Journal of Classical Analysis
Volume 13, Number 2 (2018), 141–149 doi:10.7153/jca-2018-13-10

# ON THE CONVERGENCE OF SERIES WITH RECURSIVELY DEFINED TERMS

N. S. HOANG

*Abstract.* We investigate the asymptotic behavior of a sequence $(x_n)_{n=0}^{\infty}$ defined recursively by $x_{n+1} = f(x_n)$, $n \geqslant 0$ where $f : [0,\infty) \to [0,\infty)$ is a continuous function. A fundamental criterion on the function $f(x)$ for estimating the rate of decay of $x_n$ as $n$ tends to $\infty$ and for testing convergence of the series $\sum_{n=0}^{\infty} x_n$ is proposed and justified. Criteria for testing absolute and conditional convergence of $\sum_{n=0}^{\infty} x_n$ when $f(x)$ is not a non-negative function are also formulated and proved.

## 1. Introduction

In this paper we study the decay rate of a sequence $(x_n)_{n=0}^{\infty}$ and the convergence of the series $\sum_{n=0}^{\infty} x_n$ where

$$x_{n+1} := f(x_n), \qquad n \geqslant 0, \quad x_0 > 0, \tag{1}$$

and $f : [0,\infty) \to [0,\infty)$ is a continuous function satisfying the inequality $0 < f(x) < x$, $\forall x \in (0,b]$ for some $b > 0$.

The convergence of the series $\sum_{n=0}^{\infty} x_n$ was studied earlier in [2]. In particular, in [2] the following result was formulated and proved:

PROPOSITION 1. *Consider a differentiable function $f : (0,\infty) \to (0,\infty)$ with the property that $0 < f(x) < x$ for all $x \in (0,\infty)$ and a sequence $(x_n)_{n=0}^{\infty}$ with the properties*

*a)* $\lim_{n\to\infty} x_n = 0$, *with $x_n > 0$ for all $n \in \mathbb{N}$;*

*b)* $x_{n+1} = f(x_n)$*;*

*c) the limit*

$$\lim_{x\to 0} \frac{x^a - f^a(x)}{x^a f^a(x)} = \frac{1}{k^a}$$

*holds for some $a > 0$, $k > 0$.*

*Then*

*i)* $\lim_{n\to\infty} n^{\frac{1}{a}} x_n = k$.

*ii) The series $\sum_{n=0}^{\infty} x_n$ diverges if $a \geqslant 1$.*

---

*Mathematics subject classification* (2010): 40A05, 40F05.
*Keywords and phrases*: Convergence, series, rate of decay, comparison test.


Paper JCA-13-10

*iii) The series* $\sum_{n=0}^{\infty} x_n$ *converges if* $a < 1$.

Using Proposition 1, the authors were able to estimate the rate of decay of $(x_n)_{n=0}^{\infty}$ as well as to test the convergence of $\sum_{n=0}^{\infty} x_n$ for several choices of $f(x)$ (see [2]). However, the application of Proposition 1 requires an appropriate choice of $a$ for the limit in c) to hold and this is not a trivial task in practice. Our goal in this paper is to derive a criterion simpler than Proposition 1 for estimating the rate of decay of $(x_n)_{n=0}^{\infty}$ and for testing the convergence of the series $\sum_{n=0}^{\infty} x_n$. Moreover, we also study the convergence of $\sum_{n=0}^{\infty} x_n$ when $f(x)$ is not necessary a non-negative function, i.e., $f(x)$ can take both positive and negative values.

REMARK 1. Condition a) $\lim_{n\to\infty} x_n = 0$ in Proposition 1 follows from the inequality $0 < f(x) < x$, $\forall x \in (0,\infty)$, the definition $x_{n+1} = f(x_n)$, and the continuity of $f(x)$. Indeed, we have $0 < x_{n+1} = f(x_n) < x_n$, $n \geqslant 0$. Thus, there exists $L \geqslant 0$ such that $\lim_{n\to\infty} x_n = L$, as the sequence $(x_n)_{n=0}^{\infty}$ is decreasing and bounded below by zero. Therefore, $L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(L)$. From the inequality $f(x) < x$, $\forall x > 0$, one concludes that $L = 0$, i.e., $\lim_{n\to\infty} x_n = 0$.

