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# ON ABSOLUTE MATRIX SUMMABILITY FACTORS OF INFINITE SERIES

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*Abstract.* In the present paper, a general theorem dealing with $|A,p_n;\delta|_k$ summability method of infinite series has been proved by using almost increasing sequences. Some results have also been given.

## 1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums $(s_n)$. Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad (n\to\infty), \quad (P_{-i}=p_{-i}=0, \quad i \geqslant 1). \tag{1}$$

Let $A=(a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then $A$ defines the sequence-to-sequence transformation, mapping the sequence $s=(s_n)$ to $As=(A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n=0,1,\ldots \tag{2}$$

The series $\sum a_n$ is said to be summable $|A,p_n;\delta|_k$, $k \geqslant 1$ and $\delta \geqslant 0$, if (see [10])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |\bar{\Delta} A_n(s)|^k < \infty, \tag{3}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $\delta=0$, then $|A,p_n;\delta|_k$ summability reduces to $|A,p_n|_k$ summability (see [18]). If we take $\delta=0$, $a_{nv}=\frac{p_v}{P_n}$, then we get $|\bar{N},p_n|_k$ summability (see [2]). Furthermore, if we take $\delta=0$, $a_{nv}=\frac{p_v}{P_n}$ and $p_n=1$ for all values of $n$, then $|A,p_n;\delta|_k$ summability reduces to $|C,1|_k$ summability (see [7]).

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## 2. Known result

In [3], Bor has proved the following theorem for $\left|\bar{N},p_n\right|_k$ summability factors of infinite series using positive non-decreasing sequence.

THEOREM 1. *Let* $(X_n)$ *be a positive non-decreasing sequence and let there be sequences* $(\beta_n)$ *and* $(\lambda_n)$ *such that*

$$|\Delta\lambda_n| \leqslant \beta_n, \tag{4}$$

$$\beta_n \to 0 \quad as \quad n\to\infty, \tag{5}$$

$$\sum_{n=1}^{\infty} n\,|\Delta\beta_n|\,X_n < \infty, \tag{6}$$

$$|\lambda_n|\,X_n = O(1). \tag{7}$$

*If*

$$\sum_{v=1}^{n} \frac{|s_v|^k}{v} = O(X_n) \quad as \quad n\to\infty \tag{8}$$

*and* $(p_n)$ *is a sequence such that*

$$P_n = O(np_n), \tag{9}$$

$$P_n\Delta p_n = O(p_np_{n+1}), \tag{10}$$

*then the series* $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ *is summable* $|\bar{N},p_n|_k$, $k\geqslant 1$.

REMARK 1. It should be noted that, from the hypotheses of Theorem 1, $(\lambda_n)$ is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

## 3. Main result

A positive sequence $(b_n)$ is said to be almost increasing if there exists a positive increasing sequence $(c_n)$ and two positive constants $K$ and $L$ such that $Kc_n \leqslant b_n \leqslant Lc_n$ (see [1]). Many works on almost increasing sequences have been done (see [4]–[6], [11]–[17]). The purpose of this paper is to generalize Theorem 1 for $|A,p_n;\delta|_k$ summability. Before giving the main theorem, we must first introduce some further notations.

Given a normal matrix $A=(a_{nv})$, we associate two lower semimatrices $\bar{A}=(\bar{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n,v=0,1,\ldots \tag{11}$$

and

$$\hat{a}_{00}=\bar{a}_{00}=a_{00}, \quad \hat{a}_{nv}=\bar{a}_{nv}-\bar{a}_{n-1,v}, \quad n=1,2,\ldots \tag{12}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{i=0}^{n} \bar{a}_{ni} a_i \tag{13}$$

and

$$\bar{\Delta} A_n(s) = \sum_{i=0}^{n} \hat{a}_{ni} a_i. \tag{14}$$

Now, we shall prove the following theorem.

THEOREM 2. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \; n = 0, 1, \ldots, \tag{15}$$

$$a_{n-1,v} \geqslant a_{nv}, \; for \; n \geqslant v+1, \tag{16}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{17}$$

$$\mid \hat{a}_{n,v+1} \mid = O(v \mid \Delta_v(\hat{a}_{nv}) \mid), \tag{18}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v \hat{a}_{nv}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{p_v}{P_v}\right) \quad as \quad m \to \infty, \tag{19}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\right) \quad as \quad m \to \infty. \tag{20}$$

*Let $(X_n)$ be an almost increasing sequence. If conditions (4)–(7) and (9)–(10) of Theorem 1 and*

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|s_v|^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{21}$$

*are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $|A, p_n; \delta|_k$, $k \geqslant 1$ and $0 \leqslant \delta < 1/k$.*

We should give the following lemmas for the proof of Theorem 2.

LEMMA 1. ([8]) *If $(X_n)$ is an almost increasing sequence, then under the conditions (5)–(6), we have*

$$n X_n \beta_n = O(1) \quad as \quad n \to \infty, \tag{22}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{23}$$

LEMMA 2. ([9]) *If the conditions (9) and (10) are satisfied, then we have*

$$\Delta\left(\frac{P_n}{n p_n}\right) = O\left(\frac{1}{n}\right). \tag{24}$$

## 4. Proof of Theorem 2

Let $(M_n)$ denotes the $A$-transform of the series $\sum \frac{a_n \lambda_n P_n}{n p_n}$. Then, we have

$$\bar{\Delta} M_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v \lambda_v P_v}{v p_v}$$

by (13) and (14). By applying Abel's transformation, we get

$$\begin{aligned}
\bar{\Delta} M_n &= \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_{nv} \lambda_v P_v}{v p_v} \right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_{nv} \lambda_v P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v(\hat{a}_{nv})}{v p_v} s_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} \Delta \lambda_v P_{v+1}}{(v+1) p_{v+1}} s_v \\
&\quad + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_v \Delta \left( \frac{P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 2, by Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \mid M_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

