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MEANS OF INFINITE SETS I

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Abstract. We open a new field on how one can define means on infinite sets. We investigate many different ways on how such means can be constructed. One method is based on sequences of ideals, other deals with accumulation points, one uses isolated points, other deals with average using integral, other with limit of average on surroundings and one deals with evenly distributed samples. We study various properties of such means and their relations to each other.

1. Introduction

As well known one can calculate the (weighted) arithmetic mean of finitely many numbers, some infinite series (see [7] Chapter V.), certain measurable functions (see [7] Chapter VI., [2] Chapter VI.1.), and there is a straightforward generalization for sets with finite positive Lebesgue measure (see Def 1.1).

In this paper we are going to study the ways of how can one generalize the arithmetic mean for an infinite bounded subset of $\mathbb{R}$ i.e. roughly speaking we are going to study the means of sets.

In this paper our aim is to find reasonably good and natural means for infinite bounded sets. Then study their properties and relations among them. We are going to present many methods where in some of them we deal with countable sets only.

Most of the methods described here can be easily generalized to quasi-arithmetic means as well or to more general means, however we are not going to deal such generalizations now. In this paper we focus on arithmetic type means only.

We are planning a second paper on this topic (see [8]). While this current paper mainly deals with constructing means and investigate their properties, the second paper is going to focus mainly on building and analysing concepts of this new field.

1.1. Basic notions and notations

Throughout this paper function $\mathscr{A}()$ will denote the arithmetic mean of any number of variables.

DEFINITION 1.1. If $H \subset \mathbb{R}$ is bounded, Lebesgue measurable, $\lambda(H) > 0$ then set

$$Avg(H) = \frac{\int_H x \, d\lambda}{\lambda(H)}.$$

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For $K \subset \mathbb{R}$, $y \in \mathbb{R}$ set $K^{y-} = K \cap (-\infty, y]$, $K^{y+} = K \cap [y, +\infty)$.

If $H \subset \mathbb{R}$, $\varepsilon > 0$ we use the notation $S(H,\varepsilon) = \bigcup_{x \in H} S(x,\varepsilon)$ where $S(x,\varepsilon) = \{y : |x-y| < \varepsilon\}$. Clearly $S(H,\varepsilon) = \{y : \exists x \in H\ |x-y| < \varepsilon\}$.

Let T_s denote the reflection to point $s \in \mathbb{R}$ that is $T_s(x) = 2s - x$ $(x \in \mathbb{R})$. $H \subset \mathbb{R}$ is called symmetric if there is $s \in \mathbb{R}$ such that $T_s(H) = H$.

If $H \subset \mathbb{R}, x \in \mathbb{R}$ then set $H + x = \{h + x : h \in H\}$. Similarly $\alpha H = \{\alpha h : h \in H\}$ $(\alpha \in \mathbb{R})$.

$cl(H), H'$ will denote the closure and accumulation points of $H \subset \mathbb{R}$ respectively. Let $\underline{\lim} H = \inf H'$, $\overline{\lim} H = \sup H'$ for infinite bounded H.

DEFINITION 1.2. A **generalized mean** is a function $\mathscr{K} : C \to \mathbb{R}$ where $C \subset P(\mathbb{R})$ consists of some (finite or infinite) bounded subsets of $\mathbb{R}$ and $\inf H \leqslant \mathscr{K}(H) \leqslant \sup H$ holds for all $H \in C$. We call $\mathscr{K}$ an **ordinary mean** if C consists of finite sets only.

We will verify that Avg is a generalized mean. Another simple example on all bounded sets is

$$\mathscr{M}^{lis}(H) = \begin{cases} \mathscr{A}(H) & \text{if } H \text{ is finite} \\ \frac{\underline{\lim} H + \overline{\lim} H}{2} & \text{otherwise.} \end{cases}$$

In the definition the required condition is an obvious generalization of internality on finite sets. Hence we will refer to this condition ($H \in Dom(\mathscr{K}) \Rightarrow \inf H \leqslant \mathscr{K}(H) \leqslant \sup H$) as **internality** too.

In this paper when we use the term "mean" we always refer to a generalized mean. Usually $\mathscr{K}, \mathscr{M}$ will denote means and $Dom(\mathscr{K})$ denotes the domain of $\mathscr{K}$.

2. Basic attributes of generalized means

2.1. Basic properties

Throughout these subsections $\mathscr{K}$ will denote a generalized mean.

Usually we expect $Dom(\mathscr{K})$ to be closed under finite union and intersection. Moreover closed under translation, reflection and contraction/dilation.

Some of our means will be the extension of $\mathscr{A}$ that is for finite sets it gives the arithmetic mean of the elements. Nevertheless we also consider means having domain consists of infinite sets only.

DEFINITION 2.1.

- $\mathscr{K}$ is **strong internal** if for all infinite $H \in Dom(\mathscr{K})$

$$\underline{\lim} H \leqslant \mathscr{K}(H) \leqslant \overline{\lim} H.$$

- $\mathscr{K}$ is **monotone** if $\sup H_1 \leqslant \inf H_2$ implies that $\mathscr{K}(H_1) \leqslant \mathscr{K}(H_1 \cup H_2) \leqslant \mathscr{K}(H_2)$. $\mathscr{K}$ is **strong monotone** if $\mathscr{K}$ is strong internal and $\overline{\lim} H_1 \leqslant \underline{\lim} H_2$ implies that $\mathscr{K}(H_1) \leqslant \mathscr{K}(H_1 \cup H_2) \leqslant \mathscr{K}(H_2)$.

- The mean is **translation invariant** if $x \in \mathbb{R}, H \in Dom(\mathscr{K})$ then $H + x \in Dom(\mathscr{K})$, $\mathscr{K}(H+x) = \mathscr{K}(H) + x$.
- $\mathscr{K}$ is **point-symmetric** if $H \in Dom(\mathscr{K})$ is bounded and symmetric and $T_s(H) = H$ holds then $\mathscr{K}(H) = s$.
- $\mathscr{K}$ is **homogeneous** if $H \in Dom(\mathscr{K})$ then $\alpha H \in Dom(\mathscr{K})$, $\mathscr{K}(\alpha H) = \alpha \mathscr{K}(H)$.
- $\mathscr{K}$ is **finite-independent** if $H \in Dom(\mathscr{K})$ is infinite, V is finite then $H \cup V, H - V \in Dom(\mathscr{K})$ and $\mathscr{K}(H) = \mathscr{K}(H \cup V) = \mathscr{K}(H - V)$.

PROPOSITION 2.2. *If $\mathscr{K}$ is strong internal and $H' = \{h\}$ then $\mathscr{K}(H) = h$.*

Proof. $h = \underline{\lim} H \leqslant \mathscr{K}(H) \leqslant \overline{\lim} H = h$. □

PROPOSITION 2.3. *If $\mathscr{K}$ is finite-independent then $\mathscr{K}$ is strongly internal.*

Proof. Let $H \subset \mathbb{R}$, $\varepsilon > 0$. Then $\mathscr{K}(H) = \mathscr{K}(H \cap (-\infty, \overline{\lim} H + \varepsilon]) \leqslant \overline{\lim} H + \varepsilon$ because we left out finitely many points. Since ε was arbitrary we get that $\mathscr{K}(H) \leqslant \overline{\lim} H$. Similar argument can be applied to $\underline{\lim}$. □

2.2. Some other properties

DEFINITION 2.4.