REMARK 2. The differentiability assumption on $f(x)$ in Proposition 1 was not used in its proof (see [2]). It suffices to assume that $f(x)$ is continuous.

## 2. Main results

The following theorem gives a fundamental criterion for estimating the rate of decay of $(x_n)_{n=0}^{\infty}$ and for testing the convergence of the series $\sum_{n=0}^{\infty} x_n$.

THEOREM 1. *Let* $f : [0,b] \to [0,b]$, *where* $b = constant > 0$, *be a continuous function. Assume that* $f(x)$ *satisfies simultaneously the inequality* $0 < f(x) < x$, $\forall x \in (0,b]$ *and the equation*

$$f(x) = x - cx^{1+\alpha} + o(x^{1+\alpha}) \quad as \quad x \to 0, \qquad \alpha > 0, \quad c > 0. \tag{2}$$

*Let* $(x_n)_{n=0}^{\infty}$ *be defined recursively by* (1) *where* $x_0 \in (0,b]$. *Then*

$$\lim_{n\to\infty} n^{\frac{1}{\alpha}} x_n = (c\alpha)^{-\frac{1}{\alpha}}. \tag{3}$$

*Consequently,*

- *The series* $\sum_{n=0}^{\infty} x_n$ *converges if* $0 < \alpha < 1$.
- *The series* $\sum_{n=0}^{\infty} x_n$ *diverges if* $\alpha \geqslant 1$.

*Proof.* From the Taylor series expansion

$$(1+x)^{\alpha} = 1 + \alpha x + o(x)$$

and equation (2), we have

$$\begin{aligned} f^{\alpha}(x) &= \left(x - cx^{1+\alpha} + o(x^{1+\alpha})\right)^{\alpha} = \left(x\left[1 - cx^{\alpha} + o(x^{\alpha})\right]\right)^{\alpha} \\ &= x^{\alpha}\left(1 - cx^{\alpha} + o(x^{\alpha})\right)^{\alpha} = x^{\alpha}\left(1 - \alpha cx^{\alpha} + o(x^{\alpha})\right). \end{aligned} \tag{4}$$

Thus,

$$\begin{aligned} \lim_{x\to 0} \frac{x^{\alpha} - f^{\alpha}(x)}{x^{\alpha} f^{\alpha}(x)} &= \lim_{x\to 0} \frac{x^{\alpha} - x^{\alpha}\left(1 - \alpha cx^{\alpha} + o(x^{\alpha})\right)}{x^{\alpha}x^{\alpha}\left(1 - \alpha cx^{\alpha} + o(x^{\alpha})\right)} \\ &= \lim_{x\to 0} \frac{x^{\alpha}\alpha cx^{\alpha} + o(x^{2\alpha})}{x^{2\alpha}\left(1 - \alpha cx^{\alpha} + o(x^{\alpha})\right)} = \alpha c. \end{aligned} \tag{5}$$

This and item i) in Proposition 1 imply (3).

It follows from (3) and the $p$-series test that the series $\sum_{n=0}^{\infty} x_n$ is convergent if $\alpha \in (0,1)$ and is divergent if $\alpha \geqslant 1$. Theorem 1 is proved. □

REMARK 3. Let $f(x)$ be as in Theorem 1. If $f(x)$ is analytic at zero and $\lim_{x\to 0} \frac{f(x)}{x} = 1$, then, from a Maclaurin series expansion of $f(x)$, it is clear that equation (2) holds for $\alpha \geqslant 1$. Thus, by Theorem 1, the series $\sum_{n=0}^{\infty} x_n$ where $(x_n)_{n=0}^{\infty}$ is defined by (1) is divergent.

The following result is similar to the comparison test for series and is useful for our study of convergence of the series $\sum_{n=0}^{\infty} x_n$.

