ON ABSOLUTE MATRIX SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract. In the present paper, a general theorem dealing with $|A, p_n; \delta|_k$ summability method of infinite series has been proved by using almost increasing sequences. Some results have also been given.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums $(s_n)$. Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \quad \text{as} \quad (n \to \infty), \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then $A$ defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{\nu=0}^{n} a_{nv}s_\nu, \quad n = 0, 1, \ldots$$

The series $\sum a_n$ is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [10])

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{P_{n-1}} \right)^{\delta + k - 1} |\Delta A_n(s)|^k < \infty,$$

where

$$\Delta A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $\delta = 0$, then $|A, p_n; \delta|_k$ summability reduces to $|A, p_n|_k$ summability (see [18]). If we take $\delta = 0$, $a_{nv} = \frac{p_v}{p_n}$, then we get $|N, p_n|_k$ summability (see [2]). Furthermore, if we take $\delta = 0$, $a_{nv} = \frac{p_v}{p_n}$ and $p_n = 1$ for all values of $n$, then $|A, p_n; \delta|_k$ summability reduces to $|C, 1|_k$ summability (see [7]).


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2. Known result

In [3], Bor has proved the following theorem for \( \left| N, p_n \right|_k \) summability factors of infinite series using positive non-decreasing sequence.

**Theorem 1.** Let \((X_n)\) be a positive non-decreasing sequence and let there be sequences \((\beta_n)\) and \((\lambda_n)\) such that

\[
|\Delta \lambda_n| \leq \beta_n, \quad \beta_n \to 0 \quad \text{as} \quad n \to \infty, \quad (4)
\]

\[
\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \quad (6)
\]

\[
|\lambda_n| X_n = O(1). \quad (7)
\]

If

\[
\sum_{v=1}^{n} \frac{|s_v|^k}{v} = O(X_n) \quad \text{as} \quad n \to \infty \quad (8)
\]

and \((p_n)\) is a sequence such that

\[
P_n = O(np_n), \quad (9)
\]

\[
P_n \Delta p_n = O(p_n p_{n+1}), \quad (10)
\]

then the series \(\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{p_n}\) is summable \(\left| N, p_n \right|_k, k \geq 1\).

**Remark 1.** It should be noted that, from the hypotheses of Theorem 1, \((\lambda_n)\) is bounded and \(\Delta \lambda_n = O(1/n)\) (see [3]).

3. Main result

A positive sequence \((b_n)\) is said to be almost increasing if there exists a positive increasing sequence \((c_n)\) and two positive constants \(K\) and \(L\) such that \(Kc_n \leq b_n \leq Lc_n\) (see [1]). Many works on almost increasing sequences have been done (see [4]–[6], [11]–[17]). The purpose of this paper is to generalize Theorem 1 for \(|A, p_n; \delta|_k\) summability. Before giving the main theorem, we must first introduce some further notations.

Given a normal matrix \(A = (a_{nv})\), we associate two lower semimatrices \(\bar{A} = (\bar{a}_{nv})\) and \(\hat{A} = (\hat{a}_{nv})\) as follows:

\[
\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \quad (11)
\]

and

\[
\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots \quad (12)
\]
It may be noted that $\Lambda$ and $\hat{\Lambda}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{i=0}^{n} \bar{a}_{ni}a_i$$  \hspace{1cm} (13)

and

$$\tilde{\Lambda}A_n(s) = \sum_{i=0}^{n} \hat{a}_{ni}a_i.$$  \hspace{1cm} (14)

Now, we shall prove the following theorem.

