

FURTHER INVESTIGATIONS ON SOME RESULTS OF YU

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Abstract. With the help of weighted sharing of values, we investigate the uniqueness of rational function of a meromorphic functions sharing a small function with its differential polynomial. Our results will extend and improve a number of result in the direction of Yu [18]. Specifically we stress on the improvement of two recent results of Charak and Lal [8] and Li, Yang and Liu [14]. We have exhibited several examples to justify our certain claims.

1. Introduction, definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane \mathbb{C} . If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

Throughout the paper the standard notations from Nevanlinna's theory of value distribution of meromorphic functions is used, as in [10]. We recall that $T(r, f)$ denotes the Nevanlinna characteristic function of the non-constant meromorphic function and $N(r, a; f)$ ($\bar{N}(r, a; f)$) denotes the counting function (reduced counting function) of a -points of meromorphic functions f .

A meromorphic function a is said to be a small function of f provided that $T(r, a) = S(r, f)$, that is $T(r, a) = o(T(r, f))$ as $r \rightarrow \infty$, outside of a possible exceptional set of finite linear measure.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

We also recall that if $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)}$$

is called Nevanlinna deficiency of the value a and by ramification index we mean

$$\Theta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{T(r, f)}.$$

Throughout this paper, we use the symbol, $\chi_m = \begin{cases} 0, & \text{if } m = 0, \\ 1, & \text{if } m \geq 1. \end{cases}$ We start the discussion on the following result of R. Brück [6] who first considered the uniqueness problem of an entire function sharing one value with its derivative.

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THEOREM A. [6] *Let f be a non-constant entire function. If f and f' share the value 1 CM and if $N(r, 0; f') = S(r, f)$ then $\frac{f' - 1}{f - 1}$ is a nonzero constant.*

In fact, Brück obtained the above result to justify his famous conjecture, corresponding to the uniqueness for one CM shared value of entire function with its first derivative [6]:

CONJECTURE 1. Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and f' share a finite value a CM, then $\frac{f' - a}{f - a} = c$, where c is a non zero constant.

Later many authors like Zhang [19], Yang [16], Gundersen and Yang [9] et al. ponder over the different aspect of the conjecture and obtained different results. Next we recall the following definition known as weighted sharing of values which has a remarkable influence on the subsequent results of Brück conjecture.

DEFINITION 1. [11, 12] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

The definition implies that if f, g share a value a with weight k then z_0 is an a -point of f with multiplicity $m (\leq k)$ if and only if it is an a -point of g with multiplicity $m (\leq k)$ and z_0 is an a -point of f with multiplicity $m (> k)$ if and only if it is an a -point of g with multiplicity $n (> k)$, where m is not necessarily equal to n .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) , then f, g share (a, p) for any integer $p, 0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If a is a small function we define that f and g share a IM or a CM or with weight l according as $f - a$ and $g - a$ share $(0, 0)$ or $(0, \infty)$ or $(0, l)$ respectively.

Though we use the standard notations and definitions of the value distribution theory available in [10], we explain some definitions and notations which are used in the paper.

DEFINITION 2. [13] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

- (i) $N(r, a; f | \geq p)$ ($\overline{N}(r, a; f | \geq p)$) denotes the counting function (reduced counting function) of those a -points of f whose multiplicities are not less than p .
- (ii) $N(r, a; f | \leq p)$ ($\overline{N}(r, a; f | \leq p)$) denotes the counting function (reduced counting function) of those a -points of f whose multiplicities are not greater than p .

DEFINITION 3. [17] For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f | \geq 2) + \dots + \overline{N}(r, a; f | \geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

DEFINITION 4. [20] For a positive integer p and $a \in \mathbb{C} \cup \{\infty\}$ we put

$$\delta_p(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_p(r, a; f)}{T(r, f)}$$

Clearly $0 \leq \delta(a; f) \leq \delta_p(a; f) \leq \delta_{p-1}(a; f) \leq \dots \leq \delta_2(a; f) \leq \delta_1(a; f) = \Theta(a; f)$

DEFINITION 5. [1] Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. Let z_0 be a a -point of f with multiplicity p , a a -point of g with multiplicity q . We denote by $\overline{N}_L(r, a; f)$ the counting function of those a -points of f and g where $p > q$, by $N_E^1(r, a; f)$ the counting function of those a -points of f and g where $p = q = 1$ and by $\overline{N}_E^2(r, a; f)$ the counting function of those a -points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^1(r, a; g)$, $\overline{N}_E^2(r, a; g)$.

DEFINITION 6. [11, 12] Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a -points of f whose multiplicities differ from the multiplicities of the corresponding a -points of g .

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

In 2003, Yu [18] tried to solve the conjecture in a different way than that was done earlier. He considered the uniqueness problem of an entire or meromorphic functions with its derivative sharing a small function a and obtained the following two results.

THEOREM B. [18] *Let f be a non-constant entire function, $a \in S(f)$ and $a \neq 0, \infty$. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0; f) > \frac{3}{4}$ then $f \equiv f^{(k)}$.*

THEOREM C. [18] *Let f be a non-constant non-entire meromorphic function, $a \in S(f)$ and $a \neq 0, \infty$. If*

- i) f and a have no common poles.
- ii) $f - a$ and $f^{(k)} - a$ share the value 0 CM.
- iii) $4\delta(0; f) + 2(8 + k)\Theta(\infty; f) > 19 + 2k$

then $f \equiv f^{(k)}$, where k is a positive integer.