THEOREM 2. *Let $f, g : [0,b] \to [0,\infty)$ be continuous functions. Assume that either $f$ or $g$ is increasing and that the following inequalities hold*

$$0 \leqslant g(x) < x \quad \textit{and} \quad g(x) \leqslant f(x), \qquad \forall x \in (0,b]. \tag{6}$$

*Let $x_0$ and $y_0$ be in $(0,b]$ and let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be recursively defined by*

$$x_{n+1} := f(x_n), \quad y_{n+1} := g(y_n), \quad n \geqslant 0. \tag{7}$$

*Then*

*i) If $\sum_{n=0}^{\infty} x_n$ is convergent, then $\sum_{n=0}^{\infty} y_n$ is convergent.*

*ii) If $\sum_{n=0}^{\infty} y_n$ is divergent, then $\sum_{n=0}^{\infty} x_n$ is divergent.*

*Proof.* Assume that $y_0 \leqslant x_0$. We claim that

$$0 \leqslant y_n \leqslant x_n, \qquad \forall n \geqslant 0. \tag{8}$$

We prove inequality (8) by induction. First, inequality (8) holds true for $n = 0$ by our assumption. Assume that inequality (8) holds true for some $n \geqslant 0$. If $g$ is increasing, then from our induction assumption that $y_n \leqslant x_n$ and the second inequality in (6), one gets

$$0 \leqslant y_{n+1} = g(y_n) \leqslant g(x_n) \leqslant f(x_n) = x_{n+1}.$$

If $f$ is increasing instead of $g$, then from the second inequality in (6) and our induction assumption that $y_n \leqslant x_n$, one obtains

$$0 \leqslant y_{n+1} = g(y_n) \leqslant f(y_n) \leqslant f(x_n) = x_{n+1}.$$

The inequality $0 \leqslant y_{n+1} \leqslant x_{n+1}$ and the induction principle imply (8).

From inequality (8) one gets $0 < \sum_{n=0}^{\infty} y_n \leqslant \sum_{n=0}^{\infty} x_n$. Conclusions i) and ii) of the theorem follow from this inequality.

Now consider the case when $y_0 > x_0$. Since $\lim_{n\to 0} y_n = 0$ as we mentioned before in Remark 1, there exists $n_0 > 0$ such that $0 < y_{n_0} \leqslant x_0$. From this inequality and similar arguments as above, one can prove by induction that $0 < y_{n+n_0} \leqslant x_n$, $\forall n \geqslant 0$. This implies

$$0 < \sum_{n=n_0}^{\infty} y_n = \sum_{n=0}^{\infty} y_{n+n_0} \leqslant \sum_{n=0}^{\infty} x_n.$$

Since the convergence of a series does not depend on the first finitely many terms, this inequality implies conclusions i) and ii) of the theorem. Theorem 2 is proved. □

From Theorem 2 we have the following corollary.

COROLLARY 1. *Let* $g : [0,b] \to [0,\infty)$ *be continuous and satisfy the inequality* $0 \leqslant g(x) < x$, $\forall x \in (0,b]$. *Let* $(y_n)_{n=0}^{\infty}$ *be recursively defined by* $y_{n+1} = g(y_n)$, $n \geqslant 0$, *and* $y_0 \in (0,b]$. *Then*

*i) If*

$$g(x) \geqslant x - c_1 x^{1+\alpha} \quad as \quad x \to 0, \qquad c_1 > 0, \quad \alpha \geqslant 1,$$

*then the series* $\sum_{n=0}^{\infty} y_n$ *is divergent.*

*ii) If*

$$g(x) \leqslant x - c_2 x^{1+\beta} \quad as \quad x \to 0, \qquad c_2 > 0, \quad \beta \in (0,1),$$

*then the series* $\sum_{n=0}^{\infty} y_n$ *is convergent.*

*Proof.* Let us prove part ii) of Corollary 1. Define

$$f(x) := x - \frac{c_2}{2} x^{1+\beta}, \qquad x \geqslant 0. \tag{9}$$

Choose $\delta \in (0,b]$ sufficiently small such that the function $f(x)$ is positive and increasing on $(0,\delta]$ and the following inequality holds

$$g(x) \leqslant x - \frac{c_2}{2} x^{1+\beta} = f(x), \qquad \forall x \in (0,\delta]. \tag{10}$$