**Theorem 2.** Let $A = (a_{nv})$ be a positive normal matrix such that

$$\sigma_{n0} = 1, \ n = 0, 1, \ldots,$$  \hspace{1cm} (15)

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1,$$  \hspace{1cm} (16)

$$a_{nn} = O\left(\frac{p_n}{p_n}\right),$$  \hspace{1cm} (17)

$$|\hat{a}_{n,v+1}| = O\left(\frac{\delta k}{|\Delta v|}\right),$$  \hspace{1cm} (18)

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta v| = O\left(\frac{\delta k}{p_v}\right)$$  \hspace{1cm} as $m \to \infty$,  \hspace{1cm} (19)

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left(\frac{\delta k}{p_v}\right)$$  \hspace{1cm} as $m \to \infty$.  \hspace{1cm} (20)

Let $(X_n)$ be an almost increasing sequence. If conditions (4)–(7) and (9)–(10) of Theorem 1 and

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} |\sigma_v|^{1/k} = O(X_n)$$ \hspace{1cm} as $n \to \infty$,  \hspace{1cm} (21)

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{p_{\lambda n}}{np_n}$ is summable $|\Lambda, p_n; \delta|$, $k \geq 1$ and $0 \leq \delta < 1/k$.

We should give the following lemmas for the proof of Theorem 2.

**Lemma 1.** ([8]) If $(X_n)$ is an almost increasing sequence, then under the conditions (5)–(6), we have

$$nX_n\beta_n = O(1)$$ \hspace{1cm} as $n \to \infty$,  \hspace{1cm} (22)

$$\sum_{n=1}^{\infty} \beta_nX_n < \infty.$$  \hspace{1cm} (23)

**Lemma 2.** ([9]) If the conditions (9) and (10) are satisfied, then we have

$$\Delta \left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right).$$  \hspace{1cm} (24)
4. Proof of Theorem 2

Let \((M_n)\) denotes the \(A\)-transform of the series \(\sum \frac{a_n \lambda_n P_n}{np_n}\). Then, we have

\[
\overline{\Delta M_n} = \sum_{v=1}^{n} a_v \frac{\lambda_v P_v}{vp_v}
\]

by (13) and (14). By applying Abel’s transformation, we get

\[
\overline{\Delta M_n} = \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_v \lambda_v P_v}{vp_v} \right) s_v + a_{nn} \frac{\lambda_n P_n}{np_n} s_n
\]

\[
= \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v (\hat{a}_v)}{vp_v} s_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} \lambda_{v+1} P_{v+1}}{(v+1)p_{v+1}} s_v
\]

\[
+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_v \Delta \left( \frac{P_v}{vp_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{np_n} s_n
\]

\[
= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}.
\]

To complete the proof of Theorem 2, by Minkowski’s inequality, it is enough to show that

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right) k + k - 1 |M_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4.
\]

First, by applying Hölder’s inequality with indices \(k\) and \(k'\), where \(k > 1\) and \(\frac{1}{k} + \frac{1}{k'} = 1\), we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) k + k - 1 \left| M_{n,1} \right|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) k + k - 1 \left\{ \sum_{v=1}^{n-1} \left( \frac{P_v}{vp_v} \right) \left| \Delta_v (\hat{a}_v) \right| \left| \lambda_v \right| \left| s_v \right| \right\}^k
\]

\[
\leq \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) k + k - 1 \left\{ \sum_{v=1}^{n-1} \left( \frac{P_v}{vp_v} \right) \left| \Delta_v (\hat{a}_v) \right| \left| \lambda_v \right| \left| s_v \right| \right\}^k
\]

\[
\times \left\{ \sum_{v=1}^{n-1} \left| \Delta_v (\hat{a}_v) \right| \right\}^{k-1}.
\]

By (11) and (12), we have

\[
\Delta_v (\hat{a}_v) = \hat{a}_v - \hat{a}_{n,v+1} = \hat{a}_v - \hat{a}_{n-1,v} - \hat{a}_{n,v+1} = \hat{a}_v - \hat{a}_{n-1,v} + \hat{a}_{n-1,v+1} = a_v - a_{n-1,v}.
\]

Thus using (11), (15) and (16)

\[
\sum_{v=1}^{n-1} \left| \Delta_v (\hat{a}_v) \right| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_v) \leq a_{nn}.
\]
Hence,

