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ON SOME INEQUALITIES CONCERNING RELATIVE $(p,q)-\varphi$ TYPE AND RELATIVE $(p,q)-\varphi$ WEAK TYPE OF ENTIRE OR MEROMORPHIC FUNCTIONS WITH RESPECT TO AN ENTIRE FUNCTION

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Abstract. The main aim of this paper is to study some growth properties of entire and meromorphic functions on the basis of relative (p,q)-φ type and relative (p,q)-φ weak type of entire and meromorphic function with respect to an entire function.

1. Introduction, definitions and notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [12, 14, 19]. We also use the standard notations and definitions of the theory of entire functions which are available in [20] and therefore we do not explain those in details. Let f be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. Since $M_f(r)$ is strictly increasing and continuous, therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} M_f^{-1}(s) = \infty$. When f is meromorphic, one may introduce another function $T_f(r)$ defined by $T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta$ known as Nevanlinna's characteristic function of f (see [12, p.4]), playing the same role as $M_f(r)$. Moreover, if f is non-constant entire then $T_f(r)$ is also strictly increasing and continuous functions of r. Therefore its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$. For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Further we assume that throughout the present paper p, q, m and n always denote positive integers. Now considering this, we introduce the definition of the (p,q)-th order and (p,q)-th lower order of an entire or meromorphic function which are as follows:

DEFINITION 1. The (p,q)-th order and (p,q)-th lower order of an entire function f are defined as:

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

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If f is a meromorphic function, then

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

The function f is said to be of regular (p,q)-growth when (p,q)-th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q)-growth are said to be of irregular (p,q)-growth.

Definition 1 avoids the restriction $p \geqslant q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [13]. Moreover for entire and meromorphic functions when $p<q$, then Definition 1 is a special case of Proposition 1.2 and Definition 1.6 of [18] respectively for $\varphi(r) = \log^{[l]} r$ where $l > p-q$.

However the above definition is very useful for measuring the growth of entire and meromorphic functions. If $p=l$ and $q=1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire or meromorphic function f. For details about generalized order one may see [17]. Also for $p=2$ and $q=1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire or meromorphic function f.

In this connection we just recall the following definition where we will give a minor modification to the original definition (see e.g. [13]):

DEFINITION 2. An entire function f is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b=1$ if $p=q$ and $b=0$ for otherwise. Moreover if $0<\rho^{(p,q)}(f)<\infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n<p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n<q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n=1,2,\cdots \end{cases}.$$

Similarly for $0<\lambda^{(p,q)}(f)<\infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n<p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n<q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n=1,2,\cdots \end{cases}.$$

Analogously one can easily verify that Definition 2 of index-pair can also be applicable to a meromorphic function f.

Now revisiting the ideas developed by Shen et al. [18] one may introduce the definition of (p,q)-φ order and (p,q)-φ lower order of entire functions in the following way:

$$\begin{matrix} \rho^{(p,q)}(f,\varphi) \\ \lambda^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} \varphi(r)},$$

where $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function.

If f is a meromorphic function, then for a non-decreasing unbounded function $\varphi : [0,+\infty) \to (0,+\infty)$, we get that

$$\begin{matrix} \rho^{(p,q)}(f,\varphi) \\ \lambda^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} \varphi(r)}.$$

In fact the above definition also avoids the restriction $p \geqslant q$ of the original definitions of (p,q)-φ order and (p,q)-φ lower order of entire and meromorphic functions introduced by Shen et al. [18]. Further for any non-decreasing unbounded function $\varphi : [0,+\infty) \to (0,+\infty)$, if we assume $\lim\limits_{r\to+\infty} \frac{\log^{[q]}\varphi(ar)}{\log^{[q]}\varphi(r)} = 1$ for all $\alpha > 0$, then for any entire function f, using the inequality $T_f(r) \leqslant \log M_f(r) \leqslant 3T_f(2r)\{cf.\ [12]\}$, one can easily verify that

$$\begin{matrix} \rho^{(p,q)}(f,\varphi) \\ \lambda^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} \varphi(r)} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} \varphi(r)}.$$

If $\varphi(r) = r$, then Definition 1 is the special case of the above definition. However, the function f is said to be of regular (p,q)-φ growth when (p,q)-φ order and (p,q)-φ lower order of f are the same. Functions which are not of regular (p,q)-φ growth are said to be of irregular (p,q)-φ growth.