It follows from equation (9), the assumption $\beta \in (0,1)$, and Theorem 1 that the series $\sum_{n=0}^{\infty} x_n$ where $(x_n)_{n=0}^{\infty}$ is defined by (1) is convergent. Since $0 \leqslant g(x) < x$, $\forall x \in (0,b]$, one gets $\lim_{n\to\infty} y_n = 0$ (see Remark 1). Thus, there exists $n_0 > 0$ such that $0 \leqslant y_n \leqslant \delta$, $\forall n \geqslant n_0$. This, inequality (10), the fact that $f$ is increasing on $(0,\delta]$ and the series $\sum_{n=0}^{\infty} x_n$ is convergent, and Theorem 2 imply that the series $\sum_{n=0}^{\infty} y_{n+n_0}$ is convergent. Thus, the series $\sum_{n=0}^{\infty} y_n$ is convergent. Part ii) of Corollary 1 is proved.

A proof of part i) can be obtained similarly by defining

$$f_0(x) := g(x), \qquad g_0(x) := x - 2c_1 x^{1+\alpha},$$

and applying Theorem 2 for the functions $g_0(x)$ and $f_0(x)$. We leave it to the reader to fill in the details. □

EXAMPLE 1. Consider the following functions

$$g_1(x) = \left|\sin\left(\frac{1}{x}\right)\right|\left(x - \frac{1}{2}x\sqrt{x}\right), \qquad x \in (0,1], \qquad g_1(0) := 0, \tag{11}$$

$$g_2(x) = x - x^2\left|\sin\left(\frac{1}{x}\right)\right|, \qquad x \in (0,1], \qquad g_2(0) := 0. \tag{12}$$

It is clear that $g_i : [0,1] \to [0,\infty)$ is continuous and $0 \leqslant g_i < x$, $\forall x \in (0,1]$, $i = 1,2$. In addition, we have

$$g_1(x) \leqslant x - \frac{1}{2}x\sqrt{x}, \qquad g_2(x) \geqslant x - x^2, \qquad x \in [0,1]. \tag{13}$$

Thus, by Corollary 1, the series $\sum_{n=0}^{\infty} y_n$ is convergent if $(y_n)_{n=0}^{\infty}$ is defined with $g(x) = g_1(x)$ and is divergent if $(y_n)_{n=0}^{\infty}$ is defined with $g(x) = g_2(x)$. Note that Theorem 1 is not applicable for the functions $g_1$ and $g_2$ in this example.

The following result allows us to study the convergence of series with terms that can be either negative or positive.

THEOREM 3. *Let $f : [0,b] \to [0,\infty)$ be increasing and let $g : [-b,b] \to \mathbb{R}$ be continuous and satisfy*

$$|g(x)| < |x|, \qquad |g(x)| \leqslant f(|x|), \qquad \forall x : |x| \in (0,b]. \tag{14}$$

*Let $x_0 \in (0,b]$ and $y_0 \in [-b,b]$ and let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be defined by*

$$x_{n+1} = f(x_n), \quad y_{n+1} = g(y_n), \quad n \geqslant 0.$$

*If $\sum_{n=0}^{\infty} x_n$ is convergent, then $\sum_{n=0}^{\infty} y_n$ is absolutely convergent.*

*Proof.* From inequality (14) and the definition of $(y_n)_{n=0}^{\infty}$, one gets

$$|y_{n+1}| = |g(y_n)| < |y_n|, \qquad n \geqslant 0. \tag{15}$$

Therefore, the sequence $(|y_n|)_{n=0}^{\infty}$ is decreasing and bounded below by $0$. Thus, there exists some $L \geqslant 0$ such that $\lim_{n\to\infty} |y_n| = L$. We claim that $L = 0$. Assume that $L > 0$. From the first inequality in (14) and the continuity of $g(x)$, one gets

$$L = \lim_{n\to\infty} |y_{n+1}| = \lim_{n\to\infty} |g(y_n)| = \begin{cases} |g(L)| < L & \text{or,} \\ |g(-L)| < L, \end{cases}$$

which is a contradiction. This contradiction implies that $L = 0$, i.e.,

$$\lim_{n\to\infty} |y_n| = 0. \tag{16}$$

Thus, there exists $n_0 > 0$ such that $|y_{n_0}| \leqslant x_0$.