- $\mathscr{K}$ is **convex** if I is a closed interval and $\mathscr{K}(H) \in I$, $L \subset I, H \cup L \in Dom(\mathscr{K})$ then $\mathscr{K}(H \cup L) \in I$.
- $\mathscr{K}$ is called **closed** if $H, cl(H) \in Dom(\mathscr{K})$ then $\mathscr{K}(cl(H)) = \mathscr{K}(H)$.
- $\mathscr{K}$ is called **accumulated** if $H, H' \in Dom(\mathscr{K})$ then $\mathscr{K}(H') = \mathscr{K}(H)$.

Obviously property "accumulated" is equivalent with that $\mathscr{K}(H) = \mathscr{K}(H') = \mathscr{K}(H'') = \mathscr{K}(H''') = \dots$ if all sets are in $Dom(\mathscr{K})$.

We will often use the following simple fact.

LEMMA 2.5. *$\mathscr{A}$ is convex.*

Proof. Let I be a closed interval, H, L are finite and $\mathscr{A}(H) \in I$, $L \subset I$. It is known that if A, B are disjoint finite sets with cardinality $|A| = a, |B| = b$

$$\mathscr{A}(A \cup B) = \frac{a\mathscr{A}(A) + b\mathscr{A}(B)}{a+b} = \frac{a}{a+b}\mathscr{A}(A) + \frac{b}{a+b}\mathscr{A}(B)$$

that is the convex combination of $\mathscr{A}(A)$ and $\mathscr{A}(B)$ hence it is between $\mathscr{A}(A)$ and $\mathscr{A}(B)$.

Now apply this to H and $L - H$. For both $\mathscr{A}(H), \mathscr{A}(L - H) \in I$ hence so is $\mathscr{A}(H \cup L)$. □

We now present two negative results on means being continuous according to the Hausdorff pseudo-metric. Let us recall the definition of Hausdorff pseudo-metric. If $X,Y \subset \mathbb{R}$ then $d_H(X,Y) = \inf\{\varepsilon \geqslant 0 : X \subset S(Y,\varepsilon), Y \subset S(X,\varepsilon)\}$.

PROPOSITION 2.6. *If $\mathscr{K}$ is strongly internal then it fails to be a continuous function according to the Hausdorff pseudo-metric.*

Proof. Let $H = \{0, \frac{1}{n} : n \in \mathbb{N}\} \cup \{1, 1+\frac{1}{n} : n \in \mathbb{N}\}$. By strong-internality $\mathscr{K}(H) \in [0,1]$. Let $L_{2k} = \{\frac{1}{n} : n \leqslant k\} \cup \{1, 1+\frac{1}{n} : n \in \mathbb{N}\}, L_{2k+1} = \{0, \frac{1}{n} : n \in \mathbb{N}\} \cup \{1+\frac{1}{n} : n \leqslant k\}$. Then clearly $L_k \to H$ in the Hausdorff metric and by strong-internality $\mathscr{K}(L_{2k}) = 1, \mathscr{K}(L_{2k+1}) = 0$ for all $k \in \mathbb{N}$. $\square$

PROPOSITION 2.7. *$\mathscr{A}$ is not a continuous function according to the Hausdorff metric.*

Proof. Let $C = \{0,1\}$, $C_n = \{\frac{1}{n}; 1+\frac{1}{n}; 1+\frac{1}{2n}\}$. Clearly $C_n \to C$ in the Hausdorff metric but $\mathscr{A}(C_n) \to \frac{2}{3}$, $\mathscr{A}(C) = \frac{1}{2}$. $\square$

3. Simple generalized means

3.1. Mean by isolated points

If the isolated points determine the set in the sense that $cl(H - H') = H$ then a mean can be defined by them using that for $\forall \delta > 0$ $H - S(H', \delta)$ is finite.

DEFINITION 3.1. If $cl(H - H') = H$ then let

$$\mathscr{M}^{iso}(H) = \lim_{\delta \to 0+0} \mathscr{A}(H - S(H', \delta))$$

if it exists.

LEMMA 3.2. *Let $(H_n), (L_n)$ be two infinite sequences of finite sets such that all sets are uniformly bounded, $\forall n\, H_n \cap L_n = \emptyset$ and $\mathscr{A}(H_n) \to a$. Moreover $\lim_{n\to\infty} \frac{|L_n|}{|H_n|} = 0$. Then $\mathscr{A}(H_n \cup L_n) \to a$.*

Proof. Clearly

$$\mathscr{A}(H_n \cup L_n) = \frac{\sum\limits_{h_i \in H_n} h_i + \sum\limits_{h_j \in L_n} h_j}{|H_n \cup L_n|} = \frac{|H_n|}{|H_n \cup L_n|} \mathscr{A}(H_n) + \frac{|L_n|}{|H_n \cup L_n|} \mathscr{A}(L_n).$$

$\mathscr{A}(L_n)$ is bounded, $\frac{|L_n|}{|H_n \cup L_n|} \to 0$ and $\frac{|H_n|}{|H_n \cup L_n|} \to 1$ give the statement. $\square$

THEOREM 3.3. *$\mathscr{M}^{iso}$ is a generalized mean. Moreover it is finite-independent, strongly internal, monotone, translation invariant, point-symmetric, homogeneous, convex and closed.*

Proof. Clearly $\mathscr{M}^{iso}$ is internal since $H-S(H',\delta)\subset[\inf H,\sup H]$.

It is also finite-independent because $H-H'$ is infinite and removing (or adding) finitely many new points would not change the limit. In order to prove that let $H_n = H-S(H',\delta)$, $L_n=\{$the new points in $H-S(H',\delta)\}$. Then apply 3.2.

Strong internality then follows from 2.3.

Let us show monotonicity. If $\sup H_1\leqslant\inf H_2$ then

$$H_1\cup H_2-S((H_1\cup H_2)',\delta)=(H_1-S(H_1',\delta))\cup(H_2-S(H_2',\delta))$$

which gives that $\mathscr{A}(H_1-S(H_1',\delta))\leqslant\mathscr{A}(H_1\cup H_2-S((H_1\cup H_2)',\delta))$. When taking the limit we end up with $\mathscr{M}^{iso}(H_1)\leqslant\mathscr{M}^{iso}(H_1\cup H_2)$. The other inequality is similar.

To prove that $\mathscr{M}^{iso}$ is translation invariant, point-symmetric, homogeneous, it is enough to refer to the fact that $H-S(H',\delta)$ and $\mathscr{A}$ both have the same properties.

To verify convexity let I be a closed interval, $\mathscr{M}^{iso}(H)\in I$, $L\subset I$, $L, L\cup H\in Dom\,\mathscr{M}^{iso}$. It is known that if A,B are disjoint finite sets with cardinality $|A|=a,|B|=b$

$$\mathscr{A}(A\cup B)=\frac{a\mathscr{A}(A)+b\mathscr{A}(B)}{a+b}$$

that is the convex combination of $\mathscr{A}(A)$ and $\mathscr{A}(B)$. If we apply it for $A_\delta=H-S(H',\delta)$, $B_\delta=\big(H\cup L-S((H\cup L)',\delta)\big)-A_\delta\subset L-S(L',\delta)$ then $\mathscr{A}(A_\delta)\to p\in I$, $\mathscr{A}(B_\delta)\in I$ hence in limit $(\delta\to 0+0)$ we get that $\mathscr{A}(A_\delta\cup B_\delta)\to q\in I$ using that the limit exists because $H\cup L\in Dom\,\mathscr{M}^{iso}$.

