
H &= \left|\frac{[p,q]z^p+\sum_{n=1}^{\infty}\Lambda_n[n+p,q]a_{n+p}z^{n+p}-[p,q]z^p-[p,q]\sum_{n=1}^{\infty}b_{n+p}z^{n+p}}{\left(B+(1-\frac{\alpha}{p})(A-B)\right)\left([p,q]z^p+[p,q]\sum_{n=1}^{\infty}b_{n+p}z^{n+p}\right)-B\left([p,q]z^p+\sum_{n=1}^{\infty}\Lambda_n[n+p,q]a_{n+p}z^{n+p}\right)}\right| \\
&= \left|\frac{\sum_{n=1}^{\infty}\Lambda_n[n+p,q]a_{n+p}z^{n+p}-[p,q]\sum_{n=1}^{\infty}b_{n+p}z^{n+p}}{(1-\frac{\alpha}{p})(A-B)[p,q]z^p-\sum_{n=1}^{\infty}B\Lambda_n[n+p,q]a_{n+p}z^{n+p}+\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]\sum_{n=1}^{\infty}b_{n+p}z^{n+p}}\right| \\
&= \left|\frac{\sum_{n=1}^{\infty}\Lambda_n[n+p,q]a_{n+p}z^{n}-[p,q]\sum_{n=1}^{\infty}b_{n+p}z^{n}}{(1-\frac{\alpha}{p})(A-B)[p,q]-\sum_{n=1}^{\infty}B\Lambda_n[n+p,q]a_{n+p}z^{n}+\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]\sum_{n=1}^{\infty}b_{n+p}z^{n}}\right| \\
&\leqslant \frac{\sum_{n=1}^{\infty}\Lambda_n[n+p,q]\left|a_{n+p}\right|+[p,q]\sum_{n=1}^{\infty}\left|b_{n+p}\right|}{(1-\frac{\alpha}{p})(A-B)[p,q]-\sum_{n=1}^{\infty}B\Lambda_n[n+p,q]\left|a_{n+p}\right|-\left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]\sum_{n=1}^{\infty}\left|b_{n+p}\right|}
\end{aligned} \tag{9}$$

Since $g(z) \in \mathscr{S}_p^*(1/2)$, by the Corrolary 1 $G(z)$ is in the class $\mathscr{S}_p^*$ having representation (8); then

$$\left|b_{p+n}\right| \leqslant p+n \tag{10}$$

and $H < 1$, where we have used (6) which completes the direct part.

Conversely, let $f \in \mathcal{K}_{p,q}(\alpha,\delta,A,B)$ be given by (1). Then from (5) we have for $z \in \mathbb{D}$, that

$$\left| \frac{\frac{z\partial_q F_\delta(z)}{[p,q]G(z)} - 1}{B + (1-\frac{\alpha}{p})(A-B) - B\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}} \right|$$

$$= \left| \frac{\sum_{n=1}^{\infty} \Lambda_n [n+p,q] a_{n+p} z^n - [p,q]\sum_{n=1}^{\infty} b_{n+p} z^n}{(1-\frac{\alpha}{p})(A-B)[p,q] - \sum_{n=1}^{\infty} B\Lambda_n [n+p,q] a_{n+p} z^n + \left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]\sum_{n=1}^{\infty} b_{n+p} z^n} \right|.$$

Since $|\operatorname{Re} z| \leqslant |z|$, we have

$$\operatorname{Re}\left\{ \frac{\sum_{n=1}^{\infty} \Lambda_n [n+p,q] a_{n+p} z^n - [p,q]\sum_{n=1}^{\infty} b_{n+p} z^n}{(1-\frac{\alpha}{p})(A-B)[p,q] - \sum_{n=1}^{\infty} B\Lambda_n [n+p,q] a_{n+p} z^n + \left(B+(1-\frac{\alpha}{p})(A-B)\right)[p,q]\sum_{n=1}^{\infty} b_{n+p} z^n} \right\} < 1 \qquad (11)$$

Now choose values of z on the real axis so, that $\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}$ is real. Upon clearing the denominator in (11) and letting $z \to 1^-$ through real values, we obtain (6). □

THEOREM 3. *Let $f \in \mathcal{K}_{p,q}(\alpha,\delta,A,B)$ and be of the form (1). Then*

$$|a_{p+n}| \leqslant \frac{[p,q]^2}{[p+n,q]\left((1-\delta)[p,q]+\delta[p+n,q]\right)} \left(p+n+\frac{(p-\alpha)(A-B)(n-1)(2p+n)}{2p} \right).$$