Now in order to compare the growth of entire or meromorphic functions having the same (p,q)-φ order, one may introduce the concepts of (p,q)-φ type and (p,q)-φ lower type in the following manner:

DEFINITION 3. Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The (p,q)-φ type and the (p,q)-φ lower type of an entire function f with non-zero finite (p,q)-φ order $\rho^{(p,q)}(f,\varphi)$ are defined as:

$$\begin{matrix} \sigma^{(p,q)}(f,\varphi) \\ \overline{\sigma}^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(r)}{\left[\log^{[q-1]} \varphi(r)\right]^{\rho^{(p,q)}(f,\varphi)}}.$$

If f is a meromorphic function, then

$$\begin{matrix} \sigma^{(p,q)}(f,\varphi) \\ \overline{\sigma}^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-2]} T(r)}{\left[\log^{[q-1]} \varphi(r)\right]^{\rho^{(p,q)}(f,\varphi)}}.$$

Likewise, to compare the growth of entire and meromorphic functions having the same (p,q)-φ lower order, one can also introduce the concepts of (p,q)-φ weak type $\overline{\tau}^{(p,q)}(f,\varphi)$ and the growth indicator $\tau^{(p,q)}(f,\varphi)$ of an entire or meromorphic function f in the following manner:

DEFINITION 4. Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The (p,q)-φ weak type $\overline{\tau}^{(p,q)}(f,\varphi)$ and the growth indicator $\tau^{(p,q)}(f,\varphi)$ of an entire function f with non-zero finite (p,q)-φ lower order $\lambda^{(p,q)}(f,\varphi)$ are defined as:

$$\begin{matrix}\tau^{(p,q)}(f,\varphi)\\ \overline{\tau}^{(p,q)}(f,\varphi)\end{matrix} = \lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p-1]}M_f(r)}{\left[\log^{[q-1]}\varphi(r)\right]^{\lambda^{(p,q)}(f,\varphi)}}.$$

If f is a meromorphic function, then

$$\begin{matrix}\tau^{(p,q)}(f,\varphi)\\ \overline{\tau}^{(p,q)}(f,\varphi)\end{matrix} = \lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p-2]}M_f(r)}{\left[\log^{[q-1]}\varphi(r)\right]^{\lambda^{(p,q)}(f,\varphi)}}.$$

In particular, $\sigma^{(p,q)}(f,r)=\sigma^{(p,q)}(f)$ and $\overline{\sigma}^{(p,q)}(f,r)=\overline{\sigma}^{(p,q)}(f)$ are the (p,q)-th type and the (p,q)-th lower type of an entire or meromorphic function f. Similarly, $\tau^{(p,q)}(f,r)=\tau^{(p,q)}(f)$ and $\overline{\tau}^{(p,q)}(f,r)=\overline{\tau}^{(p,q)}(f)$.

Also if $p=l$, $q=1$ and $\varphi(r)=r$, then we write $\sigma^{(l,1)}(f)=\sigma^{(l)}(f)$, $\overline{\sigma}^{(l,1)}(f)=\overline{\sigma}^{(l)}(f)$, and $\overline{\tau}^{(l,1)}(f)=\overline{\tau}^{(l)}(f)$ where $\sigma^{(l)}(f)$, $\overline{\sigma}^{(l)}(f)$ and $\overline{\tau}^{(l)}(f)$ are known as generalized type, generalized lower type and generalized weak type respectively. Also $\sigma(f)$, $\overline{\sigma}(f)$ and $\overline{\tau}(f)$ are respectively known as type, lower type and weak type of f.