To show that $\mathscr{M}^{iso}$ is closed it is enough to mention that H and $cl(H)$ have the same set of isolated points. □

EXAMPLE 3.4. For $H=\{0,1\}\cup\{\frac{1}{n}:n\in\mathbb{N}\}\cup\{1+\frac{1}{2^n}:n\in\mathbb{N}\}$, $\mathscr{M}^{iso}(H)=0$.

Proof. Evidently $H'=\{0,1\}$. If $\delta=\frac{1}{k}$ then

$$H-S(H',\delta)=\{\frac{1}{n}:n<k\}\cup\{1+\frac{1}{2^n}:2^n<k\}.$$

If we apply 3.2 for $H_k=\{\frac{1}{n}:n<k\}$, $L_k=\{1+\frac{1}{2^n}:2^n<k\}$ then we get the statement. □

EXAMPLE 3.5. $\mathscr{M}^{iso}(H)$ does not exist always.

Proof. Define a set in the following way. Let $H_1=\{1.7\}$. If $H_1,\dots,H_{n-1}$ are already defined then let H_n consists of some finitely many points such that

$$H_n\subset\begin{cases}\left(\frac{1}{n+1},\frac{1}{n}\right) & \text{if n is even}\\ 1+\left(\frac{1}{n+1},\frac{1}{n}\right) & \text{if n is odd}\end{cases}$$

and $\mathscr{A}(H_1\cup\ldots\cup H_n)\leqslant\frac{1}{4}$ when n is even, $\mathscr{A}(H_1\cup\ldots\cup H_n)\geqslant\frac{3}{4}$ when n is odd. Then let $H=\bigcup\limits_{i=1}^{\infty}H_i$.

We then ended up with an infinite set $H \subset [0,2]$ such that $H' = \{0,1\}$ and $\mathscr{A}(H - S(H',\delta))$ can be smaller than $\frac{1}{4}$ or greater than $\frac{3}{4}$ depending on δ hence the limit does not exists. □

THEOREM 3.6. *$\mathscr{M}^{iso}(H)$ is not accumulated.*

Proof. It is easy to construct a set $H \subset [0,1]$ such that $H' = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\forall \delta\ \mathscr{A}(\{h \in H : h > \delta\}) \geqslant 0.5$. For that set we get $\mathscr{M}^{iso}(H) \geqslant 0.5$, $\mathscr{M}^{iso}(H') = 0$.

For constructing such set let $H_1 = \{1.5\}$. If $H_1, \dots, H_{n-1}$ are already defined then let H_n consists of some points such that $H_n \subset S\big(\{\frac{1}{k} : k < n\}, \frac{1}{n}\big), \forall k < n\ H_n \cap S(\frac{1}{k}, \frac{1}{n}) \neq \emptyset$ and $\mathscr{A}(H_1 \cup \ldots \cup H_n) \geqslant 0.5$. Obviously it can be done since we can add as many points around 1 as we want. Then let $H = \bigcup\limits_{i=1}^{\infty} H_i$. □

3.2. Mean by accumulation points

Let us recall the classic definition. $H^{(0)} = H$, $H^{(1)} = H'$ where H' denotes the accumulation points of H. Then $H^{(n+1)} = (H^{(n)})'$ $(n \geqslant 0)$.

Assume that H is infinite bounded. Then there are two cases. Either there is $n \in \mathbb{N}$ such that $H^{(n)} = \emptyset$ or $\forall n \in \mathbb{N}\ H^{(n)} \neq \emptyset$. We can define a mean in the first case.

DEFINITION 3.7. Let $H \subset \mathbb{R}$. Let $lev(H) = n \in \mathbb{N} \cup \{0\}$ if $H^{(n+1)} = \emptyset$ and $H^{(n)} \neq \emptyset$. Otherwise let $lev(H) = +\infty$.

DEFINITION 3.8. Let $H \subset \mathbb{R}$, $lev(H) = n$. Set $\mathscr{M}^{acc}(H) = \mathscr{A}\left(H^{(n)}\right)$.

In this sense we may say that the last level accumulation points determine the mean and nothing else. Roughly speaking the last accumulation points store the only ”weights” of the set.

LEMMA 3.9. $lev(H \cup K) = \max\{lev(H), lev(K)\}$.

Proof. It is known that $(H \cup K)' = H' \cup K'$. From that by induction we get that $(H \cup K)^{(n)} = H^{(n)} \cup K^{(n)}$. Which implies that $(H \cup K)^{(n)} = \emptyset$ iff $H^{(n)} = \emptyset$ and $K^{(n)} = \emptyset$.

Now let $m = \max\{lev(H), lev(K)\}$. Then $(H \cup K)^{(m)} \neq \emptyset$ and $(H \cup K)^{(m+1)} = \emptyset$ which gives the statement. □

LEMMA 3.10. *If $lev(H) < lev(K)$ then $\mathscr{M}^{acc}(H \cup K) = \mathscr{M}^{acc}(K)$.*

Proof. By 3.9 $lev(H \cup K) = lev(K)$. Then $(H \cup K)^{(lev(K))} = K^{(lev(K))}$. □

LEMMA 3.11. $lev(H \cap K) \leqslant \min\{lev(H), lev(K)\}$.

Proof. It is known that if $A \subset B$ then $A' \subset B'$ and then by induction $A^{(n)} \subset B^{(n)}$. Apply it for $H \cap K$ and H and then K. □

THEOREM 3.12. *$\mathscr{M}^{acc}$ is strongly internal, finite-independent, translation invariant, point-symmetric, homogeneous, convex, closed and accumulated generalized mean.*

Proof. First of all we remark that the definition of $\mathscr{M}^{acc}(H)$ makes sense since $H^{(n)}$ is finite when $H^{(n+1)}=\emptyset$.

$\mathscr{M}^{acc}$ is strongly internal because $\underline{\lim} H=\min H'$, $\overline{\lim} H=\max H'$. This gives that $\mathscr{M}^{acc}$ is a generalized mean.

$\mathscr{M}^{acc}$ is finite-independent since H' does not change if remove or add finitely many points to H.

It is translation invariant, point-symmetric, homogeneous since the accumulation operator has the same properties.

To verify convexity let I be a closed interval, $\mathscr{M}^{acc}(H)\in I$, $L\subset I$, $L,L\cup H\in Dom\ \mathscr{M}^{acc}$. Let $lev(H)=n, lev(L)=k$. Now we have three cases: $n<k$, $n>k$, $n=k$. Using 3.10 the first two are obviously implies that $\mathscr{M}^{acc}(H\cup L)\in I$. For the third 2.5 gives the statement.

$\mathscr{M}^{acc}$ is closed because $cl(H)'=H'$.

$\mathscr{M}^{acc}$ is accumulated since $lev(H')=lev(H)-1$ and

$$H^{(lev(H))}=(H')^{(lev(H)-1)}=(H')^{(lev(H'))}. \quad \square$$

THEOREM 3.13. *If either $lev(H)\neq lev(K)$ or $lev(H)=lev(K)=n$ and $H^{(n)}\cap K^{(n)}=\emptyset$ then $\mathscr{M}^{acc}(H\cup K)\in[\mathscr{M}^{acc}(H),\mathscr{M}^{acc}(K)]$.*

Proof. The first case is obvious. For the second case apply

$$\mathscr{A}(A\cup B)=\frac{a\mathscr{A}(A)+b\mathscr{A}(B)}{a+b}$$

when $A\cap B=\emptyset$ and $a=|A|,b=|B|$. $\square$

EXAMPLE 3.14. $H\cap K=\emptyset$ does not imply that $\mathscr{M}^{acc}(H\cup K)\in [\mathscr{M}^{acc}(H),\mathscr{M}^{acc}(K)]$.