First, by applying Hölder's inequality with indices $k$ and $k'$, where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |M_{n,1}|^k &\leqslant \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left\{ \sum_{v=1}^{n-1} \left( \frac{P_v}{v p_v} \right) |\Delta_v(\hat{a}_{nv})| \, |\lambda_v| |s_v| \right\}^k \\
&\leqslant \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left\{ \sum_{v=1}^{n-1} \left( \frac{P_v}{v p_v} \right)^k |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |s_v|^k \right\} \\
&\quad \times \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right\}^{k-1}.
\end{aligned}$$

By (11) and (12), we have

$$\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} = a_{nv} - a_{n-1,v}.$$

Thus using (11), (15) and (16)

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \leqslant a_{nn}.$$

Hence,

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,1}|^k &\leqslant \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k \frac{1}{v^k}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\right\}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k \frac{1}{v^k}|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{v^k}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\lambda_v|\frac{|s_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{i=1}^{v}\left(\frac{P_i}{p_i}\right)^{\delta k}\frac{|s_i|^k}{i}+O(1)|\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|s_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)|\lambda_m|X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

By using (9) and Hölder's inequality, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||s_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||s_v|^k\right\}\\
&\quad\times\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v\,|s_v|^k\right\}\\
&= O(1)\sum_{v=1}^{m}\beta_v\,|s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k} v\beta_v\frac{|s_v|^k}{v}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{i=1}^{v}\left(\frac{P_i}{p_i}\right)^{\delta k}\frac{|s_i|^k}{i}+O(1)m\beta_m\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|s_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v+O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)m\beta_m X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

Since $\Delta\left(\frac{P_v}{vp_v}\right)=O\left(\frac{1}{v}\right)$ by (24), as in $M_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v||s_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v|^k|s_v|^k\\
&\quad\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v|^k|s_v|^k\\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\lambda_v|\frac{|s_v|^k}{v}\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypothese of Theorem 2 and Lemma 1.

Finally, as in $M_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{n^k}\left(\frac{P_n}{p_n}\right)^k|\lambda_n|^{k-1}|\lambda_n||s_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}|\lambda_n|\frac{|s_n|^k}{n}\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

## 5. Conclusions

If we take $(X_n)$ as a positive non-decreasing sequence, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 2, then we get Theorem 1. In this case, the condition (21) reduces to the condition (8). Also, the conditions (15)–(20) are automatically satisfied. Also, if we take $\delta = 0$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of $n$, then we get a result for $|C,1|_k$ summability.

REFERENCES


[1] N. K. BARI AND S. B. STEČKIN, *Best approximations and differential properties of two conjugate functions*, Trudy. Moskov. Mat. Obšč. **5** (1956), 483–522 (in Russian).

[2] H. BOR, *On two summability methods*, Math. Proc. Cambridge Philos. Soc. **97** (1985), 147–149.

[3] H. BOR, *A note on $|\bar{N}, p_n|_k$ summability factors of infinite series*, Indian J. Pure Appl. Math. **18** (1987), 330–336.

[4] H. BOR, *On absolute Riesz summability factors*, Adv. Stud. Contemp. Math. (Pusan), **3** (2) (2001), 23–29.

[5] H. BOR, *A note on absolute Riesz summability factors*, Math. Inequal. Appl. **10** (2007), 619–625.

[6] H. BOR, *A new application of almost increasing sequences*, J. Comput. Anal. Appl. **10** (2008), 17–23.

[7] T. M. FLETT, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. **7** (1957), 113–141.

[8] S. M. MAZHAR, *A note on absolute summability factors*, Bull. Inst. Math. Acad. Sinica **25** (1997), 233–242.

[9] K. N. MISHRA AND R. S. L. SRIVASTAVA, *On $|\bar{N}, p_n|$ summability factors of infinite series*, Indian J. Pure Appl. Math. **15** (1984), 651–656.

[10] H. S. ÖZARSLAN AND H. N. ÖĞDÜK, *Generalizations of two theorems on absolute summability methods*, Aust. J. Math. Anal. Appl. **1** (2004), Article 13, 7 pp.

[11] H. S. ÖZARSLAN, *A new application of almost increasing sequences*, Miskolc Math. Notes. **14** (2013), 201–208.

[12] H. S. ÖZARSLAN, *On generalized absolute matrix summability*, Asia Pacific J. Math. **1** (2) (2014), 150–156.

[13] H. S. ÖZARSLAN, *A new application of absolute matrix summability*, C. R. Acad. Bulgare Sci. **68** (2015), 967–972.

[14] H. S. ÖZARSLAN, *A new study on generalized absolute matrix summability*, Commun. Math. Appl. **4** (2016), 303–309.

[15] H. S. ÖZARSLAN, *A new application of generalized almost increasing sequences*, Bull. Math. Anal. Appl. **8** (2016), 9–15.

[16] H. S. ÖZARSLAN AND A. KARAKAŞ, *A new result on the almost increasing sequences*, J. Comp. Anal. Appl. **22** (2017), 989–998.

[17] H. S. ÖZARSLAN AND B. KARTAL, *A generalization of a theorem of Bor*, J. Inequal. Appl. **179** (2017), 1–8.

[18] W. T. SULAIMAN, *Inclusion theorems for absolute matrix summability methods of an infinite series. IV*, Indian J. Pure Appl. Math. **34** (11) (2003), 1547–1557.


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