Mainly the growth investigation of entire or meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire or meromorphic function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2, 15]) will come. Extending this notion, Sánchez Ruiz et al. [16] gave the definitions of relative (p,q)-th order and relative (p,q)-th lower order of an entire function with respect to another entire function and Debnath et al. [4] introduced the definitions of relative (p,q)-th order and relative (p,q)-th lower order of a meromorphic function with respect to another entire function in the light of index-pair. Now in order to make some progress in the study of relative order,one may introduce the definitions of relative (p,q)-φ order and relative (p,q)-φ lower order of an entire or meromorphic function with respect to another entire function in the following way:

DEFINITION 5. Let $\varphi:[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function. The relative (p,q)-φ order and the relative (p,q)-φ lower order of an entire function f with respect to another entire function g are defined as

$$\begin{matrix}\rho_g^{(p,q)}(f,\varphi)\\ \lambda_g^{(p,q)}(f,\varphi)\end{matrix} = \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}M_g^{-1}\left(M_f(r)\right)}{\log^{[q]}\varphi(r)}.$$

If f is meromorphic and g is entire, then

$$\begin{matrix}\rho_g^{(p,q)}(f,\varphi)\\ \lambda_g^{(p,q)}(f,\varphi)\end{matrix} = \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}T_g^{-1}\left(T_f(r)\right)}{\log^{[q]}\varphi(r)}.$$

Further if relative (p,q)-φ order and the relative (p,q)-φ lower order of f with respect to g are the same, then f is called a function of regular relative (p,q)-φ growth with respect to g. Otherwise, f is said to be irregular relative (p,q)-φ growth.with respect to g.

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as relative (p,q)-φ type and relative (p,q)-φ lower type of entire or meromorphic functions with respect to another entire function which are as follows:

DEFINITION 6. Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The relative (p,q)-φ type and the relative (p,q)-φ lower type of an entire function f with respect to another entire function g having non-zero finite relative (p,q)-φ order $\rho_g^{(p,q)}(f,\varphi)$ are defined as:

$$\begin{matrix} \sigma_g^{(p,q)}(f,\varphi) \\ \overline{\sigma}_g^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_g^{-1}\left(M_f(r)\right)}{\left[\log^{[q-1]}\varphi(r)\right]^{\rho_g^{(p,q)}(f,\varphi)}}.$$

If f is meromorphic and g is entire, then

$$\begin{matrix} \sigma_g^{(p,q)}(f,\varphi) \\ \overline{\sigma}_g^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left[\log^{[q-1]}\varphi(r)\right]^{\rho_g^{(p,q)}(f,\varphi)}}.$$

Analogously, to determine the relative growth of f having same non zero finite relative (p,q)-φ lower order with respect to g, one can introduce the definition of relative (p,q)-φ weak type $\overline{\tau}_g^{(p,q)}(f)$ and the growth indicator $\tau_g^{(p,q)}(f)$ of f with respect to g of finite positive relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f)$ in the following way:

DEFINITION 7. Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The relative (p,q)-φ weak type $\overline{\tau}_g^{(p,q)}(f,\varphi)$ and the growth indicator $\tau_g^{(p,q)}(f,\varphi)$ of an entire function f with respect to another entire function g having non-zero finite relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f,\varphi)$ are defined as:

$$\begin{matrix} \tau_g^{(p,q)}(f,\varphi) \\ \overline{\tau}_g^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_g^{-1}\left(M_f(r)\right)}{\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_g^{(p,q)}(f,\varphi)}}.$$

If f is meromorphic and g is entire, then

$$\begin{matrix} \tau_g^{(p,q)}(f,\varphi) \\ \overline{\tau}_g^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_g^{(p,q)}(f,\varphi)}}.$$