Proof. To show that it is easy to construct sets such that $H\cap K=\emptyset$ and $H'=\{-2,-1,3\}$, $K'=\{-1,1\}$. Then $\mathscr{M}^{acc}(H)=\mathscr{M}^{acc}(K)=0$ while $\mathscr{M}^{acc}(H\cup K)=\mathscr{A}(\{-2,-1,1,3\})=\frac{1}{4}$. $\square$

3.3. Means by ideals

We define a generalized mean with respect to ideals for sets which are not in the ideal in question.

Let us recall the definition of an ideal. $\mathscr{I}\subset P(\mathbb{R})$ is an ideal if $A,B\in\mathscr{I}$ implies that $A\cup B\in\mathscr{I}$ and $B\in\mathscr{I},A\subset B$ implies that $A\in\mathscr{I}$.

DEFINITION 3.15. Let $\mathscr{I}$ be an ideal. We call $\mathscr{I}$
translation invariant if $H \in \mathscr{I}, x \in \mathbb{R}$ implies $H + x \in \mathscr{I}$,
symmetric if $H \in \mathscr{I}, x \in \mathbb{R}$ implies $\{x + y : x - y \in H\} \in \mathscr{I}$,
homogeneous if $H \in \mathscr{I}, \alpha \in \mathbb{R}$ implies $\alpha H \in \mathscr{I}$.

Evidently the regularly used ideals (e.g. finite sets, countable sets, category 1 sets, sets with Lebesgue measure 0) all have these properties.

DEFINITION 3.16. Let $\mathscr{I}$ be an ideal and $H \subset \mathbb{R}$, $H \notin \mathscr{I}$ be bounded. Set $\overline{\lim}^{\mathscr{I}} H = \inf\{x : H^{x+} \in \mathscr{I}\}$. Similarly $\underline{\lim}^{\mathscr{I}} H = \sup\{x : H^{x-} \in \mathscr{I}\}$.

If $\mathscr{I} = \{\emptyset\}$ then $\overline{\lim}^{\mathscr{I}} = \sup, \underline{\lim}^{\mathscr{I}} = \inf$. If $\mathscr{I} = \{\text{finite sets}\}$ then $\overline{\lim}^{\mathscr{I}} = \overline{\lim}, \underline{\lim}^{\mathscr{I}} = \underline{\lim}$. If $\mathscr{I} = \{\text{countable sets}\}$ then $\underline{\lim}^{\mathscr{I}}, \overline{\lim}^{\mathscr{I}}$ are the minimal/maximal consendation points of H. If $\mathscr{I} = \{\text{sets with Lebesgue measure } 0\}$ then $\underline{\lim}^{\mathscr{I}}, \overline{\lim}^{\mathscr{I}}$ are the inf/sup of Lebesgue density points of H.

PROPOSITION 3.17. *If $\mathscr{I}$ is a σ-ideal, $H \notin \mathscr{I}$ then* $\overline{\lim}^{\mathscr{I}} H = \min\{x : H^{x+} \in \mathscr{I}\}$, $\underline{\lim}^{\mathscr{I}} H = \max\{x : H^{x-} \in \mathscr{I}\}$. $\square$

PROPOSITION 3.18. *If $\mathscr{I}_1 \subset \mathscr{I}_2$, $H \notin \mathscr{I}_2$ then* $\underline{\lim}^{\mathscr{I}_1} H \leqslant \underline{\lim}^{\mathscr{I}_2} H \leqslant \overline{\lim}^{\mathscr{I}_2} H \leqslant \overline{\lim}^{\mathscr{I}_1} H$. $\square$

DEFINITION 3.19. If $\mathscr{I}$ is an ideal, $H \notin \mathscr{I}$ then $\mathscr{M}^{\mathscr{I}}(H) = \frac{\underline{\lim}^{\mathscr{I}}(H) + \overline{\lim}^{\mathscr{I}}(H)}{2}$.

THEOREM 3.20. *Let $\mathscr{I}$ is an ideal. Then $\mathscr{M}^{\mathscr{I}}$ is a monotone, convex generalized mean. If $\mathscr{I}$ is translation invariant, point-symmetric, homogeneous then the mean $\mathscr{M}^{(\mathscr{I})}$ has all these properties as well. If $\{$finite sets$\} \subset \mathscr{I}$ then it is finite-independent and strong internal.*

Proof. $\inf H \leqslant \underline{\lim}^{\mathscr{I}}(H) \leqslant \sup H$, $\inf H \leqslant \overline{\lim}^{\mathscr{I}}(H) \leqslant \sup H$ gives that $\mathscr{M}^{\mathscr{I}}$ is internal i.e. a mean.

If $\sup H_1 \leqslant \inf H_2$ then $\underline{\lim}^{\mathscr{I}}(H_1) \leqslant \underline{\lim}^{\mathscr{I}}(H_1 \cup H_2)$, $\overline{\lim}^{\mathscr{I}}(H_1) \leqslant \overline{\lim}^{\mathscr{I}}(H_1 \cup H_2)$ which yields that $\mathscr{M}^{\mathscr{I}}(H_1) \leqslant \mathscr{M}^{\mathscr{I}}(H_1 \cup H_2)$. The other part of monotonicity can be handled similarly.

To verify convexity let I be a closed interval, $\mathscr{M}^{\mathscr{I}}(H) \in I$, $L \subset I$, $L, L \cup H \in Dom\ \mathscr{M}^{\mathscr{I}}$. Clearly if $x > \max I$ then $H^{x+} \in \mathscr{I}$ and because of $L \subset I$ we get that $(H \cup L)^{x+} \in \mathscr{I}$ which gives that $\overline{\lim}^{\mathscr{I}}(H \cup L) \leqslant \max I$. The other inequality is similar.

If $\mathscr{I}$ is translation invariant, point-symmetric, homogeneous then so are $\underline{\lim}^{\mathscr{I}}, \overline{\lim}^{\mathscr{I}}$ and then so is $\mathscr{M}^{\mathscr{I}}$.

If $\{$finite sets$\} \subset \mathscr{I}$ then evidently $\mathscr{M}^{\mathscr{I}}$ is finite-independent hence it is strong internal by 2.3. $\square$

If $\mathscr{I} = \{\text{finite sets}\}$ then $\mathscr{M}^{(\mathscr{I})} = \mathscr{M}^{lis}$. If $\mathscr{I} = \{\emptyset\}$ then $\mathscr{M}^{(\mathscr{I})}(H) = \frac{\inf H + \sup H}{2}$ that is clearly not strong internal.

DEFINITION 3.21. Let $(\mathscr{I}_n)$ be a sequence of ideals such that $\mathscr{I}_0 = \{\text{finite sets}\}$ and $\mathscr{I}_0 \subset \mathscr{I}_1 \subset \mathscr{I}_2 \subset \dots$. The mean associated to this sequence is definied by

$$\mathscr{M}^{(\mathscr{I}_n)}(H) = \begin{cases} \mathscr{A}(H) & \text{if } H \text{ is finite} \\ \frac{\underline{\lim}^{\mathscr{I}_n} H + \overline{\lim}^{\mathscr{I}_n} H}{2} & \text{if } H \in \mathscr{I}_{n+1} - \mathscr{I}_n \\ \lim_{n\to\infty} \frac{\underline{\lim}^{\mathscr{I}_n} H + \overline{\lim}^{\mathscr{I}_n} H}{2} & \text{if } H \notin \bigcup_0^\infty \mathscr{I}_n. \end{cases}$$

REMARK 3.22. (1) Because of Proposition 3.18 the limit in the last defining line always exists.