Recently in connection with the Yu's [18] result, Zhang and Lü [21] considered the uniqueness of the n -th power of a meromorphic function sharing a small function with its k -th derivative and proved the following theorem.

THEOREM D. [21] *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\neq 0, \infty)$ be a small function with respect to f . Suppose $f^n - a$ and $f^{(k)} - a$ share $(0, s)$. If $s = \infty$ and*

$$(3 + k)\Theta(\infty; f) + 2\Theta(0; f) + \delta_{2+k}(0; f) > 6 + k - n \tag{1.1}$$

or $s = 0$ and

$$(6 + 2k)\Theta(\infty; f) + 4\Theta(0; f) + 2\delta_{2+k}(0; f) > 12 + 2k - n \quad (1.2)$$

then $f^n \equiv f^{(k)}$

Next we give the following definition.

DEFINITION 7. [2, 3] Let $n_{0j}, n_{1j}, \dots, n_{kj}$ be non negative integers.

The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \dots (f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d(M_j) = \sum_{i=0}^k n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^k (i+1)n_{ij}$.

The sum $P[f] = \sum_{j=1}^t b_j M_j[f]$ is called a differential polynomial generated by f of degree $\bar{d}(P) = \max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, \dots, t$.

The numbers $\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$. $P[f]$ is said to be homogeneous if $\bar{d}(P) = \underline{d}(P) = d$ (say). $P[f]$ is called a Linear Differential Polynomial generated by f if $\bar{d}(P) = 1$. Otherwise $P[f]$ is called Non-linear Differential Polynomial. We denote by $Q = \max\{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = \max\{n_{1j} + 2n_{2j} + \dots + kn_{kj} : 1 \leq j \leq t\}$.

DEFINITION 8. [4] For any two positive integers n and $r \leq 3$,

$$\mu_r = \min\{r, n\} \quad \text{and} \quad \mu_r^* = (r+1) - \mu_r.$$

In [5], Bhoosnurmath and Kabbur considered the uniqueness of a meromorphic function f and its differential polynomial $\mathcal{P}[f]$. Motivated by such an uniqueness result later Charak and Lal [7] asked the following natural question

Is it true that $p(f) \equiv \mathcal{P}[f]$, when $p(f)$ with $p(0) = 0$ and $\mathcal{P}[f]$ share (a, s) , where a is a small meromorphic function of f ?

In [7], Charak and Lal investigated and answered the above question affirmatively as follows.

THEOREM E. *Let f be a non-constant meromorphic function, $a (\neq 0, \infty)$ be a meromorphic small function and $p(z)$ be of degree $n \geq 1$ with $p(0) = 0$. Let $\mathcal{P}[f]$ be a differential polynomial of f . Suppose $p(f) - a$ and $\mathcal{P}[f] - a$ share $(0, s)$. If $s \geq 2$ and*

$$\begin{aligned} & (Q+3)\Theta(\infty; f) + 2n\Theta(0; p(f)) + \bar{d}(\mathcal{P})\delta(0, f) \\ & > Q + 3 + 2\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}) + n, \end{aligned} \quad (1.3)$$

or, if $s = 1$ and

$$\begin{aligned} & \left(Q + \frac{7}{2}\right)\Theta(\infty; f) + \frac{5n}{2}\Theta(0; p(f)) + \bar{d}(\mathcal{P})\delta(0, f) \\ & > Q + \frac{7}{2} + 2\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}) + \frac{3n}{2}, \end{aligned} \quad (1.4)$$

or, if $s = 0$ and

$$\begin{aligned} & (2Q + 6)\Theta(\infty; f) + 4n\Theta(0; p(f)) + 2\bar{d}(\mathcal{P})\delta(0, f) \\ & > 2Q + 6 + 4\bar{d}(\mathcal{P}) - 2\underline{d}(\mathcal{P}) + 3n, \end{aligned} \tag{1.5}$$

then $p(f) \equiv \mathcal{P}[f]$.

Very recently, in this direction, for homogeneous differential polynomial, Li, Yang and Liu [14] obtained the following result.

THEOREM F. [14] *Let f be a non-constant meromorphic function and $\mathcal{P}[f]$ be a non-constant homogeneous differential polynomial of degree d and weight Γ satisfying $\Gamma \geq (k + 2)d - 2$. Let $a(z)$ be a small meromorphic function of f such that $a(z) \neq 0, \infty$. Suppose that $f - a$ and $\mathcal{P}[f] - a$ share $(0, s)$. If $s \geq 2$ and*

$$3\Theta(\infty; f) + d\delta_{2+\Gamma-d}(0, f^d) + \delta_2(0, f) + \delta(a, f) > 4 \tag{1.6}$$

or, $s = 1$ and

$$\begin{aligned} & \frac{7 + \Gamma - d}{2}\Theta(\infty; f) + \frac{d}{2}\delta_{1+\Gamma-d}(0, f^d) + d\delta_{2+\Gamma-d}(0, f^d) + \delta_2(0, f) + \delta(a, f) \\ & > \frac{\Gamma + 9}{2} \end{aligned} \tag{1.7}$$

or, $s = 0$ and

$$\begin{aligned} & [2(\Gamma - d) + 6]\Theta(\infty; f) + d\delta_{1+\Gamma-d}(0, f^d) + d\delta_{2+\Gamma-d}(0, f^d) + \delta_2(0, f) \\ & + \Theta(0; f) + \delta(a, f) > 2\Gamma + 8 \end{aligned} \tag{1.8}$$

then $\frac{\mathcal{P}[f] - a}{f - a} = C$, where C is a non-zero constant.