If we consider $\varphi(r)=r$, then $\rho_g^{(p,q)}(f,r)$ $(\lambda_g^{(p,q)}(f,r))$, $\sigma_g^{(p,q)}(f,r)$ $(\overline{\sigma}_g^{(p,q)}(f,r))$ and $\overline{\tau}_g^{(p,q)}(f,r)$ are respectively known as relative (p,q)-th order (relative (p,q)-th lower order), relative (p,q)-th type (relative (p,q)-th lower type) and relative (p,q)-th weak type of f with respect to g. Further for $\varphi(r)=r$, we simplify to denote $\rho_g^{(p,q)}(f,r)$ $(\lambda_g^{(p,q)}(f,r))$, $\sigma_g^{(p,q)}(f,r)$ $(\overline{\sigma}_g^{(p,q)}(f,r))$ and $\tau_g^{(p,q)}(f,r)$ $(\overline{\tau}_g^{(p,q)}(f,r))$ by $\rho_g^{(p,q)}(f)$ $(\lambda_g^{(p,q)}(f))$, $\sigma_g^{(p,q)}(f)$ $(\overline{\sigma}_g^{(p,q)}(f))$ and $\tau_g^{(p,q)}(f)$ $(\overline{\tau}_g^{(p,q)}(f))$ respectively.

However the main aim of this paper is to investigate some growth properties of entire and meromorphic functions using relative (p,q)-φ order, relative (p,q)-φ type and relative (p,q)-φ weak type which improve and extend some earlier results (see, e.g., [5]-[11]). Through out the paper we consider that all the growth indicators are non-zero finite.

2. Main results

First of all, we recall one related known property which will be needed in order to prove our results, as we see in the following lemma.

LEMMA 1. [3] *Let f be a meromorphic function and g, h be any two entire functions such that $0<\lambda_h^{(m,q)}(f,\varphi)\leqslant\rho_h^{(m,q)}(f,\varphi)<\infty$ and $0<\lambda_h^{(m,p)}(g)\leqslant\rho_h^{(m,p)}(g)<\infty$. Then*

$$\frac{\lambda_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}\leqslant\lambda_g^{(p,q)}(f,\varphi)\leqslant\min\left\{\frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)},\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}\right\}$$
$$\leqslant\max\left\{\frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)},\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}\right\}\leqslant\rho_g^{(p,q)}(f,\varphi)\leqslant\frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}.$$

From the conclusion of the above result, one may write $\rho_g^{(p,q)}(f,\varphi)=\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}$ and $\lambda_g^{(p,q)}(f,\varphi)=\frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}$ when $\lambda_h^{(m,p)}(g)=\rho_h^{(m,p)}(g)$. Similarly $\rho_g^{(p,q)}(f,\varphi)=\frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}$ and $\lambda_g^{(p,q)}(f,\varphi)=\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}$ when $\lambda_h^{(m,q)}(f,\varphi)=\rho_h^{(m,q)}(f,\varphi)$.

REMARK 1. [3] If we take "f be an entire function" instead of "f be a meromorphic function" in Lemma 1 and the other conditions remain the same then one can easily derive the same conclusion of Lemma 1.

Now we present the main results of the paper with their proofs.

THEOREM 1. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0<\rho_h^{(m,q)}(f,\varphi)<\infty$ and $0<\lambda_h^{(m,p)}(g)\leqslant\rho_h^{(m,p)}(g)<\infty$. Then*

$$\max\left\{\left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}},\left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}$$

$$\leqslant\sigma_g^{(p,q)}(f,\varphi)\leqslant\left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Proof. Let us consider that $\varepsilon(>0)$ is arbitrary number. Now from the definitions of $\sigma_h^{(m,q)}(f,\varphi)$ and $\overline{\sigma}_h^{(m,q)}(f,\varphi)$, we have for all sufficiently large values of r that

$$T_f(r)\leqslant T_h\left(\exp^{[m-1]}\left(\left(\sigma_h^{(m,q)}(f,\varphi)+\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}\right)\right),\tag{1}$$

$$T_f(r)\geqslant T_h\left(\exp^{[m-1]}\left(\left(\overline{\sigma}_h^{(m,q)}(f,\varphi)-\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}\right)\right)\tag{2}$$

and also for a sequence of values of r tending to infinity, we get that

$$T_f(r)\geqslant T_h\left(\exp^{[m-1]}\left(\left(\sigma_h^{(m,q)}(f,\varphi)-\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}\right)\right),\tag{3}$$

$$T_f(r)\leqslant T_h\left(\exp^{[m-1]}\left(\left(\overline{\sigma}_h^{(m,q)}(f,\varphi)+\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}\right)\right).\tag{4}$$