Let us prove by induction that

$$|y_{n+n_0}| \leqslant x_n, \qquad n \geqslant 0. \tag{17}$$

Inequality (17) holds for $n = 0$. Assume that it is true for some $n \geqslant 0$. From the definition of $(y_n)_{n=0}^{\infty}$, the second inequality in (14) and the assumption that $f(x)$ is increasing, we get

$$|y_{n+1+n_0}| = |g(y_{n+n_0})| \leqslant f(|y_{n+n_0}|) \leqslant f(x_n) = x_{n+1}.$$

Thus, inequality (17) holds by induction principle. From inequality (17) and the convergence of $\sum_{n=0}^{\infty} x_n$, one obtains

$$0 \leqslant \sum_{n=n_0}^{\infty} |y_n| = \sum_{n=0}^{\infty} |y_{n+n_0}| \leqslant \sum_{n=0}^{\infty} x_n < \infty.$$

This implies that the series $\sum_{n=0}^{\infty} |y_n|$ is convergent or, equivalently, the series $\sum_{n=0}^{\infty} y_n$ is absolutely convergent. $\square$

The following result is a corollary of Theorem 3.

COROLLARY 2. *Let $g : [-b,b] \to \mathbb{R}$ be continuous and satisfy the following conditions:*

$$|g(x)| < |x|, \qquad \forall x : |x| \in (0,b], \tag{18}$$

$$|g(x)| \leqslant |x| - c|x|^{1+\alpha} + o(|x|^{1+\alpha}) \quad \textit{as} \quad x \to 0, \quad c > 0, \;\; \alpha \in (0,1). \tag{19}$$

*Let $(y_n)_{n=0}^{\infty}$ be defined by $y_{n+1} = g(y_n)$, $n \geqslant 0$, $y_0 \in [-b,b]$. Then the series $\sum_{n=0}^{\infty} y_n$ is absolutely convergent.*

*Proof.* Define

$$f(x) := x - \frac{cx^{1+\alpha}}{2}, \qquad x \geqslant 0. \tag{20}$$

It is clear that

$$0 < |g(x)| \leqslant |x| - \frac{c|x|^{1+\alpha}}{2} = f(|x|), \qquad \forall x : |x| \in (0,\delta], \tag{21}$$

and the function $f(x)$ is increasing on $(0,\delta]$, if $\delta > 0$ is sufficiently small. If $x_0 \in (0,\delta]$, then the series $\sum_{n=0}^{\infty} x_n$, with $(x_n)_{n=0}^{\infty}$ defined by equation (1), is convergent by Theorem 1 since $\alpha \in (0,1)$.

From similar arguments as in Theorem 3, we get $\lim_{n\to\infty} y_n = 0$ (see (16)). Thus, there exists $N_0 > 0$ such that $y_n \in [0,\delta)$, $\forall n \geqslant N_0$. This, inequality (21), the convergence of $\sum_{n=0}^{\infty} x_n$, and Theorem 3 imply that the series $\sum_{n=0}^{\infty} y_{N_0+n}$ is absolutely convergent. Thus, the series $\sum_{n=0}^{\infty} y_n$ is absolutely convergent. Corollary 2 is proved. □

If $\limsup_{x\to 0}\left|\frac{g(x)}{x}\right| < 1$, then inequality (19) holds for any given $c > 0$ and $\alpha \in (0,1)$. Thus, from Corollary 2 one gets the following result:

COROLLARY 3. *Let $g : [-b,b] \to \mathbb{R}$ be a continous function satisfying the inequality $|g(x)| < |x|$, $\forall x \in [-b,b]$. Assume that*

$$\limsup_{x\to 0}\left|\frac{g(x)}{x}\right| < 1. \tag{22}$$

*Let $(y_n)_{n=0}^{\infty}$ be defined by $y_{n+1} = g(y_n)$, $n \geqslant 0$, where $y_0 \in [-b,b]$. Then the series $\sum_{n=0}^{\infty} y_n$ is absolutely convergent.*

Let us discuss an application of Corollary 2.