Specially when $s = 0$ i.e., when f and $\mathcal{P}[f]$ share $(a, 0)$, then $f \equiv \mathcal{P}[f]$.

From the above discussions, we observe that till date in the above theorems, the factor f^n can not be extended up to a n -th degree polynomial expression of a meromorphic function $P_n(f)$ without any restriction. Now since f^n or $P_n(f)$ with $P_n(0) = 0$ are nothing but a part of a rational function $\mathcal{R}(f)$, it is quite natural to extend the above mentioned theorems up to a relation between a $\mathcal{R}(f)$ of a meromorphic function f and a general differential polynomial $\mathcal{P}[f]$ generated by f .

Henceforth we denote by $\mathcal{R}(f)$ as defined in Lemma 1 and so we mean $\lambda = \max\{m, n\}$, p_i ($1 \leq i \leq u$) and q_j ($1 \leq j \leq l$) are positive integers. Let $P_n(f) = a_n \prod_{i=1}^u (f - d_i)^{p_i}$, $1 \leq u \leq n$ and $P_m(z) = b_m \prod_{j=1}^l (f - c_j)^{q_j}$, $1 \leq l \leq m$ respectively, where u and l are two positive integers. Let $c_0 \neq c_j$ ($j = 1, \dots, l$) be a complex constant.

We now define for any positive integer $r \leq 3$,

$$\mu_r^i = \min\{p_i, r\} \quad \text{and} \quad \mu_r^{*i} = (r + 1) - \mu_r^i, \quad \forall \quad i = 1, \dots, u.$$

Clearly, for $i = 1$, this coincides with the Definition 8.

Let us define $l^* = \begin{cases} \chi_m, & \text{if } m=0, \\ l\chi_m, & \text{if } m \geq 1. \end{cases}$

Now it is natural to consider the above mentioned problems in more general setting in a compact and convenient way. A question arises as follows:

QUESTION 1. Is it possible $\mathcal{R}(f) \equiv \mathcal{P}[f]$, when $\mathcal{R}(f) - a$ and $\mathcal{P}[f] - a$ share $(0, s)$, where a is a small function of a meromorphic function f ?

To find out the possible answer of the above question is the main motivation of writing this paper. The following is the main result of this paper.

THEOREM 1. (Main) *Let f be a non-constant meromorphic function and $m(\geq 0)$, $n(\geq 1)$ are integers and $a \equiv a(z) (\neq 0, \infty)$ be a meromorphic small function of f . Let $\mathcal{P}[f]$ be a differential polynomial containing at least one derivative. Suppose that $\mathcal{R}(f) - a$ and $\mathcal{P}[f] - a$ share $(0, s)$. If $2 \leq s < \infty$ and*

$$\begin{aligned} & (Q+3)\Theta(\infty; f) + \bar{d}(\mathcal{P})\delta(0; f) + \sum_{j=0}^{l^*} \chi_j \left\{ \Theta_{(2)}(c_j; f) + \Theta(c_j; f) \right\} + \sum_{i=1}^u \mu_2^i \delta_{\mu_2^{*i}}(d_i; f) \\ & > Q + 3 + 2\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}) + 2l^* + \sum_{i=1}^u \mu_2^i - \lambda, \end{aligned} \quad (1.9)$$

or, if $s = 1$ and

$$\begin{aligned} & \left(Q + \frac{7}{2} \right) \Theta(\infty; f) + \bar{d}(\mathcal{P})\delta(0; f) + \sum_{j=0}^{l^*} \chi_j \left\{ \Theta_{(2)}(c_j; f) + \frac{3}{2}\Theta(c_j; f) \right\} + \sum_{i=1}^u \mu_2^i \delta_{\mu_2^{*i}}(d_i; f) \\ & + \frac{1}{2} \sum_{i=1}^u \Theta(d_i; f) > Q + \frac{7}{2} + 2\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}) + \frac{5l^*}{2} + \sum_{i=1}^u \mu_2^i + \frac{1}{2}u - \lambda, \end{aligned} \quad (1.10)$$

or, if $s = 0$ and

$$\begin{aligned} & (2Q+6)\Theta(\infty; f) + 2\bar{d}(\mathcal{P})\delta(0; f) + \sum_{j=0}^{l^*} \chi_j \left\{ \Theta_{(2)}(c_j; f) + 3\Theta(c_j; f) \right\} + \sum_{i=1}^u \mu_2^i \delta_{\mu_2^{*i}}(d_i; f) \\ & + 2 \sum_{i=1}^u \Theta(d_i; f) > 2Q + 6 + 4\bar{d}(\mathcal{P}) - 2\underline{d}(\mathcal{P}) + 4l^* + \sum_{i=1}^u \mu_2^i + 2u - \lambda, \end{aligned} \quad (1.11)$$

then $\mathcal{R}(f) \equiv \mathcal{P}[f]$.

Now putting $m = 0$ in Theorem 1, we can immediately deduce the following corollary which improve the result of Charak and Lal [8].