Proof. The $f \in A_p$ belonging to the class $\mathcal{K}_{p,q}(\alpha,\delta,A,B)$ satisfies

$$\frac{t^p z^{p+1}\partial_q F_\delta(z)}{[p,q]g(z)g(tz)} \prec \frac{1+[B+(1-\frac{\alpha}{p})(A-B)]z}{1+Bz}.$$

If

$$G(z) = \frac{g(z)g(tz)}{t^p z^p}$$

and

$$h(z) = \frac{z\partial_q F_\delta(z)}{[p,q]G(z)}, \qquad (12)$$

and it is of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n,$$

since

$$h(z) \prec \frac{1+[B+(1-\frac{\alpha}{p})(A-B)]z}{1+Bz} = 1 + \frac{(p-\alpha)(A-B)}{p}z + \dots.$$

Then by Lemma 1 we get

$$|d_n| \leqslant \frac{(p-\alpha)(A-B)}{p}. \qquad (13)$$

Now putting the series expansions of $h(z), G(z)$ and $f(z)$ in (12), simplify and compare the coefficients of z^{p+n} on both sides

$$\frac{(1-\delta)[p,q]+\delta[p+n,q]}{[p,q]^2}[p+n,q]a_{p+n} = b_{p+n}+b_{p+n-1}d_1+b_{p+n-2}d_2+\ldots+b_{p+1}d_{n-1}.$$

Taking modulus on both sides, using the triangle's inequality and then by (13) and (10) we obtain

$$\frac{(1-\delta)[p,q]+\delta[p+n,q]}{[p,q]^2}[p+n,q]\left|a_{p+n}\right| \leqslant n+p+\frac{(p-\alpha)(A-B)}{p}\sum_{i=2}^{n-1}(p+i),$$

which implies that

$$\left|a_{p+n}\right| \leqslant \frac{[p,q]^2}{[p+n,q]\left((1-\delta)[p,q]+\delta[p+n,q]\right)}\left(p+n+\frac{(p-\alpha)(A-B)(n-1)(2p+n)}{2p}\right),$$

where $|a_1| = 1$. □

THEOREM 4. *Assume $f \in \mathscr{K}_{p,q}(\alpha,\delta,A,B)$ has the form (1). Then for $|z| = r$*

$$\frac{[p,q](1-Cr)r^{p-1}}{(1-Br)(1+r)^{2p}} \leqslant \left|\partial_q F_\delta(z)\right| \leqslant \frac{[p,q](1+Cr)r^{p-1}}{(1+Br)(1-r)^{2p}}$$

where $C = B+(1-\frac{\alpha}{p})(A-B)$.

Proof. Suppose that $f \in \mathscr{K}_{p,q}(\alpha,\delta,A,B)$. Then we can write

$$\frac{z\partial_q F_\delta(z)}{[p,q]G(z)} \prec \frac{1+Cz}{1+Bz}.$$

Accordingly with $|z| = r$

$$\left|\frac{z\partial_q F_\delta(z)}{[p,q]G(z)} - \frac{1-CBr^2}{1-B^2r^2}\right| \leqslant \frac{(C-B)r}{1-B^2r^2},$$

routine simplifications give us

$$\frac{1-Cr}{1-Br} \leqslant \left|\frac{z\partial_q F_\delta(z)}{[p,q]G(z)}\right| \leqslant \frac{1+Cr}{1+Br}. \tag{14}$$

Because $G(z) \in \mathscr{S}_p^*$, thus

$$\frac{r^p}{(1+r)^{2p}} \leqslant |G(z)| \leqslant \frac{r^p}{(1-r)^{2p}}. \tag{15}$$

Now by replacing (15) in (14) we obtain the required result. □

THEOREM 5. *Let* $f \in \mathcal{K}_{p,q}(\alpha,\delta,A,B)$ *has the form* (1). *Then for* $|z| = r$

$$r^p(1-\tau_1) \leqslant |f(z)| \leqslant r^p(1+\tau_1),$$

where

$$\tau_1 = \frac{[p,q]^2((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))}{p(1+B)[1+p,q]((1-\delta)[p,q]+\delta[p+1,q])}.$$

Proof. Consider

$$\begin{aligned} |f(z)| &= \left| z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \right| \\ &\leqslant |z^p| + \sum_{n=1}^{\infty} |a_{n+p}| \; |z|^{n+p} \\ &= r^p + \sum_{n=1}^{\infty} |a_{n+p}| \; r^{n+p}. \end{aligned}$$

As $|z| = r < 1$, so $r^{n+p} < r^p$ and

$$|f(z)| \leqslant r^p \left(1 + \sum_{n=1}^{\infty} |a_{n+p}|\right) \tag{16}$$

Similarly

$$|f(z)| \geqslant r^p \left(1 - \sum_{n=1}^{\infty} |a_{n+p}|\right) \tag{17}$$

Since (6), this implies that

$$\begin{aligned} &\sum_{n=1}^{\infty} \big(p(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])|a_{n+p}| \\ &\quad + (p(1+A)-\alpha(A-B))(p+n)[p,q]^2\big) \leqslant (p-\alpha)(A-B)[p,q]^2. \end{aligned}$$