Similarly from the definitions of $\sigma_h^{(m,p)}(g)$ and $\overline{\sigma}_h^{(m,p)}(g)$, it follows for all sufficiently large values of r that

$$T_g(r)\leqslant T_h\left(\exp^{[m-1]}\left(\left(\sigma_h^{(m,p)}(g)+\varepsilon\right)\left(\log^{[p-1]}r\right)^{\rho_h^{(m,p)}(g)}\right)\right)$$

$$\text{i.e., } T_h(r)\geqslant T_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}r}{\left(\sigma_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right)\text{ and}\tag{5}$$

$$T_h(r)\leqslant T_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}r}{\left(\overline{\sigma}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right).\tag{6}$$

Also for a sequence of values of r tending to infinity, we obtain that

$$T_h(r)\leqslant T_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}r}{\left(\sigma_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right)\text{ and}\tag{7}$$

$$T_h(r) \geqslant T_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\overline{\sigma}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right). \tag{8}$$

Further from the definitions of $\tau_h^{(m,q)}(f,\varphi)$ and $\overline{\tau}_h^{(m,q)}(f,\varphi)$, we have for all sufficiently large values of r that

$$T_f(r) \leqslant T_h\left(\exp^{[m-1]}\left(\left(\tau_h^{(m,q)}(f,\varphi)+\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\lambda_h^{(m,q)}(f,\varphi)}\right)\right), \tag{9}$$

$$T_f(r) \geqslant T_h\left(\exp^{[m-1]}\left(\left(\overline{\tau}_h^{(m,q)}(f,\varphi)-\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\lambda_h^{(m,q)}(f,\varphi)}\right)\right) \tag{10}$$

and also for a sequence of values of r tending to infinity, we get that

$$T_f(r) \geqslant T_h\left(\exp^{[m-1]}\left(\left(\tau_h^{(m,q)}(f,\varphi)-\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\lambda_h^{(m,q)}(f,\varphi)}\right)\right), \tag{11}$$

$$T_f(r) \leqslant T_h\left(\exp^{[m-1]}\left(\left(\overline{\tau}_h^{(m,q)}(f,\varphi)+\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\lambda_h^{(m,q)}(f,\varphi)}\right)\right). \tag{12}$$

Similarly from the definitions of $\tau_h^{(m,p)}(g)$ and $\overline{\tau}_h^{(m,p)}(g)$, it follows for all sufficiently large values of r that

$$T_h(r) \geqslant T_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\tau_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right) \text{ and} \tag{13}$$

$$T_h(r) \leqslant T_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\overline{\tau}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right) \tag{14}$$

Also for a sequence of values of r tending to infinity, we obtain that

$$T_h(r) \leqslant T_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\tau_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right) \text{ and} \tag{15}$$

$$T_h(r) \geqslant T_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right). \tag{16}$$

Now from (3) and in view of (13), we get for a sequence of values of r tending to infinity that

$$\log^{[p-1]} T_g^{-1}(T_f(r))$$
$$\geqslant \log^{[p-1]} T_g^{-1}\left(T_h\left(\exp^{[m-1]}\left(\left(\sigma_h^{(m,q)}(f,\varphi)-\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}\right)\right)\right)$$

i.e.,

$$\log^{[p-1]} T_g^{-1}\left(T_f(r)\right) \geqslant \left(\frac{\left(\sigma_h^{(m,q)}(f,\varphi)-\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}}{\left(\tau_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

i.e.,

$$\frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left(\log^{[q-1]}\varphi(r)\right)^{\frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}}} \geqslant \left(\frac{\left(\sigma_h^{(m,q)}(f,\varphi)-\varepsilon\right)}{\left(\tau_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}.$$

Since in view of Lemma 1 $\frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)} \geqslant \rho_g^{(p,q)}(f,\varphi)$, and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r\to+\infty}\frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left(\log^{[q-1]}\varphi(r)\right)^{\rho_g^{(p,q)}(f,\varphi)}} \geqslant \left(\frac{\sigma_h^{(m,q)}(f,\varphi)-\varepsilon}{\tau_h^{(m,p)}(g)+\varepsilon}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

i.e.,

$$\sigma_g^{(p,q)}(f,\varphi) \geqslant \left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}. \tag{17}$$

Analogously from (2) and (16), we get that

$$\sigma_g^{(p,q)}(f,\varphi) \geqslant \left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \tag{18}$$

as in view of Lemma 1 it follows that $\frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)} \geqslant \rho_g^{(p,q)}(f,\varphi)$.