COROLLARY 1. *Let f be a non-constant meromorphic function and $n(\geq 1)$ be an integer and $a \equiv a(z) (\neq 0, \infty)$ be a meromorphic small function of f . Let $\mathcal{P}[f]$ be*

a differential polynomial containing atleast one derivative. Suppose that $p(f) - a$ and $\mathcal{P}[f] - a$ share $(0, s)$. If $2 \leq s < \infty$ and

$$\begin{aligned} & (Q+3)\Theta(\infty; f) + \bar{d}(\mathcal{P})\delta(0; f) + \sum_{i=1}^u \mu_2^i \delta_{\mu_2^{*i}}(d_i; f) \\ & > Q+3 + 2\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}) + \sum_{i=1}^u \mu_2^i - n, \end{aligned} \tag{1.12}$$

or, if $s = 1$ and

$$\begin{aligned} & \left(Q + \frac{7}{2}\right)\Theta(\infty; f) + \bar{d}(\mathcal{P})\delta(0; f) + \sum_{i=1}^u \mu_2^i \delta_{\mu_2^{*i}}(d_i; f) + \frac{1}{2} \sum_{i=1}^u \Theta(d_i; f) \\ & > Q + \frac{7}{2} + 2\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}) + \sum_{i=1}^u \mu_2^i + \frac{1}{2}u - n, \end{aligned} \tag{1.13}$$

or, if $s = 0$ and

$$\begin{aligned} & (2Q+6)\Theta(\infty; f) + 2\bar{d}(\mathcal{P})\delta(0; f) + \sum_{i=1}^u \mu_2^i \delta_{\mu_2^{*i}}(d_i; f) + 2 \sum_{i=1}^u \Theta(d_i; f) \\ & > 2Q+6 + 4\bar{d}(\mathcal{P}) - 2\underline{d}(\mathcal{P}) + \sum_{i=1}^u \mu_2^i + 2u - n, \end{aligned} \tag{1.14}$$

then $p(f) \equiv \mathcal{P}[f]$.

If we consider a meromorphic function f and its homogeneous differential polynomial $\mathcal{P}[f]$ sharing a small function then we get the following result from Corollary 1.

COROLLARY 2. *Let f be a non-constant meromorphic function and $a \equiv a(z) (\not\equiv 0, \infty)$ be a meromorphic small function of f . Let $\mathcal{P}[f]$ be a homogeneous differential polynomial containing atleast one derivative. Suppose that $f - a$ and $\mathcal{P}[f] - a$ share $(0, s)$. If $2 \leq s < \infty$ and*

$$(\Gamma + 3 - d)\Theta(\infty; f) + d \delta(0; f) + \delta_2(0; f) > \Gamma + 3,$$

or, if $s = 1$ and

$$\left(\Gamma + \frac{7}{2} - d\right)\Theta(\infty; f) + d \delta(0; f) + \delta_2(0; f) + \frac{1}{2}\Theta(0; f) > \Gamma + 4,$$

or, if $s = 0$ and

$$(2(\Gamma - d) + 6)\Theta(\infty; f) + 2d \delta(0; f) + \delta_2(0; f) + 2\Theta(0; f) > 2\Gamma + 8,$$

then $f \equiv \mathcal{P}[f]$.

REMARK 1. If we compare the conditions of Corollary 2 and Theorem F, we see that

$$(\Gamma + 3 - d)\Theta(\infty; f) + d\delta(0; f) + \delta_2(0; f) > \Gamma + 3$$

implies

$$3\Theta(\infty; f) + d\delta(0; f) + \delta_2(0; f) > 3 + d \geq 4$$

and

$$\begin{aligned} & 3\Theta(\infty; f) + d\delta(0; f) + \delta_2(0; f) \\ & < 3\Theta(\infty; f) + d\delta_{2+\Gamma-d}(0; f^d) + \delta_2(0; f) + \delta(a; f). \end{aligned}$$

Again

$$\left(\Gamma + \frac{7}{2} - d\right)\Theta(\infty; f) + d\delta(0; f) + \delta_2(0; f) + \frac{1}{2}\Theta(0; f) > \Gamma + 4$$

implies

$$\frac{7+\Gamma-d}{2}\Theta(\infty; f) + d\delta(0; f) + \delta_2(0; f) + \frac{1}{2}\Theta(0; f) > \frac{\Gamma}{2} + 4 + \frac{d}{2} \geq \frac{\Gamma+9}{2}$$

and

$$\begin{aligned} & \frac{7+\Gamma-d}{2}\Theta(\infty; f) + d\delta(0; f) + \delta_2(0; f) + \frac{1}{2}\Theta(0; f) \\ & < \frac{7+\Gamma-d}{2}\Theta(\infty; f) + \frac{d}{2}\delta_{1+\Gamma-d}(0; f^d) + d\delta_{2+\Gamma-d}(0; f^d) + \delta_2(0; f) + \delta(a; f). \end{aligned}$$

Also we see that for the case of *IM* sharing

$$\begin{aligned} & (2(\Gamma - d) + 6)\Theta(\infty; f) + 2d\delta(0; f) + \delta_2(0; f) + 2\Theta(0; f) \\ & < [2(\Gamma - d) + 6]\Theta(\infty; f) + d\delta_{1+\Gamma-d}(0; f^d) + d\delta_{2+\Gamma-d}(0; f^d) + \delta_2(0; f) + \Theta(0; f) + \delta(a; f). \end{aligned}$$

It is clear from Remark 1.1 that Corollary 2 is a direct extension and improvement of Theorem F.

The following examples show that the condition $a \neq 0$ is necessary in Theorem 1.

EXAMPLE 1. Let $\mathcal{R}(f) = \frac{f^n}{f^2 - 1}$ with $n \geq 14$ and $\mathcal{P}[f] = -ff'$, where $f = \frac{e^z}{e^z - 1}$. It is clear that $\mathcal{R}(f)$ and $\mathcal{P}[f]$ share $(0, \infty)$ and all the conditions (1.9)–(1.11) in Theorem 1 are satisfied but $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

EXAMPLE 2. Let $\mathcal{R}(f) = \frac{f^n}{f^m - a}$, where $n > m + 8$, $a \in \mathbb{C} - \{0\}$ and $\mathcal{P}[f] = f^2 + ff'$, where $f = \frac{1}{e^z - 1}$. It is clear that $\mathcal{R}(f)$ and $\mathcal{P}[f]$ share $(0, \infty)$ and all the conditions (1.9)–(1.11) in Theorem 1 are satisfied but $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

The following examples show that the deficiency conditions (1.9)–(1.11) in Theorem 1 can't be removed when $\mathcal{P}[f]$ is a homogeneous differential polynomial.