But

$$\begin{aligned} &(p(1+A)-\alpha(A-B))(p+1)[p,q]^2 + p(1+B)[1+p,q]((1-\delta)[p,q] \\ &\quad +\delta[p+1,q]) \sum_{n=1}^{\infty} |a_{n+p}| \\ &\leqslant \sum_{n=1}^{\infty} \big(p(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])|a_{n+p}| \\ &\quad + (p(1+A)-\alpha(A-B))(p+n)[p,q]^2\big) \\ &\leqslant (p-\alpha)(A-B)[p,q]^2, \end{aligned}$$

which gives

$$\sum_{n=1}^{\infty}\left|a_{n+p}\right| \leqslant \frac{[p,q]^2((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))}{p(1+B)[1+p,q]((1-\delta)[p,q]+\delta[p+1,q])}$$

By putting the right hand side expression value in (16) and (17) we conclude the assertion. □

Here, and in what follows, we denote by $\mathscr{C}_p(\beta)$ the class of p-valent convex functions of order β where $0 \leqslant \beta < 1$.

THEOREM 6. *Let $f \in \mathscr{K}_{p,q}(\alpha,\delta,A,B)$. Then $f \in \mathscr{C}_p(\alpha)$ for $|z| < r_1$, where*

$$r_1 = \left(\frac{p^2(p-\beta)(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])}{(p+n)(n+p-\beta)((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2}\right)^{\frac{1}{n}},$$

and $n \in N$.

Proof. Let $f \in \mathscr{K}_{p,q}(\alpha,\delta,A,B)$. To prove that $f \in \mathscr{C}_p(\beta)$ we only have to show:

$$\left|\frac{zf''(z)-(p-1)f'(z)}{zf''(z)+(1-2\beta+p)f'(z)}\right| < 1.$$

Using (1) along with some simplifications

$$\sum_{n=1}^{\infty}\frac{(p+n)(n+p-\beta)}{p(p-\beta)}\left|a_{n+p}\right||z|^n < 1. \tag{18}$$

From (6), we obtain that

$$\sum_{n=1}^{\infty} p(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])\left|a_{n+p}\right|$$

$$\leqslant ((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{p(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])}{((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2}\left|a_{n+p}\right| < 1.$$

The inequality (18) will be satisfied if the following holds

$$\sum_{n=1}^{\infty}\frac{(p+n)(n+p-\beta)}{p(p-\beta)}\left|a_{n+p}\right||z|^n$$
$$< \sum_{n=1}^{\infty}\frac{p(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])}{((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2}\left|a_{n+p}\right|,$$

which implies that

$$|z|^n < \frac{p^2(p-\beta)(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])}{(p+n)(n+p-\beta)((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2},$$

and so

$$|z| < \left(\frac{p^2 (p-\beta)(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])}{(p+n)(n+p-\beta)((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2} \right)^{\frac{1}{n}} = r_1.$$

□

We denote by $\mathscr{S}_p^*(\beta)$ the class of p-valent Starlike functions of order β where $0 \leqslant \beta < 1$.

THEOREM 7. *Let $f \in \mathscr{K}_{p,q}(\alpha,\delta,A,B)$. Then $f \in \mathscr{S}_p^*(\beta)$ for $|z| < r_2$, where*

$$r_2 = \left(\frac{(p-\beta)\,p(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])}{(n+p-\beta)((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2} \right)^{\frac{1}{n}},$$

and $n \in N$.

Proof. We know that $f \in \mathscr{S}_p^*(\beta)$ if and only if

$$\left| \frac{zf'(z)-pf(z)}{zf'(z)+(p-2\beta)f(z)} \right| \leqslant 1.$$

Using (1) upon reducing the material we conclude

$$\sum_{n=1}^{\infty} \left(\frac{n+p-\beta}{p-\beta} \right) |a_{n+p}|\,|z|^n < 1. \tag{19}$$

Now from (6) we obtain that

$$\sum_{n=1}^{\infty} \frac{p(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])}{((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2} |a_{n+p}| < 1.$$

Inequality (19) is valid if

$$\sum_{n=1}^{\infty} \left(\frac{n+p-\beta}{p-\beta} \right) |a_{n+p}|\,|z|^n$$
$$< \sum_{n=1}^{\infty} \frac{p(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])}{((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2} |a_{n+p}|.$$