(2) The definition works for a finite sequence of ideals as well (simply set $\mathscr{I}_n = \mathscr{I}_k$ if $n \geqslant k$ for a certain k).

(3) We can omit the condition that $\mathscr{I}_0 = \{\text{finite sets}\}$. In that case $\mathscr{M}^{(\mathscr{I}_n)}$ remains undefined for infinite sets $H \in \mathscr{I}_1$.

The next theorem can be proved like 3.20.

THEOREM 3.23. *Let $(\mathscr{I}_n)$ be a sequence of ideals such that $\mathscr{I}_0 = \{\text{finite sets}\}$ and $\mathscr{I}_0 \subset \mathscr{I}_1 \subset \mathscr{I}_2 \subset \dots$. Then $\mathscr{M}^{\mathscr{I}}$ is a monotone generalized mean. If $\mathscr{I}_n$ is translation invariant, point-symmetric, homogeneous then the mean $\mathscr{M}^{(\mathscr{I}_n)}$ has all these properties as well.* □

4. Properties of generalized *Avg*

We can generalize *Avg* in the following way.

DEFINITION 4.1. Let μ^s denote the s-dimensional Hausdorff measure in $\mathbb{R}$ $(0 \leqslant s \leqslant 1)$. If $0 < \mu^s(H) < +\infty$ and H is μ^s measurable (i.e. H is an s-set) then set

$$Avg(H) = \frac{\int_H x \, d\mu^s}{\mu^s(H)}.$$

If for a given s we restrict *Avg* for s-sets then we will use the notation Avg^s.

Clearly $Avg = Avg^0 = \mathscr{A}$ for finite sets and we get back the original definition of Avg^1 for sets with positive Lebesgue measure (Definition 1.1).

THEOREM 4.2. *Avg is translation invariant, point-symmetric, homogeneous.*

Proof. All properties are a consequence of the theorem on integral by substitution. Let us see them one by one.

We show that *Avg* is translation invariant: Let $h(x) = x + r$ $(r \in \mathbb{R})$, H be an s-set $(0 \leqslant s \leqslant 1)$. Then

$$\frac{\int_{h(H)} x d\mu^s}{\mu^s(h(H))} = \frac{\int_H x \circ h(x) d\mu^s}{\mu^s(H)} = \frac{\int_H x + r d\mu^s}{\mu^s(H)} = \frac{\int_H x d\mu^s}{\mu^s(H)} + \frac{\int_H r d\mu^s}{\mu^s(H)} = Avg(H) + r$$

where we also used the straightforward fact that $\mu^s(H+r)=\mu^s(H)$.

We prove that *Avg* is point-symmetric: By translation invariance it is enough to handle the case when H is symmetric for 0: if H is symmetric for $p\in\mathbb{R}$ then $Avg(H-p)+p=Avg(H)$ and $H-p$ is symmetric for 0 and if $Avg(H-p)=0$ then it would give that $Avg(H)=p$. Let $h(x)=-x$. Then

$$\int_{H^{0-}} x d\mu^s = \int_{h^{-1}(H^{0-})} x\circ h(x) d\mu^s = \int_{H^{0+}} -x d\mu^s = -\int_{H^{0+}} x d\mu^s$$

which implies that $Avg(H)=0$.

Now we verify that *Avg* is homogeneous: Let $h(x)=\alpha x$ $(\alpha\in\mathbb{R})$. Then

$$\frac{\int\limits_{h(H)} x d\mu^s}{\mu^s(h(H))} = \frac{\int\limits_{h(H)} x d\mu^s}{\int\limits_{h(H)} 1 d\mu^s} = \frac{\int\limits_H x\circ h(x)\cdot\alpha d\mu^s}{\int\limits_H 1\circ h(x)\cdot\alpha d\mu^s} = \frac{\alpha^2\int\limits_H x d\mu^s}{\alpha\int\limits_H 1 d\mu^s} = \alpha Avg(H). \quad \square$$

We can show now that *Avg* is a generalized mean. For that we prove that it is strong-internal in the following stronger sense.

PROPOSITION 4.3. *Let $H\subset\mathbb{R}$ be a bounded s-set ($0\leqslant s\leqslant 1$). Then $\underline{\lim}^{\mathscr{I}} H < Avg(H) < \overline{\lim}^{\mathscr{I}} H$ where $\mathscr{I}=\{H\subset\mathbb{R} : \mu^s(H)=0\}$.*

Proof. By being translation invariant we can assume that $\underline{\lim}^{\mathscr{I}} H=0$. We have to prove that $Avg(H)>0$ that is equivalent with $\int\limits_H x\, d\mu^s>0$. Clearly there is $n\in\mathbb{N}$ such that $\mu^s(H^{\frac{1}{n}+})>0$. Then $0<\frac{1}{n}\mu^s(H^{\frac{1}{n}+})\leqslant \int\limits_{H^{\frac{1}{n}+}} x\, d\mu^s \leqslant \int\limits_H x\, d\mu^s$.

The other inequality can be handled similarly. $\square$

LEMMA 4.4. *If H_1,H_2 are s-sets and $H_1\cap H_2=\emptyset$ then*

$$Avg(H_1\cup H_2)=\frac{\mu^s(H_1)Avg(H_1)+\mu^s(H_2)Avg(H_2)}{\mu^s(H_1)+\mu^s(H_2)}$$

Proof.

$$Avg(H_1\cup H_2)=\frac{\int\limits_{H_1} x d\mu^s+\int\limits_{H_2} x d\mu^s}{\mu^s(H_1)+\mu^s(H_2)}=\frac{\mu^s(H_1)Avg(H_1)+\mu^s(H_2)Avg(H_2)}{\mu^s(H_1)+\mu^s(H_2)}. \quad \square$$

THEOREM 4.5. *Avg is strong monotone for s-sets with $s>0$.*

Proof. Let H_1 be an s_1-set, H_2 be an s_2-set ($0<s_1,s_2\leqslant 1$) and $p=\overline{\lim} H_1 \leqslant \underline{\lim} H_2 = r$. Evidently $\mu^{s_1}(H_1^{p+})=0$ and $\mu^{s_2}(H_2^{r-})=0$. Hence $Avg(H_1)\leqslant p$ and $Avg(H_2)\geqslant r$.

If $s_1 < s_2$ then $Avg(H_1 \cup H_2) = Avg(H_2) = r \geqslant p \geqslant Avg(H_1)$.

If $s_2 < s_1$ then $Avg(H_1 \cup H_2) = Avg(H_1) = p \leqslant r \leqslant Avg(H_2)$.

If $s_1 = s_2 = s$ then we can assume that H_1, H_2 are disjoint because removing a set with 0 measure does not change Avg i.e. $Avg(H_1) = Avg(H_1 - H_1^{p+})$ and $Avg(H_2) = Avg(H_2 - H_2^{r-})$. Then by 4.4

$$Avg(H_1 \cup H_2) = \frac{\mu^s(H_1)Avg(H_1) + \mu^s(H_2)Avg(H_2)}{\mu^s(H_1) + \mu^s(H_2)}$$

which implies that $Avg(H_1) \leqslant Avg(H_1 \cup H_2) \leqslant Avg(H_2)$. □

EXAMPLE 4.6. Avg is not closed and not accumulated either.