EXAMPLE 3. Let $\mathcal{R}(f) = \frac{f^2 + 5f - 4}{5(f^2 + 1)}$ and $\mathcal{P}[f] = \frac{1}{5}f'$, where $f = e^z$. It is clear that $\mathcal{R}(f) - \frac{1}{5}$ and $\mathcal{P}[f] - \frac{1}{5}$ share $(0, \infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied and $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

EXAMPLE 4. Let $\mathcal{R}(f) = \frac{2f + 1}{f}$ and $\mathcal{P}[f] = f - f'$, where $f = \frac{e^z - 1}{e^z + 1}$. It is clear that $\mathcal{R}(f) - 1$ and $\mathcal{P}[f] - 1$ share $(0, \infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied and $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

EXAMPLE 5. For $c \neq 0$, let $\mathcal{R}(f) = \frac{(c + 1)f}{f - 1}$ and $\mathcal{P}[f] = f'$, where $f = e^{-z}$. It is clear that $\mathcal{R}(f) - c$ and $\mathcal{P}[f] - c$ share $(0, \infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied and $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

EXAMPLE 6. Let $\mathcal{R}(f) = f^3 + c - 1$, where $c \in \mathbb{C} - \{0\}$ and $\mathcal{P}[f] = \frac{c}{\beta}f^2f'$, where $f = e^{\beta z}$, $\beta \in \mathbb{C} - \{0\}$. It is clear that $\mathcal{R}(f) - c = e^{3\beta z} - 1$ and $\mathcal{P}[f] - c = c(e^{3\beta z} - 1)$ share $(0, \infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied and $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

EXAMPLE 7. Let $f(z) = -\sin(\alpha z) + a - \frac{a}{\alpha^{4k}}$, $k \in \mathbb{N}$; where $\alpha \neq 0, \alpha^{4k} \neq 1$ and $a \in \mathbb{C} - \{0\}$. Let $\mathcal{R}(f) = f$ and $\mathcal{P}[f] = f^{(4k)}$. Then $\mathcal{P}[f] = -\alpha^{4k} \sin(\alpha z)$. Here $\delta(0; f) = 0$. Also it is clear that $\mathcal{R}(f)$ and $\mathcal{P}[f]$ share (a, ∞) but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied, hence $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

EXAMPLE 8. Let $f(z) = e^{Nz}$, where N is a non-zero integer. For a natural number n we define

$$\mathcal{R}(f) = -N^{2n} \sum_{r=0}^{2n-1} (-1)^r \binom{2n}{r} f^{2n-r} \quad \text{and} \quad \mathcal{P}[f] = f^{(2n)}.$$

Then it is clear that

$$\mathcal{R}(f) - N^{2n} = -N^{2n}(e^{Nz} - 1)^{2n} \quad \text{and} \quad \mathcal{P}[f] - N^{2n} = N^{2n}(e^{Nz} - 1).$$

Thus we see that $\mathcal{R}(f)$ and $\mathcal{P}[f]$ share $(N^{2n}, 0)$. Here $\Theta(\infty; f) = 1$ and $\delta_q(0; f) = 1, \forall q \in \mathbb{N}$.

Thus the condition (1.11) in Theorem 1 is not satisfied and $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

The following examples show that the deficiency conditions (1.9)–(1.11) in Theorem 1 can't be removed when $\mathcal{P}[f]$ is non homogeneous differential polynomial.

EXAMPLE 9. Let $\mathcal{R}(f) = \frac{(f+1)^3 - (af^2+b)}{af^2+b}$, where $a, b \in \mathbb{C}$ with $a \neq 0$, $b \neq 1$ and $\mathcal{P}[f] = f^2 f' + 3f f' + 3f$, where $f = e^z$. It is clear that $\mathcal{R}(f) + 1 = \frac{(e^z+1)^3}{af^2+b}$ and $\mathcal{P}[f] + 1 = (e^z+1)^3$ share $(0, \infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied, hence $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

EXAMPLE 10. Let $\mathcal{R}(f) = \frac{f(f+ac+2)}{af+1}$, where $a, c \in \mathbb{C} - \{0\}$ and $\mathcal{P}[f] = f f' + 2f$, where $f = e^z$. It is clear that $\mathcal{R}(f) - c = \frac{e^{2z} + 2e^z - c}{ae^z + 1}$ and $\mathcal{P}[f] - c = e^{2z} + 2e^z - c$ share $(0, \infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied, hence $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

EXAMPLE 11. Let $\mathcal{R}(f) = f^3 + 2f^2 + f - 1$ and $\mathcal{P}[f] = f^2 + 2f'$, where $f = e^z$. It is clear that $\mathcal{R}(f) + 1 = e^z(e^z+1)^2$ and $\mathcal{P}[f] + 1 = (e^z+1)^2$ share $(0, \infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied, hence $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

2. Useful lemmas

In this section we present some lemmas which will be needed in the sequel. Let \mathcal{F}, \mathcal{G} be two non-constant meromorphic functions. Henceforth we shall denote by \mathcal{H} the following function.