This gives

$$|z|^n < \frac{(p-\beta)\,p(1+B)[n+p,q]((1-\delta)[p,q]+\delta[p+n,q])}{(n+p-\beta)((p-\alpha)(A-B)-(p(1+A)-\alpha(A-B))(p+1))[p,q]^2},$$

and hence

$$|z| < \left(\frac{(p-\beta)\, p\,(1+B)\,[n+p,q]\,((1-\delta)[p,q]+\delta[p+n,q])}{(n+p-\beta)\,((p-\alpha)\,(A-B)-(p(1+A)-\alpha(A-B))\,(p+1))\,[p,q]^2} \right)^{\frac{1}{n}} = r_2,$$

Thus we obtain the required result. □

REFERENCES

[1] I. ALDAWISH AND M. DARUS, *Starlikeness of q-differential operator involving quantum calculus*, Korean J. Math. **22** (4), 699–709 (2014).

[2] H. ALDWEBY AND M. DARUS, *Some subordination results on q-analogue of Ruscheweyh differential operator*, Abstr. Appl. Anal., Vol. 2014, Article ID 958563, 6 pages 1–6 (2013).

[3] A. ARAL, *On the generalized Picard and Gauss Weierstrass singular integrals*, J. Comput. Anal. Appl. **8** (3), 249–261 (2006).

[4] A. ARAL, V. GUPTA AND R. P. AGARWAL, *Applications of q-Calculus in Operator Theory*, Springer-Verlag New York, 2013.

[5] A. ARAL AND V. GUPTA, *Generalized q-Baskakov operators*, Math. Slovaca **61** (4), 619–634 (2011).

[6] A. ARAL AND V. GUPTA, *On q-Baskakov type operators*, Demonstr. Math. **42** (1), 109–122 (2009).

[7] G. A. ANASTASSIU AND S. G. GAL, *Geometric and approximation properties of generalized singular integrals*, J. Korean Math. Soci. **23** (2), 425–443 (2006).

[8] J. DZIOK, G. MURUGUSUNDARAMOORTHY AND J. SOKOŁ, *On certain class of meromorphic functions with positive coefcients*, Acta Math. Sci. Ser. B **32** (4), 1–16 (2012).

[9] M. R. GANIGI AND B. A. URALEGADDI, *New criteria for meromorphic univalent functions*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **33** (81), 9–13 (1989).

[10] A. HUDA AND M. DARUS, *Integral operator defined by q-analogue of Liu-Srivastava operator*, Studia Universitatis Babes-Bolyai Series Mathematica **58** (4), 529–537 (2013).

[11] F. H. JACKSON, *On q-definite integrals*, The Quarterly Journal of Pure and Applied Mathematics **41**, 193–203 (1910).

[12] F. H. JACKSON, *On q-functions and a certain difference operator*, Earth and Environmental Science Transactions of The Royal Society of Edinburgh **46** (2), 253–281 (1909).

[13] W. JANOWSKI, *Some extremal problems for certain families of analytic functions*, Ann. Polon. Math. **28**, 297–326 (1973).

[14] S. KANAS AND D. RĂDUCANU, *Some class of analytic functions related to conic domains*, Math. Slovaca. **64** (5), 1183–1196 (2014).

[15] S. MAHMMOD AND J. SOKÓŁ, *New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator*, Results Math. **71** (4), 1345–1357 (2017).

[16] A. MOHAMMED AND M. DARUS, *A generalized operator involving the q-hypergeometric function*, Mat. vesn. **65** (4), 454–465 (2013).

[17] T. M. SEOUDY AND M. K. AOUF, *Coefficient estimates of new classes of q-starlike and q-convex functions of complex order*, J. Math. Inequal. **10** (1), 135–145 (2016).

[18] H. M. SRIVASTAVA AND D. BANSAL, *Close-to-convexity of a certain family of q-Mittag-Leffler functions*, J. Nonlinear Var. Anal. **1** (1), 61–69 (2017).

[19] H. M. SRIVASTAVA, *Univalent functions, fractional calculus and associated generalized hypergeometric functions*, in Univalent Functions, Fractional Calculus, and Their Applications (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), pp. 329–354, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[20] C. POMMERENKE, *On meromorphic starlike functions*, Pacific J. Math. **13**, 221–235 (1963).

[21] W. ROGOSINSK, *On the coefficients of subordinate functions*, Proc. London Math. Soc. **48** (2), 48–82 (1943).

[22] B. A. URALEGADDI AND C. SOMANATHA, *Certain diferential operators for meromorphic functions*, Houston J. Math. **17** (2), 279–284, (1991).

[23] W. C. ROYSTER, *Meromorphic starlike multivalent functions*, Trans. Amer. Math. Soc. **107** (1963), 300–308.

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