Again in view of (6), we have from (1) for all sufficiently large values of r that

$$\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)$$
$$\leqslant \log^{[p-1]} T_g^{-1}\left(T_h\left(\exp^{[m-1]}\left(\left(\sigma_h^{(m,q)}(f,\varphi)+\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}\right)\right)\right)$$

i.e.,

$$\log^{[p-1]} T_g^{-1}\left(T_f(r)\right) \leqslant \left(\frac{\left(\sigma_h^{(m,q)}(f,\varphi)+\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}}{\left(\overline{\sigma}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

i.e.,

$$\frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left(\log^{[q-1]}\varphi(r)\right)^{\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}}} \leqslant \left(\frac{\left(\sigma_h^{(m,q)}(f,\varphi)+\varepsilon\right)}{\left(\overline{\sigma}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Since in view of Lemma 1 it follows that $\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)} \leqslant \rho_g^{(p,q)}(f,\varphi)$ and $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left(\log^{[q-1]}\varphi(r)\right)^{\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}}} \leqslant \left(\frac{\sigma_h^{(m,q)}(f,\varphi)+\varepsilon}{\overline{\sigma}_h^{(m,p)}(g)-\varepsilon}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

i.e.,

$$\sigma_g^{(p,q)}(f,\varphi) \leqslant \left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}. \tag{19}$$

Thus the theorem follows from (17), (18) and (19). □

The conclusion of the following theorem can be carried out from (6) and (9); (9) and (14) respectively after applying the same technique of Theorem 1 and with the help of Lemma 1. Therefore its proof is omitted.

THEOREM 2. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0<\lambda_h^{(m,q)}(f,\varphi)<\infty$ and $0<\lambda_h^{(m,p)}(g)\leqslant\rho_h^{(m,p)}(g)<\infty$. Then*

$$\sigma_g^{(p,q)}(f,\varphi) \leqslant \min\left\{\left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}.$$

Similarly in the line of Theorem 1 and with the help of Lemma 1, one may easily carry out the following theorem from pairwise inequalities numbers (10) and (13); (7) and (9); (6) and (12) respectively and therefore its proofs is omitted:

THEOREM 3. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0<\lambda_h^{(m,q)}(f,\varphi)\leqslant\rho_h^{(m,q)}(f,\varphi)<\infty$ and $0<\lambda_h^{(m,p)}(g)\leqslant\rho_h^{(m,p)}(g)<\infty$. Then*

$$\left(\frac{\overline{\tau}_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\leqslant\overline{\tau}_g^{(p,q)}(f,\varphi)$$

$$\leqslant\min\left\{\left(\frac{\overline{\tau}_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}},\left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}.$$

THEOREM 4. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0<\rho_h^{(m,q)}(f,\varphi)<\infty$ and $0<\lambda_h^{(m,p)}(g)\leqslant\rho_h^{(m,p)}(g)<\infty$. Then*

$$\overline{\tau}_g^{(p,q)}(f,\varphi)\geqslant\max\left\{\left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}},\left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}.$$

With the help of Lemma 1, the conclusion of the above theorem can be carried out from (2), (5) and (2),(13) respectively after applying the same technique of Theorem 1 and therefore its proof is omitted.