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1} \right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1} \right). \quad (2.1)$$

LEMMA 1. [15] *Let f be a non-constant meromorphic function and let*

$$\mathcal{R}(f) = \frac{\sum_{k=0}^n a_k f^k}{\sum_{j=0}^m b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, \mathcal{R}(f)) = \lambda T(r, f) + S(r, f),$$

where $\lambda = \max\{n, m\}$.

LEMMA 2. [5] *Let f be a meromorphic function and $\mathcal{P}[f]$ be a differential polynomial. Then*

$$m \left(r, \frac{\mathcal{P}[f]}{f \bar{d}(\mathcal{P})} \right) \leq (\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P})) m \left(r, \frac{1}{f} \right) + S(r, f). \quad (2.2)$$

$$N\left(r, \frac{\mathcal{P}[f]}{f\bar{d}(\mathcal{P})}\right) \leq (\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}))N\left(r, \frac{1}{f}\right) + Q\left[\bar{N}(r, \infty; f) + \bar{N}(r, 0; f)\right] + S(r, f). \tag{2.3}$$

$$N\left(r, \frac{1}{\mathcal{P}[f]}\right) \leq Q\bar{N}(r, \infty; f) + (\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}))m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f\bar{d}(\mathcal{P})}\right) + S(r, f). \tag{2.4}$$

3. Proof of the theorem

Proof of Theorem 1. Let $\mathcal{F} = \frac{\mathcal{R}(f)}{a}$ and $\mathcal{G} = \frac{\mathcal{P}[f]}{a}$. Then $\mathcal{F} - 1 = \frac{\mathcal{R}(f) - a}{a}$ and $\mathcal{G} - 1 = \frac{\mathcal{P}[f] - a}{a}$. Since $\mathcal{R}(f) - a$ and $\mathcal{P}[f] - a$ share $(0, s)$ it follows that \mathcal{F}, \mathcal{G} share $(1, s)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $\mathcal{H} \neq 0$.

Then from (2.1) we get

$$\begin{aligned} &N(r, \infty; \mathcal{H}) \tag{3.1} \\ &\leq \bar{N}(r, \infty; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f | \geq 2) + \bar{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + \bar{N}(r, 0; \mathcal{F} | \geq 2) + \bar{N}(r, 0; \mathcal{G} | \geq 2) \\ &\quad + \bar{N}_0(r, 0; \mathcal{F}') + \bar{N}_0(r, 0; \mathcal{G}') + \bar{N}(r, 0; a) + \bar{N}(r, \infty; a). \end{aligned}$$

where $\bar{N}_0(r, 0; \mathcal{F}')$ is the reduced counting function of those zeros of \mathcal{F}' which are not the zeros of $\mathcal{F}(\mathcal{F} - 1)$ and $\bar{N}_0(r, 0; \mathcal{G}')$ is similarly defined. Let z_0 be a simple zero of $\mathcal{F} - 1$.

By the *Second Fundamental Theorem*, we get

$$\begin{aligned} &T(r, \mathcal{F}) + T(r, \mathcal{G}) \tag{3.2} \\ &\leq 2\bar{N}(r, \infty; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f) + \bar{N}(r, 0; \mathcal{F}) + \bar{N}(r, 0; \mathcal{G}) + \bar{N}(r, 1; \mathcal{F}) + \bar{N}(r, 1; \mathcal{G}) \\ &\quad - N_0(r, 0; \mathcal{F}') - N_0(r, 0; \mathcal{G}') + S(r, f), \end{aligned}$$

where $N_0(r, 0; \mathcal{F}')$ is the counting function of those zeros of \mathcal{F}' which are not the zeros of $\mathcal{F}(\mathcal{F} - 1)$ and $N_0(r, 0; \mathcal{G}')$ is similarly defined.

Subcase 1.1. Let $s \geq 1$. Then from (2.1), we get

$$\begin{aligned} &N_E^1(r, 1; \mathcal{F}) \tag{3.3} \\ &\leq N(r, \mathcal{H}) + S(r, f) \\ &\leq \bar{N}(r, \infty; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f | \geq 2) + \bar{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + \bar{N}(r, 0; \mathcal{F} | \geq 2) + \bar{N}(r, 0; \mathcal{G} | \geq 2) \\ &\quad + \bar{N}_0(r, 0; \mathcal{F}') + \bar{N}_0(r, 0; \mathcal{G}') + S(r, f) \end{aligned}$$

and

$$\begin{aligned}
& \bar{N}(r, 1; \mathcal{F}) + \bar{N}(r, 1; \mathcal{G}) \tag{3.4} \\
&= N_E^1(r, 1; \mathcal{F}) + \bar{N}_E^2(r, 1; \mathcal{F}) + \bar{N}_L(r, 1; \mathcal{F}) + \bar{N}_L(r, 1; \mathcal{G}) + \bar{N}(r, 1; \mathcal{G}) + S(r, f) \\
&\leq \bar{N}(r, \infty; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f \geq 2) + \bar{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + \bar{N}(r, 0; \mathcal{F} \geq 2) \\
&\quad + \bar{N}(r, 0; \mathcal{G} \geq 2) + \bar{N}_0(r, 0; \mathcal{F}') + \bar{N}_0(r, 0; \mathcal{G}') + \bar{N}_E^2(r, 1; \mathcal{F}) + \bar{N}_L(r, 1; \mathcal{F}) \\
&\quad + \bar{N}_L(r, 1; \mathcal{G}) + \bar{N}(r, 1; \mathcal{G}) + S(r, f).
\end{aligned}$$