Proof. Let $H = [0,1] \cup ([1,2] \cap \mathbb{Q})$. Then $Avg(H) = Avg^1(H) = 0.5$ while $Avg(cl(H)) = Avg^1(cl(H)) = 1$. □

EXAMPLE 4.7. Symmetry gives $Avg(C) = \frac{1}{2}$ where C is the Cantor set.

THEOREM 4.8. *Avg is convex.*

Proof. Let I be a closed interval, $Avg(A) \in I$, $C \subset I$, $C, C \cup A \in Dom\, Avg$. Let A be an s-set, C be an r-set ($0 \leqslant s, r \leqslant 1$).

If $s < r$ then $A \cup C$ is an r-set and $Avg(A \cup C) = Avg^r(A \cup C) = Avg^r(C) = Avg(C) \in I$. If $r < s$ then $A \cup C$ is an s-set and $Avg(A \cup C) = Avg^s(A \cup C) = Avg^s(A) = Avg(A) \in I$.

Let now $s = r$. If $\mu^s(C - A) = 0$ then the statement is obvious. Let us suppose $\mu^s(C - A) > 0$. By 4.4

$$Avg(A \cup C) = Avg(A \cup^* (C - A)) =$$

$$Avg(A)\frac{\mu^s(A)}{\mu^s(A) + \mu^s(C - A)} + Avg(C - A)\frac{\mu^s(C - A)}{\mu^s(A) + \mu^s(C - A)} \in I$$

because it is a convex combination of $Avg(A)$ and $Avg(C - A)$ and both are in I. □

5. Mean by ε-neighbourhoods of the set

We are going to approximate the set by ε-neighbourhoods and as they have positive Lebesgue measure, take Avg of those as an approximation of the mean of the set.

DEFINITION 5.1. Let $H \subset \mathbb{R}$ arbitrary. Set

$$LAvg(H) = \lim_{\delta \to 0+0} Avg(S(H, \delta))$$

if the limit exists.

PROPOSITION 5.2. *$LAvg(H) = LAvg(cl(H))$ i.e. $LAvg$ is closed.*

Proof. It follows from the fact that $S(cl(H),\delta) = S(H,\delta)$. $\square$

This shows that $Avg(H) \neq LAvg(H)$ in general since Avg is not closed.

THEOREM 5.3. *Let $H \subset \mathbb{R}$ be a finite set. Then $LAvg(H) = \mathscr{A}(H)$.*

Proof. Let $\delta < \frac{1}{2}\min\{|x-y| : x,y \in H, x \neq y\}$. Then

$$AvgS(H,\delta) = \frac{\sum_{x_i \in H} 2\delta x_i}{|H|2\delta} = \frac{\sum_{x_i \in H} x_i}{|H|} = \mathscr{A}(H). \qquad \square$$

THEOREM 5.4. *LAvg is finite-independent for infinite sets.*

Proof. It is enough to prove that for a single point p since from that by induction we can get the statement. If p is an accumulation point then we are done by 5.2. Let us assume that p is an isolated point and $S(p,\delta_0) \cap S(H-\{p\},\delta_0) = \emptyset$. H is infinite and bounded hence contains an infinite sequence $(h_n) \subset H$ consisting of distinct points. It is enough to show that $\lim\limits_{\delta \to 0+0} \frac{\lambda(S(p,\delta))}{\lambda(S(H,\delta))} = 0$ because from that the statement follows since

$$Avg(S(H,\delta)) = \frac{\lambda(S(p,\delta))}{\lambda(S(H,\delta))} Avg(S(p,\delta)) + \frac{\lambda(S(H-\{p\},\delta))}{\lambda(S(H,\delta))} Avg(S(H-\{p\},\delta))$$

whenever $\delta < \delta_0$ i.e. when $S(p,\delta) \cap S(H-\{p\},\delta) = \emptyset$.

For that let $K > 0$. Find $L \subset H$ such that $|L| = K$. Then find $\delta_1 < \delta_0$ such that $l_1, l_2 \in L$ implies that $|l_1 - l_2| < 2\delta_1$. Let $\delta < \delta_1$. Then

$$\frac{\lambda(S(p,\delta))}{\lambda(S(H,\delta))} < \frac{2\delta}{\lambda(S(L,\delta))} = \frac{2\delta}{2\delta K} = \frac{1}{K}$$

which gives that we required when $K \to \infty$. $\square$

PROPOSITION 5.5. *LAvg is strongly internal.*

Proof. By 2.3 and 5.4 it is enough to show that $LAvg$ is internal.

For that let $m = \inf H$, $\varepsilon > 0$. If $\delta < \varepsilon$ then $Avg(S(H,\delta)) > m - \varepsilon$. Because it is true for all ε then $LAvg(H) \geqslant m$. The other inequality is similar. $\square$

THEOREM 5.6. *LAvg is translation invariant, point-symmetric and homogeneous.*

Proof. translation invariance comes from $S(H+r,\delta) = S(H,\delta)+r$ and Avg being translation invariant (4.2).

Symmetry follows from $S(-H,\delta) = -S(H,\delta)$ and Avg being point-symmetric (4.2).

For proving that $LAvg$ is homogeneous let $\alpha \in \mathbb{R}$. Then

$$Avg(S(\alpha H,\delta)) = Avg(\alpha S(H,\frac{1}{\alpha}\delta)) = \alpha Avg(S(H,\frac{1}{\alpha}\delta)).$$

When $\delta \to 0+0$ then the left hand side tends to $LAvg(\alpha H)$ while the right hand side tends to $\alpha LAvg(H)$. $\square$

LEMMA 5.7. *Let $H \subset \mathbb{R}$ be compact. Then $\forall \varepsilon > 0\ \exists \delta_0 > 0$ such that $\delta < \delta_0$ implies $\lambda(S(H,\delta)) < \lambda(H) + \varepsilon$.*

Proof. For $\frac{\varepsilon}{2}$ there are open intervals I_i $(i \in \mathbb{N})$ such that $H \subset \bigcup_1^\infty I_i$ and $\sum_1^\infty \lambda(I_i) < \lambda(H) + \frac{\varepsilon}{2}$. H being compact yields that finitely many covers H as well, e.g. $H \subset \bigcup_1^n I_i$. If we set $\delta_0 = \frac{\varepsilon}{4n}$ then $\delta < \delta_0$ implies that $\lambda(S(H,\delta)) \leqslant \sum_1^n \lambda(S(I_i,\delta)) < \lambda(H) + \frac{\varepsilon}{2} + 2n\frac{\varepsilon}{4n} = \lambda(H) + \varepsilon$. $\square$

THEOREM 5.8. *Let $H \subset \mathbb{R}$ be bounded, Lebesgue measurable and $\lambda(H) > 0$. Then $Avg(H) = LAvg(H)$ iff $\lambda(cl(H) - H) = 0$ or $Avg(cl(H) - H) = Avg(H)$.*

Proof. Let us assume first that $\lambda(cl(H) - H) = 0$ i.e. $\lambda(cl(H)) = \lambda(H)$. Then clearly $Avg(H) = Avg(cl(H))$ and by 5.2 $LAvg(cl(H)) = LAvg(H)$. Hence it is enough to prove the statement for compact sets.