THEOREM 5. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0<\rho_h^{(m,q)}(f,\varphi)<\infty$ and $0<\lambda_h^{(m,p)}(g)\leqslant\rho_h^{(m,p)}(g)<\infty$. Then*

$$\left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\leqslant\overline{\sigma}_g^{(p,q)}(f,\varphi)$$

$$\leqslant\min\left\{\left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}},\left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}.$$

Proof. From (2) and in view of (13), we get for all sufficiently large values of r that

$$\log^{[p-1]}T_g^{-1}\left(T_f(r)\right)$$

$$\geqslant\log^{[p-1]}T_g^{-1}\left(T_h\left(\exp^{[m-1]}\left(\left(\overline{\sigma}_h^{(m,q)}(f,\varphi)-\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}\right)\right)\right)$$

i.e.,

$$\log^{[p-1]}T_g^{-1}\left(T_f(r)\right)\geqslant\left(\frac{\left(\overline{\sigma}_h^{(m,q)}(f,\varphi)-\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}}{\left(\tau_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

i.e.,

$$\frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left(\log^{[q-1]}\varphi(r)\right)^{\frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}}} \geqslant \left(\frac{\left(\overline{\sigma}_h^{(m,q)}(f,\varphi)-\varepsilon\right)}{\left(\tau_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}.$$

Since in view of Lemma 1 $\frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)} \geqslant \rho_g^{(p,q)}(f,\varphi)$, and $\varepsilon(>0)$ is arbitrary, we get from above that

$$\liminf_{r\to+\infty}\frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left(\log^{[q-1]}\varphi(r)\right)^{\frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}}} \geqslant \left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)-\varepsilon}{\tau_h^{(m,p)}(g)+\varepsilon}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

i.e.,

$$\overline{\sigma}_g^{(p,q)}(f,\varphi) \geqslant \left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}. \tag{20}$$

Further in view of (7), we get from (1) for a sequence of values of r tending to infinity that

$$\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)$$
$$\leqslant \log^{[p-1]} T_g^{-1}\left(T_h\left(\exp^{[m-1]}\left(\left(\sigma_h^{(m,q)}(f,\varphi)+\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}\right)\right)\right)$$

i.e.,

$$\log^{[p-1]} T_g^{-1}\left(T_f(r)\right) \leqslant \left(\frac{\left(\sigma_h^{(m,q)}(f,\varphi)+\varepsilon\right)\left(\log^{[q-1]}\varphi(r)\right)^{\rho_h^{(m,q)}(f,\varphi)}}{\left(\sigma_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

i.e.,

$$\frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left(\log^{[q-1]}\varphi(r)\right)^{\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}}} \leqslant \left(\frac{\left(\sigma_h^{(m,q)}(f,\varphi)+\varepsilon\right)}{\left(\sigma_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Again as in view of Lemma 1, $\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)} \leqslant \rho_g^{(p,q)}(f,\varphi)$ and $\varepsilon(>0)$ is arbitrary, therefore we get from above that

$$\liminf_{r\to+\infty}\frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left(\log^{[q-1]}\varphi(r)\right)^{\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}}} \leqslant \left(\frac{\sigma_h^{(m,q)}(f,\varphi)+\varepsilon}{\sigma_h^{(m,p)}(g)-\varepsilon}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

i.e.,

$$\overline{\sigma}_g^{(p,q)}(f,\varphi) \leqslant \left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}. \tag{21}$$

Similarly from (4) and (6), we get that i.e.,

$$\overline{\sigma}_g^{(p,q)}(f,\varphi) \leqslant \left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \tag{22}$$

as in view of Lemma 1 it follows that $\frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)} \leqslant \rho_g^{(p,q)}(f,\varphi)$.

Thus the theorem follows from (20), (21) and (22). □

THEOREM 6. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \lambda_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \leqslant \rho_h^{(m,p)}(g) < \infty$. Then*

$$\overline{\sigma}_g^{(p,q)}(f,\varphi) \leqslant \min\left\{\left(\frac{\overline{\tau}_h^{(m,q)}(f,\varphi)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}},\right.$$
$$\left.\left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (6) and (12); (7) and (9); (12) and (14); (9) and (15) respectively after applying the same technique of Theorem 5 and with the help of Lemma 1. Therefore its proof is omitted.