Subcase 1.1.a. Let $s \geq 2$. In this case proceeding same way as in [7, Subcase 1.2, Proof of Theorem 1] we get,

$$\begin{aligned}
& T(r, \mathcal{F}) \tag{3.5} \\
&\leq 3\bar{N}(r, \infty; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f \geq 2) + N_2(r, 0; \mathcal{F}) \\
&\quad + N(r, 0; \mathcal{P}[f]) + S(r, f).
\end{aligned}$$

Using (2.4) of Lemma 2 in (3.5), we obtained

$$\begin{aligned}
& T(r, F) \\
&\leq (Q+3)\bar{N}(r, \infty; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f \geq 2) + \sum_{i=1}^u \mu_2^i N_{\mu_2^{si}}(r, d_i; f) \\
&\quad + (\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}))T(r, f) + \bar{d}(\mathcal{P})N(r, 0; f) + S(r, f)
\end{aligned}$$

i.e., for any $\varepsilon > 0$, we get

$$\begin{aligned}
& \left\{ (Q+3)\Theta(\infty; f) + \sum_{j=0}^{l^*} \chi_j \left\{ \Theta(c_j; f) + \Theta_{(2)}(c_j; f) \right\} + \bar{d}(\mathcal{P})\delta(0; f) \right. \\
& \quad \left. + \sum_{i=1}^u \mu_2^i \delta_{\mu_2^{si}}(d_i; f) \right\} T(r, f) \\
&\leq \left\{ Q+3 + 2\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}) + 2l^* + \sum_{i=1}^u \mu_2^i - \lambda + \varepsilon \right\} T(r, f) + S(r, f),
\end{aligned}$$

which contradicts (1.9).

Subcase 1.1.b. Let $s = 1$

Then proceeding same way as in [7, Subcase 1.1, Proof of Theorem 1] we get,

$$\begin{aligned}
& \bar{N}(r, 1; \mathcal{F}) + \bar{N}(r, 1; \mathcal{G}) \tag{3.6} \\
&\leq \frac{3}{2}\bar{N}(r, \infty; f) + \frac{1}{2} \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f \geq 2) + \frac{1}{2}\bar{N}(r, 0; F) \\
&\quad + \bar{N}(r, 0; \mathcal{F} \geq 2) + \bar{N}(r, 0; \mathcal{G} \geq 2) + T(r, \mathcal{G}) + \bar{N}_0(r, 0; \mathcal{F}') + \bar{N}(r, 0; \mathcal{G}') + S(r, f).
\end{aligned}$$

and then from (3.2), we get as above

$$\begin{aligned}
 & T(r, \mathcal{F}) \tag{3.7} \\
 & \leq \frac{7}{2} \overline{N}(r, \infty; f) + \frac{3}{2} \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f \geq 2) + N_2(r, 0; \mathcal{F}) + \frac{1}{2} \overline{N}(r, 0; \mathcal{F}) \\
 & \quad + N(r, 0; \mathcal{P}[f]) + S(r, f)
 \end{aligned}$$

Now using (2.4) in Lemma 2, we get from (3.7)

$$\begin{aligned}
 & T(r, \mathcal{F}) \\
 & \leq \left(Q + \frac{7}{2} \right) \overline{N}(r, \infty; f) + \frac{3}{2} \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f) + \sum_{j=0}^{l^*} \overline{N}(r, c_j; f \geq 2) + \sum_{i=1}^u \mu_2^i N_{\mu_2^{*i}}(r, d_i; f) \\
 & \quad + \frac{1}{2} \sum_{i=1}^u \overline{N}(r, d_i; f) + (\overline{d}(\mathcal{P}) - \underline{d}(\mathcal{P})) T(r, f) + \overline{d}(\mathcal{P}) N(r, 0; f) + S(r, f)
 \end{aligned}$$

i.e., for any $\varepsilon > 0$, we get

$$\begin{aligned}
 & \left\{ \left(Q + \frac{7}{2} \right) \Theta(\infty; f) + \sum_{j=0}^{l^*} \chi_j \left\{ \Theta_{(2)}(c_j; f) + \frac{3}{2} \Theta(c_j; f) \right\} + \sum_{i=1}^u \mu_2^i \delta_{\mu_2^{*i}}(d_i; f) \right. \\
 & \quad \left. + \frac{1}{2} \sum_{i=1}^u \Theta(d_i; f) + \overline{d}(\mathcal{P}) \delta(0; f) \right\} T(r, f) \\
 & \leq \left\{ Q + \frac{7}{2} + 2\overline{d}(\mathcal{P}) - \underline{d}(\mathcal{P}) + \frac{5l^*}{2} + \sum_{i=1}^u \mu_2^i + \frac{1}{2} u - \lambda + \varepsilon \right\} T(r, f) + S(r, f),
 \end{aligned}$$

which contradicts (1.10).

Subcase 1.2. Let $s = 0$.