Let $\varepsilon > 0$ be given. By 5.7 $\forall \varepsilon_0 > 0\ \exists \delta_0 > 0$ such that $\delta_0 < 1, \delta < \delta_0$ implies $\lambda(H) \leqslant \lambda(S(H,\delta)) < \lambda(H) + \varepsilon_0$. Let $K = \sup H + 1$. Then

$$\begin{aligned}
|Avg(H) - Avg(S(H,\delta))| &= \left| \frac{\int_H x\,d\lambda}{\lambda(H)} - \frac{\int_{S(H,\delta)} x\,d\lambda}{\lambda(S(H,\delta))} \right| = \left| \frac{\int_H x\,d\lambda}{\lambda(H)} - \frac{\int_H x\,d\lambda + \int_{S(H,\delta)-H} x\,d\lambda}{\lambda(S(H,\delta))} \right| \\
&\leqslant \left| \int_H x\,d\lambda \right| \left| \frac{1}{\lambda(H)} - \frac{1}{\lambda(S(H,\delta))} \right| + \left| \frac{\int_{S(H,\delta)-H} x\,d\lambda}{\lambda(S(H,\delta))} \right| \\
&\leqslant K\lambda(H) \left| \frac{\lambda(S(H,\delta)) - \lambda(H)}{\lambda(H)\lambda(S(H,\delta))} \right| + \left| \frac{\lambda(S(H,\delta) - H)K}{\lambda(S(H,\delta))} \right| \\
&\leqslant \frac{2K}{\lambda(H)} |\lambda(S(H,\delta)) - \lambda(H)| < \varepsilon
\end{aligned}$$

if $\varepsilon_0 < \frac{\varepsilon\lambda(H)}{2K}$.

Now assume that $\lambda(cl(H) - H) \neq 0, Avg(cl(H) - H) = Avg(H)$. Then

$$Avg(cl(H)) = \frac{\lambda(cl(H) - H)Avg(cl(H) - H) + \lambda(H)Avg(H)}{\lambda(cl(H) - H) + \lambda(H)} = Avg(H). \qquad (1)$$

Assume now that $Avg(H) = LAvg(H)$. Then apply the first assertion for $cl(H)$. We get $Avg(cl(H)) = LAvg(cl(H)) = LAvg(H)$ which yields $Avg(cl(H)) = Avg(H)$. Then (1) gives the statement. $\square$

EXAMPLE 5.9. Let $L = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{2 + \frac{1}{2^n} : n \in \mathbb{N}\}$. Then $LAvg(L) = 0$.

Proof. Let $L_1 = \{\frac{1}{n} : n \in \mathbb{N}\}, L_2 = \{2 + \frac{1}{2^n} : n \in \mathbb{N}\}$.

Let $\delta > 0$. Let us estimate where the δ surroundings $S(x,\delta)$ intersect each other on points of L_1 and L_2. They intersect on points of L_1 when $\frac{1}{n-1} - \frac{1}{n} < 2\delta$. It is $n-1 > \frac{1}{\sqrt{2\delta}}$. They intersect on points of L_2 when $\frac{1}{2^{n-1}} - \frac{1}{2^n} < 2\delta$. It is $n > -\log_2 2\delta$. Then

$$\int\limits_{S(L_1,\delta)} x\, d\lambda < (\sqrt{2\delta} + 2\delta))\sqrt{\frac{\delta}{2}} + 2\delta(1 + \frac{1}{2} + \ldots + \frac{1}{n-2}) <$$

$$< (\sqrt{2\delta} + 2\delta))\sqrt{\frac{\delta}{2}} + 2\delta(1 - \log\sqrt{2\delta}) < 2\delta(1 - \log\sqrt{2\delta})$$

if δ is small enough.

$$\lambda(S(L_1,\delta)) = \sqrt{2\delta} + 2\delta + (n-1-1)2\delta > \sqrt{2\delta} + 2\delta + \sqrt{2\delta} - 2\delta = 2\sqrt{2\delta}$$

$$\int\limits_{S(L_2,\delta)} x\, d\lambda < (2\delta + 2\delta)(2+\delta) + 2\delta(2 + \frac{1}{2^1} + \ldots + 2 + \frac{1}{2^{n-1}}) <$$

$$< (4\delta)(2+\delta) + 2\delta(2(n-1)+1) < (4\delta)(2+\delta) + 2\delta(1 - 2\log_2 2\delta) < 15\delta(1 - \log\sqrt{2\delta})$$

if δ is small enough.

$$\lambda(S(L_2,\delta)) = 4\delta + (n-1)2\delta > 4\delta + 2\delta(-\log_2 2\delta - 1) > 2\delta + 2\delta(-\log 2\delta)$$

$$0 < \frac{\int\limits_{S(L_1,\delta)} x\, d\lambda + \int\limits_{S(L_2,\delta)} x\, d\lambda}{\lambda(S(L_1,\delta)) + \lambda(S(L_2,\delta))} < \frac{17\delta(1 - \log\sqrt{2\delta})}{2\sqrt{2\delta} + 2\delta + 2\delta(-\log 2\delta)} =$$

$$= \frac{17\sqrt{\delta}(1 - \log\sqrt{2\delta})}{2\sqrt{2} + 2\sqrt{\delta} + 2\sqrt{\delta}(-\log 2\delta)} \to 0 \text{ if } \delta \to 0+0$$

using that $\lim_{x\to 0+0} x\log x = 0$. □

PROPOSITION 5.10. *LAvg is not accumulated.*

Proof. The example in 5.9 shows that since $LAvg(L') = 1$ by 5.3. □

PROBLEM 1. Prove or disprove the conjecture that *LAvg* is an extension of $\mathscr{M}^{iso}$.

6. Mean by evenly distributed sample

Now we define a mean in a way that we take finite sample points from the set and calculate their arithmetic mean and we consider it as an approximation for the mean. It is important that the sample has to be evenly distributed.

DEFINITION 6.1. Let $H\subset\mathbb{R}$, $a=\inf H, b=\sup H$. If $n\in\mathbb{N}, 0\leqslant i\leqslant n$ then set $H_{n,i}=H\cap[a+\frac{i}{n}(b-a), a+\frac{i+1}{n}(b-a))$. Let $I_n=\{0\leqslant i\leqslant n : H_{n,i}\ne\emptyset\}$.

We say that the mean of H is $k=\mathcal{M}^{eds}(H)$ if $\forall\varepsilon>0\ \exists N\in\mathbb{N}$ such that $n>N, \xi_i\in H_{n,i}\ (i\in I_n)$ implies that $|\mathcal{A}(\{\xi_i : i\in I_n\})-k|<\varepsilon$.

THEOREM 6.2. *If $H\subset\mathbb{R}$ the following statements are equivalent:*
(1) $\mathcal{M}^{eds}(H)=k$
(2) $\forall n\in\mathbb{N}$ *we select arbitrary points* $\xi_{n,i}\in H_{n,i}\ (i\in I_n)$ *then*

$$\lim_{n\to\infty}\mathcal{A}(\{\xi_{n,i} : i\in I_n\})=k$$

(3) $\forall n\in\mathbb{N}$ *we select arbitrary points* $\xi_{n,i}\in[a+\frac{i}{n}(b-a), a+\frac{i+1}{n}(b-a))\ (i\in I_n)$ *then* $\lim_{n\to\infty}\mathcal{A}(\{\xi_{n,i} : i\in I_n\})=k$
(4) $\lim_{n\to\infty}\mathcal{A}(\{a+\frac{i}{n}(b-a) : i\in I_n\})=k$.