EXAMPLE 2. Consider $g(x) := x\sin(\frac{1}{x})/(1+\sqrt{|x|})$, $x \in (0,1]$ and $g(0) := 0$. We have

$$|g(x)| = \frac{|x|\left|\sin(\frac{1}{x})\right|}{1+\sqrt{|x|}} < |x|, \qquad \forall x : |x| \in (0,1].$$

Moreover, from the equation

$$\frac{1}{1+\sqrt{|x|}} = \sum_{n=0}^{\infty}(-1)^n\left(\sqrt{|x|}\right)^n = 1 - \sqrt{|x|} + o\left(\sqrt{|x|}\right), \qquad \forall x \in (-1,1),$$

one gets

$$|g(x)| \leqslant \frac{|x|}{1+\sqrt{|x|}} = |x| - |x|^{1+\frac{1}{2}} + o(|x|^{1+\frac{1}{2}}) \qquad \text{as} \quad x \to 0. \tag{23}$$

Inequality (23) combined with Corollary 2 implies that the series $\sum_{n=0}^{\infty} y_n$ with $y_{n+1} = g(y_n)$, $n \geqslant 0$, $|y_0| \in (0,1]$ is absolutely convergent.

The following result provides a test for conditional convergence.

COROLLARY 4. *Let $g : [-b,b] \to \mathbb{R}$ be continuous and satisfy the following conditions*

$$g(x) \leqslant 0 \leqslant g(-x), \qquad \forall x \in [0,b], \tag{24}$$

$$|g(x)| < |x|, \qquad \forall x : |x| \in (0,b], \tag{25}$$

$$|g(x)| \geqslant |x| - c|x|^{1+\alpha} + o(|x|^{1+\alpha}) \quad \textit{as} \quad x \to 0, \quad c > 0, \;\; \alpha \geqslant 1. \tag{26}$$

*Let* $(y_n)_{n=0}^{\infty}$ *be defined by* $y_{n+1} = g(y_n)$, $n \geqslant 0$, $y_0 \in [-b,b]$. *Then the series* $\sum_{n=0}^{\infty} y_n$ *is conditionally convergent.*

*Proof.* It follows from inequality (24) that the series $\sum_{n=0}^{\infty} y_n$ is an alternating series. Moreover, from inequality (25) and similar arguments as in Theorem 3 (see (15) and (16)), one concludes that the sequence $(|y_n|)_{n=0}^{\infty}$ is monotonically decreasing to zero. Thus, the series $\sum_{n=0}^{\infty} y_n$ is convergent by the alternating series test.

Define

$$f(x) := x - (1+c)x^{1+\alpha}, \qquad x \geqslant 0. \tag{27}$$

From inequality (26) and equation (27) one concludes that there exists $\delta > 0$ which is sufficiently small such that $f(x)$ is increasing on $(0,\delta]$ and the following inequalities hold:

$$|g(x)| > |x| - (1+c)|x|^{1+\alpha} = f(|x|) > 0, \qquad \forall x : |x| \in (0,\delta]. \tag{28}$$

It follows from (27) and Theorem 1 that the series $\sum_{n=0}^{\infty} x_n$ is divergent for any given $x_0 \in (0,\delta]$ which will be chosen later. Since the sequence $(|y_n|)_{n=0}^{\infty}$ is monotonically decreasing to zero, there exists $n_0 > 0$ such that $|y_{n_0}| < \delta$. Let $x_0 := |y_{n_0}|$. Let us prove by induction that

$$|y_{n+n_0}| \geqslant x_n \geqslant 0, \qquad \forall n \geqslant 0. \tag{29}$$

Inequality (29) holds for $n = 0$. Assume that (29) holds for some $n \geqslant 0$. From inequality (28) and the fact that $f(x)$ is increasing on $(0,\delta]$, one gets

$$|y_{n+n_0+1}| = |g(y_{n+n_0})| > f(|y_{n+n_0}|) \geqslant f(x_n) = x_{n+1}.$$

Thus, inequality (29) holds by induction.