Similarly in the line of Theorem 1 and with the help of Lemma 1, one may easily carry out the following theorem from pairwise inequalities numbered (11) and (13); (10) and (16); (6) and (9) respectively and therefore its proof is omitted:

THEOREM 7. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \lambda_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \leqslant \rho_h^{(m,p)}(g) < \infty$. Then*

$$\max\left\{\left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f,\varphi)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}$$
$$\leqslant \tau_g^{(p,q)}(f,\varphi) \leqslant \left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

THEOREM 8. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \lambda_h^{(m,q)}(f,\varphi) \leqslant \rho_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \leqslant \rho_h^{(m,p)}(g) < \infty$. Then*

$$\tau_g^{(p,q)}(f,\varphi) \geqslant \max\left\{ \left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \right\}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (3) and (5); (2) and (8); (3) and (13); (2) and (16) respectively after applying the same technique of Theorem 5 and with the help of Lemma 1. Therefore its proof is omitted.

Now we state the following two theorems without their proofs as because those can be derived easily using the same technique or with some easy reasoning with the help of Lemma 1 and therefore left to the readers.

THEOREM 9. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \rho_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \rho_h^{(m,p)}(g) \left(= \lambda_h^{(m,p)}(g)\right) < \infty$. Then*

$$\begin{aligned}
\left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}} &\leqslant \overline{\sigma}_g^{(p,q)}(f,\varphi) \\
&\leqslant \min\left\{ \left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \right\} \\
&\leqslant \max\left\{ \left(\frac{\overline{\sigma}_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \right\} \\
&\leqslant \sigma_g^{(p,q)}(f,\varphi) \leqslant \left(\frac{\sigma_h^{(m,q)}(f,\varphi)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.
\end{aligned}$$

REMARK 2. In Theorem 9, if we will replace the conditions "$0 < \rho_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \rho_h^{(m,p)}(g) \left(= \lambda_h^{(m,p)}(g)\right) < \infty$" by "$0 < \rho_h^{(m,q)}(f,\varphi) \left(= \lambda_h^{(m,q)}(f,\varphi)\right) < \infty$ and $0 < \rho_h^{(m,p)}(g) < \infty$" respectively, then Theorem 9 remains valid with $\overline{\tau}_g^{(p,q)}(f,\varphi)$ and $\tau_g^{(p,q)}(f,\varphi)$ replaced by $\overline{\sigma}_g^{(p,q)}(f,\varphi)$ and $\sigma_g^{(p,q)}(f,\varphi)$ respectively.

THEOREM 10. (Main) *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \rho_h^{(m,q)}(f,\varphi)\left(=\lambda_h^{(m,q)}(f,\varphi)\right) < \infty$ and $0 < \lambda_h^{(m,p)}(g) < \infty$. Then*

$$\begin{aligned}\left(\frac{\overline{\tau}_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} &\leqslant \overline{\sigma}_g^{(p,q)}(f,\varphi)\\ &\leqslant \min\left\{\left(\frac{\overline{\tau}_h^{(m,q)}(f,\varphi)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}\\ &\leqslant \max\left\{\left(\frac{\overline{\tau}_h^{(m,q)}(f,\varphi)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}\\ &\leqslant \sigma_g^{(p,q)}(f,\varphi) \leqslant \left(\frac{\tau_h^{(m,q)}(f,\varphi)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}.\end{aligned}$$

REMARK 3. In Theorem 10, if we will replace the conditions "$0 < \rho_h^{(m,q)}(f,\varphi)\left(=\lambda_h^{(m,q)}(f,\varphi)\right) < \infty$ and $0 < \lambda_h^{(m,p)}(g) < \infty$" by "$0 < \lambda_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \rho_h^{(m,p)}(g)\left(=\lambda_h^{(m,p)}(g)\right) < \infty$" respectively, then Theorem 10 remains valid with $\overline{\tau}_g^{(p,q)}(f,\varphi)$ and $\tau_g^{(p,q)}(f,\varphi)$ replaced by $\overline{\sigma}_g^{(p,q)}(f,\varphi)$ and $\sigma_g^{(p,q)}(f,\varphi)$ respectively.

REMARK 4. If we take "f be an entire function" instead of "f be a meromorphic function" in the above results and the other conditions remain the same then in view of Remark 1, one can easily derive the same conclusion of the above results using the maximum modulus of entire functions.

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