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^{k+1} |M_{n,1}|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^{k+1} \left\{ \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \right) \frac{1}{v^k} |\Delta_v(\hat{a}_m)| |\hat{\lambda}_v| |s_v| \right\}^k
\]

\[= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right) \frac{1}{v^k} |\lambda_v| |s_v| \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^n |\Delta_v(\hat{a}_m)|
\]

\[= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right) \delta^n |\lambda_v| |s_v| \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^n |\Delta_v(\hat{a}_m)|
\]

\[= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right) \delta^n |\lambda_v| |s_v| \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^n |\Delta_v(\hat{a}_m)|
\]

\[= O(1) \delta^n |\lambda_v| |s_v| \sum_{v=1}^{m} \Delta |\lambda_v| X_v + O(1) |\lambda_m| X_m
\]

\[= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) |\lambda_m| X_m
\]

\[= O(1) \text{ as } m \to \infty,
\]

by virtue of the hypotheses of Theorem 2 and Lemma 1.

By using (9) and Hölder’s inequality, we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^{k+1} |M_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^{k+1} \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta_v(\lambda_v)| |s_v| \right\}^k
\]

\[= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^{k+1} \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta_v(\lambda_v)| |s_v| \right\}^k
\]

\[\times \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta_v(\lambda_v)| \right\}^{k-1}
\]

\[= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^{k+1} \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta_v(\lambda_v)| |s_v| \right\}^k
\]

\[= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^{k+1} \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta_v(\lambda_v)| |s_v| \right\}^k
\]

\[= O(1) \sum_{v=1}^{m} \beta_v |s_v|^k \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right) |\hat{a}_{n,v+1}|
\]

\[= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right) \delta^n |\lambda_v| |s_v| \sum_{v=1}^{m} \beta_v |s_v|^k
\]
by virtue of the hypotheses of Theorem 2 and Lemma 1.

Since \( \Delta \left( \frac{P_n}{P_v} \right) = O \left( \frac{1}{v} \right) \) by (24), as in \( M_{n,1} \), we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_v} \right)^{\delta k+1-k} |M_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_v} \right)^{\delta k+1-k} \left\{ \sum_{v=1}^{n-1} \frac{1}{v} |\hat{\alpha}_{n,v+1}||\hat{\lambda}_v||s_v| \right\}^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_v} \right)^{\delta k+1-k} \sum_{v=1}^{n-1} \frac{1}{v} |\hat{\alpha}_{n,v+1}||\hat{\lambda}_v||s_v|^k
\]

\[
\times \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{\alpha}_v)| \right\}^{k-1}
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_v} \right)^{\delta k+1-k} |\hat{\alpha}_{n,n-1}| \sum_{v=1}^{n-1} \frac{1}{v} |\hat{\alpha}_{n,v+1}||\hat{\lambda}_v||s_v|^k
\]

\[
= O(1) \sum_{v=1}^{m} \frac{1}{v} |\lambda_v| |\lambda_v| |s_v| |s_{v+1}| \sum_{n=v+1}^{m+1} \left( \frac{P_n}{P_v} \right)^{\delta k} |\hat{\alpha}_{n,v+1}|
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{P_v} \right)^{\delta k} |\lambda_v||s_v|^k
\]

by virtue of the hypothesis of Theorem 2 and Lemma 1.

Finally, as in \( M_{n,1} \), we have that

\[
\sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k+1-k} |M_{n,4}|^k = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k+1-k} \left( \frac{P_n}{P_n} \right)^k \left( \frac{P_n}{P_n} \right)^k |\lambda_n|^{k-1} |\lambda_n| |s_n|^k
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k} |\lambda_n||s_n|^k
\]

by virtue of the hypotheses of Theorem 2 and Lemma 1.
5. Conclusions

If we take \((X_n)\) as a positive non-decreasing sequence, \(\delta = 0\) and \(a_m = \frac{p_m}{p_n}\) in Theorem 2, then we get Theorem 1. In this case, the condition (21) reduces to the condition (8). Also, the conditions (15)–(20) are automatically satisfied. Also, if we take \(\delta = 0\), \(a_m = \frac{p_m}{p_n}\) and \(p_n = 1\) for all values of \(n\), then we get a result for \(|C, 1|_k\) summability.

REFERENCES


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