From inequality (29) and the fact that $\sum_{n=0}^{\infty} x_n$ is divergent, one concludes that the series $\sum_{n=n_0}^{\infty} |y_n|$ is divergent. Therefore, the series $\sum_{n=0}^{\infty} y_n$ is conditionally convergent. ☐

EXAMPLE 3. Consider $g(x) = -x + \sin^3 x$, $x \in [-1,1]$. From the fact that $g(x)$ is an odd function and the inequalities $0 \leqslant \sin^3 x \leqslant \sin x \leqslant x$, $\forall x \in [0,\pi]$, we have

$$|x - \sin^3 x| = \operatorname{sign}(x)(x - \sin^3 x) = |x| - |\sin^3 x|, \qquad x \in [-1,1].$$

Thus, we have

$$|x| \geqslant |x| - |\sin^3 x| = |g(x)| \geqslant |x| - |x|^3, \qquad \forall x \in [-1,1].$$

Therefore, for this function $g(x)$, inequalities (24)–(26) hold for $b = 1$, $\alpha = 2$, and $c = 1$. By Corollary 4, the series $\sum_{n=0}^{\infty} y_n$ with $y_{n+1} = g(y_n)$, $n \geqslant 0$, $|y_0| \in (0,1]$ is conditionally convergent.

## 3. Applications

We illustrate several applications of Theorem 1 for the study of the convergence of the series $\sum_{n=0}^{\infty} x_n$ where $(x_n)_{n=1}^{\infty}$ is recursively defined by (1) for several choices of $f(x)$ in [2].

Using Maclaurin series expansions, we have

$$f_1(x) := \arctan x = x - cx^{1+\alpha} + o(x^{1+\alpha}), \qquad \alpha = 2, \quad c = \frac{1}{3}, \tag{30}$$

$$f_2(x) := \ln(1+x) = x - cx^{1+\alpha} + o(x^{1+\alpha}), \qquad \alpha = 1, \quad c = \frac{1}{2}, \tag{31}$$

$$f_3(x) := \frac{x}{(\sqrt{x}+1)^2} = x - cx^{1+\alpha} + o(x^{1+\alpha}), \qquad \alpha = \frac{1}{2}, \quad c = 2, \tag{32}$$

$$f_4(x) := \sin x = x - cx^{1+\alpha} + o(x^{1+\alpha}), \qquad \alpha = 2, \quad c = \frac{1}{6}, \tag{33}$$

$$f_5(x) := x - \arcsin(\sin^2 x) = x - cx^{1+\alpha} + o(x^{1+\alpha}), \qquad \alpha = 1, \quad c = 1. \tag{34}$$

Using Theorem 1 one concludes that the series with terms defined by $f = f_3$ is convergent while the series with terms defined by other functions, i.e., $f_1$, $f_2$, $f_4$, and $f_5$ are divergent. The rate of decay of $x_n$ as $n$ tends to infinity according to (3) is $n^{-1/\alpha}(c\alpha)^{-1/\alpha}$. Similar results were obtained in [2] (see also [1], [3]) but with much longer derivations by choosing a suitable $a$ for each function and studying the limit in Proposition 1.

*Acknowledgement.* The author thanks the referee for useful comments which improve the paper.

REFERENCES

[1] T. ANDREESCU AND R. GELCA, *Putnam and Beyond*, Springer-Verlag, New York, 2007.
[2] E. R. EASDALE, J. E. FLEMING, AND B. D. SUCEAVA, *Convergence for Series With Terms Defined by a Recurrence Relation*, Amer. Math. Monthly **124** (2017) 360–364.
[3] J. E. HARRIS AND B. D. SUCEAVA, *Problem 11244*, Amer. Math. Monthly **113** (2006) 759–760.

(Received August 20, 2018)

*N. S. Hoang*
*Department of Mathematics*
*University of West Georgia*
*Carrollton, GA 30118*
*e-mail:* nhoang@westga.edu