Proof. (1)⇔(2), (3)⇒(4) are obvious. Proving (2)⇔(4)⇔(3) at the same time observe that $|A(\{a+\frac{i}{n}(b-a) : i\in I_n\}-A(\{\xi_i : i\in I_n\})|\leqslant\frac{1}{n}$. □

The following theorem verifies that $\mathcal{M}^{eds}$ is a mean.

THEOREM 6.3. *$\mathcal{M}^{eds}$ is strongly internal.*

Proof. Let $\varepsilon>0$. Then $H\cap(-\infty,\underline{\lim}H-\varepsilon)$ is finite. H being infinte implies that $\lim_{n\to\infty}|\{H_{n,i} : H_{n,i}\ne\emptyset, H_{n,i}\subset(\underline{\lim}H-\varepsilon,+\infty)\}|=\infty$. This gives $\mathcal{M}^{eds}(H)\geqslant\underline{\lim}H-\varepsilon$ by Lemma 3.2. Since ε was arbitrary $\mathcal{M}^{eds}(H)\geqslant\underline{\lim}H$. Similar argument works for $\overline{\lim}$. □

PROPOSITION 6.4. *If H is finite then $\mathcal{M}^{eds}(H)=\mathcal{A}(H)$.*

Proof. If n is big enough then each interval contains only one point. □

THEOREM 6.5. *$\mathcal{M}^{eds}$ is translation-invariant, point-symmetric and homogeneous.*

Proof. translation-invariance follows from that $\inf, \sup$ and $\mathcal{A}$ are translation-invariant.

For symmetry it is enough to show that if $0\leqslant\inf H$ then $\mathcal{M}^{eds}(-H)=-\mathcal{M}^{eds}(H)$. For that we can choose corresponding points in the way that $\xi'_{n,i}=-\xi_{n,i}$ and then we can refer to that $\mathcal{A}$ is point-symmetric.

For being homogeneous let $f(x) = \alpha x$ $(\alpha \in \mathbb{R}^+)$. Note that f takes partition of $[a,b]$ into partition of $[\alpha a, \alpha b]$ and also it takes associated points into associated points of the other partition. Similarly for f^{-1}. For completing the proof we need also that $\mathscr{A}$ is homogeneous. □

PROPOSITION 6.6. *$\mathscr{M}^{eds}$ is monotone and convex.*

Proof. Both statement is a straightforward consequence of $\mathscr{A}$ being monotone and convex. □

PROPOSITION 6.7. *If $H = H_1 \cup^* H_2$ where $\lambda(cl(H_1)) > 0$ and $\lambda(H_2) = 0, H_2$ is compact then $\mathscr{M}^{eds}(H) = \mathscr{M}^{eds}(H_1)$.*

Proof. Let $I_n^j = \{0 \leqslant i \leqslant n-1 : H_{n,i} \neq \emptyset, H_{n,i} \subset H_j\}$ $(j \in \{1,2\})$. Then by 5.7 $\lim_{n\to\infty} \frac{1}{n}|I_n^2| = 0$. While $\inf\{\frac{1}{n}|I_n^1| : n \in \mathbb{N}\} > 0$ which gives the statement using Lemma 3.2. □

The next example shows that we cannot omit compactness.

EXAMPLE 6.8. $\mathscr{M}^{eds}([0,1] \bigcup (\mathbb{Q} \cap [1,2])) = \mathscr{M}^{eds}([0,2]) = 1$ hence $Avg \neq \mathscr{M}^{eds}$.

EXAMPLE 6.9. Let $L = \{\frac{1}{k} : k \in \mathbb{N}\} \cup \{2 + \frac{1}{2^k} : k \in \mathbb{N}\}$. Then $\mathscr{M}^{eds}(L) = 0$.

Proof. Let $a = 0, b = 2.5, n \in \mathbb{N}$. Let us estimate $|\{i \in I_n : \frac{i+1}{n}(b-a) \leqslant 1\}|$ i.e. at least how many points $\xi_{n,i}$ we get that are smaller than 1. We want a lower bound. We can get that if $\frac{1}{k-1} - \frac{1}{k} > \frac{1}{n}$. For that it is sufficient that $k < \sqrt{n}$ hence there are at least $\sqrt{n}$ such points.

Now let us estimate $|\{i \in I_n : \frac{i}{n}(b-a) \geqslant 2\}|$ i.e. how many points $\xi_{n,i}$ we get that are greater than 2. We want an upper bound. We can get that if $\frac{1}{2^{k-1}} - \frac{1}{2^k} = \frac{1}{2^k} > \frac{1}{n}$ that is $k < \log_2 n$.

Now $\lim\limits_{n\to\infty} \frac{\log_2 n}{\sqrt{n}} = 0$ completes the proof by Lemma 3.2. □

THEOREM 6.10. *$\mathscr{M}^{eds}(H) \neq \mathscr{M}^{iso}(H)$.*

Proof. Let $H = \{\frac{1}{2^k} : k \in \mathbb{N}\} \cup \{2 + \frac{1}{2^k}; 2 + \frac{1}{2^k} + \frac{1}{2^{2^k}} : k \in \mathbb{N}\}$.

Clearly $\mathscr{M}^{iso}(H) = \frac{0+2+2}{3} = \frac{4}{3}$.

Let us calculate $\mathscr{M}^{eds}(H)$. If we divide $[\inf H, \sup H]$ into 2^n subintervals then what is required in order to see points $2 + \frac{1}{2^k}, 2 + \frac{1}{2^k} + \frac{1}{2^{2^k}}$ in separate intervals? It is $\frac{1}{2^{2^k}} > \frac{1}{2^n}$ that is $k < \log_2 n$. Therefore we get $n+1$ points smaller than 1 (converging to 0) and at most $n + 1 + \log_2 n$ points greater than 2 (converging to 2). This gives that $\mathscr{M}^{eds}(H) = 1$ by Lemma 3.2. □

Similar example could show that $\mathscr{M}^{eds} \not\leqslant \mathscr{M}^{iso}$ in general.

PROBLEM 2. Provide example that shows that $\mathscr{M}^{eds} \neq LAvg$.

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REFERENCES

[1] J. M. BORWEIN, P. B. BORWEIN, *The way of all means*, Amer. Math. Monthly **94** (1987), 519–522.
[2] P. S. BULLEN, *Handbook of means and their inequalities*, vol. 260 Kluwer Academic Publisher, Dordrecht, The Netherlands (2003).
[3] Z. DARÓCZY AND ZS. PÁLES, *Functional Equations – Results and Advances*, Springer Science & Business Media Dordrecht (2002).
[4] Z. DARÓCZY AND ZS. PÁLES, *On functional equations involving means*, Publ. Math. Debrecen **62** no. 3–4 (2003), 363–377.
[5] B. EBANKS, *Looking for a few good means*, Amer. Math. Monthly **119** (2012), 658–669.
[6] M. HAJJA, *Some elementary aspects of means*, International Journal of Mathematics and Mathematical Sciences, Means and Their Inequalities, Volume 2013, Article ID 698906, 1–9.
[7] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, Cambridge university press (1988).
[8] A. LOSONCZI, *Means of infinite sets II*, prepint, arXiv:1705.06344.
[9] A. LOSONCZI, *Measures by means, means by measures*, preprint, arXiv:1706.03658.
[10] GY. MAKSA, ZS. PÁLES, *Remarks on the comparison of weighted quasi-arithmetic means*, Colloquium Mathematicum **120** (2010), 77–84.

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