In this case also proceeding same way as in [7, Subcase 1.2, Proof of Theorem 1] we get,

$$\begin{aligned}
 & T(r, \mathcal{F}) \tag{3.8} \\
 & \leq 6\overline{N}(r, \infty; f) + 3 \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f \geq 2) + N_2(r, 0; \mathcal{F}) + 2\overline{N}(r, 0; \mathcal{F}) \\
 & \quad + 2N(r, 0; \mathcal{P}[f]) + S(r, f).
 \end{aligned}$$

Using (2.4) of Lemma 2 in (3.8), we obtained

$$\begin{aligned}
 & T(r, \mathcal{F}) \\
 & \leq (2Q + 6) \overline{N}(r, \infty; f) + 3 \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f \geq 2) + \sum_{i=1}^u \mu_2^i N_{\mu_2^{*i}}(r, d_i; f) \\
 & \quad + 2 \sum_{i=1}^u \overline{N}(r, d_i; f) + 2(\overline{d}(\mathcal{P}) - \underline{d}(\mathcal{P})) T(r, f) + 2\overline{d}(\mathcal{P}) \overline{N}(r, 0; f) + S(r, f)
 \end{aligned}$$

i.e., for any $\varepsilon > 0$, we get

$$\begin{aligned} & \left\{ (2Q+6)\Theta(\infty; f) + \sum_{j=0}^{l^*} \chi_j \left\{ \Theta_{(2}(c_j; f) + 3\Theta(c_j; f) \right\} + \sum_{i=1}^u \mu_2^i \delta_{\mu_2^i}(d_i; f) \right. \\ & \left. + 2 \sum_{i=1}^u \Theta(d_i; f) + 2\bar{d}(\mathcal{P})\delta(0; f) \right\} T(r, f) \\ & \leq \left\{ 2Q+6 + 4\bar{d}(\mathcal{P}) - 2\underline{d}(\mathcal{P}) + 4l^* + \sum_{i=1}^u \mu_2^i + 2u - \lambda + \varepsilon \right\} T(r, f) + S(r, f), \end{aligned}$$

which contradicts (1.11).

Case 2. Let $\mathcal{H} \equiv 0$.

On integration we get from (2.1)

$$\frac{1}{\mathcal{F}-1} \equiv \frac{\mathcal{C}}{\mathcal{G}-1} + \mathcal{D}, \quad (3.9)$$

where \mathcal{C} , \mathcal{D} are constants and $\mathcal{C} \neq 0$.

Subcase 2.1. Suppose $\mathcal{D} \neq 0, -1$.

Following the same procedure as in [7, Case (i), Proof of Theorem 1] and applying *Second Fundamental Theorem* we obtained,

$$\lambda T(r, f) \leq 2\bar{N}(r, \infty; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f) + \sum_{i=1}^u \bar{N}(r, d_i; f) + S(r, f),$$

which implies

$$2\Theta(\infty; f) + \sum_{j=0}^{l^*} \chi_j \Theta(c_j; f) + \sum_{i=1}^u \Theta(d_i; f) \leq 2 + l^* + u - \lambda,$$

which contradicts (1.9), (1.10) and (1.11).

Subcase 2.2. Let $\mathcal{D} = 0$, then proceeding exactly same way as in [7, Case (ii), Proof of Theorem 1] and applying *Second Fundamental Theorem* we obtained,

$$\begin{aligned} T(r, \mathcal{F}) & \leq (Q+1)\bar{N}(r, \infty; f) + \sum_{j=0}^{l^*} \chi_j \bar{N}(r, c_j; f) + \sum_{i=1}^u \bar{N}(r, d_i; f) \\ & \quad + (\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}))T(r, f) + \bar{d}(\mathcal{P})N(r, 0; f) + S(r, f), \end{aligned}$$

which implies

$$\begin{aligned} & (Q+1)\Theta(\infty; f) + \sum_{j=0}^{l^*} \chi_j \Theta(c_j; f) + \sum_{i=1}^u \Theta(d_i; f) + \bar{d}(\mathcal{P})\delta(0; f) \\ & \leq Q+1 + 2\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}) + l^* + u - \lambda, \end{aligned}$$

which contradicts (1.9), (1.10) and (1.11).

In this case we get, $\mathcal{F} \equiv \mathcal{G}$ i.e., $\mathcal{R}(f) \equiv \mathcal{P}[f]$.

Subcase 2.3. Let $\mathcal{D} = -1$ and if $\mathcal{C} \neq -1$, then proceeding exactly same as in [7, Case (iii), Proof of Theorem 1] and applying *Second Fundamental Theorem*, this case follows exactly the Subcase 2.2.

So let $\mathcal{C} = -1$, then we get $\mathcal{F}\mathcal{G} \equiv 1$ i.e., $\mathcal{R}(f)\mathcal{P}[f] \equiv a^2$.

From above we have $N(r, 0; f) = S(r, f)$ and $N(r, \infty; f) = S(r, f)$.

In view of the *First Fundamental Theorem* and (2.2) in Lemma 2, we get from above

$$\begin{aligned} & (\lambda + \bar{d}(\mathcal{P}))T(r, f) \\ &= T\left(r, \frac{a^2}{\mathcal{R}(f)f\bar{d}(\mathcal{P})}\right) + S(r, f) \\ &\leq T\left(r, \frac{\mathcal{P}[f]}{f\bar{d}(\mathcal{P})}\right) + S(r, f) \\ &= m\left(r, \frac{\mathcal{P}[f]}{f\bar{d}(\mathcal{P})}\right) + N\left(r, \infty; \frac{\mathcal{P}[f]}{f\bar{d}(\mathcal{P})}\right) + S(r, f) \\ &\leq (\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}))m\left(r, \frac{1}{f}\right) + N(r, \infty; \mathcal{P}[f]) + \bar{d}(\mathcal{P})N(r, 0; f) + S(r, f) \\ &= (\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P}))(T(r, f) - N(r, 0; f)) + S(r, f), \end{aligned}$$

i.e., $(\lambda + \underline{d}(\mathcal{P}))T(r, f) \leq S(r, f)$, which is impossible.

This completes the proof. \square

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