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FURTHER RESULTS ABOUT NORMAL CRITERIA AND SHARED VALUES FOR FAMILIES OF MEROMORPHIC FUNCTIONS

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Abstract. Let k be a positive integer and let $\mathscr{F}$ be a family of meromorphic functions in the domain D all of whose zeros with multiplicity at least k. Let P be a polynomial and P have at least one simple zero, $p = \deg(P) \geqslant k+2$. If, for each pair $f,g \in \mathscr{F}$, $P(f)G^m(f)$ and $P(g)G^m(g)$ share a nonzero constant b ignoring multiplicity in D, where $G(f) = P(f^{(k)}) + H(f)$ is a differential polynomial of f satisfying $\frac{w}{\deg}|_H \leqslant \frac{kmq}{l+mq}+1$ or $w(H) - \deg(H) < qk$, and $q > l \geqslant k+1$ is a positive integer, then $\mathscr{F}$ is normal in D.

1. Introduction

Let D be a domain in $\mathbb{C}$ and $\mathscr{F}$ is a family of meromorphic in D. $\mathscr{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathscr{F}$ had a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D, to a meromorphic function or ∞ (see [9], [23], [29]). Let $P(z)$ be a polynomial or a finite complex number. $\deg P(z)$ denotes the degree of the polynomial $P(z)$.

Suppose that $f(z)$, $g(z)$ are meromorphic functions in D and $a \subset \mathbb{C} \cup \{\infty\}$. If $f(z) = a$ if and only if $g(z) = a$, we say that f and g share a ignoring multiplicity (IM) (see [27]).

DEFINITION 1. Let $D \subseteq \mathbb{C}$ be an arbitrary domain, $m, l_1, l_2, \cdots, l_m$ be non-negative integers and $(0 \leqslant l_i \leqslant k)$, if

$$M(f, f', \cdots, f^{(k)}) = a(z)\prod_{i=1}^{m} f^{(l_i)},$$

where f is meromorphic and a is a holomorphic function in D ($a \not\equiv 0$), then $M(f, f', \cdots, f^{(k)})$ is called a differential monomial of degree $\deg(M) := m$ and weight $w(M) := \sum_{i=1}^{m}(1+l_i)$.

The summation $H := M_1 + \cdots + M_n$ of differential monomials M_j is called the differential polynomial. For the convenience in this article, deviating from the common definition, we definite the degree of $\deg(H) := \min\{\deg(M_1), \cdots, \deg(M_n)\}$ and weight $w(H) := \max\{w(M_1), \cdots, w(M_n)\}$, $\frac{w}{\deg}|_H = \max\left\{\frac{w(M_1)}{\deg(M_1)}, \frac{w(M_2)}{\deg(M_2)}, \cdots, \frac{w(M_n)}{\deg(M_n)}\right\}$.

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DEFINITION 2. Let q be a positive integer, and $b_j(z)$ $(j = 1,2...,q-1)$ be analytic function on a domain D.

$$P(f^{(k)})(z) = (f^{(k)}(z))^q + b_{q-1}(z)(f^{(k)}(z))^{q-1} + \cdots + b_1(z)f^{(k)}(z),$$

$$G(f) = P(f^{(k)}) + H(f, f', \cdots, f^{(k)}).$$

In 1959, Hayman [10] proposed:

Conjecture A If f is a transcendental meromorphic function, then $f^n f'$ assumes every finite non-zero complex number infinitely often for any positive integer n.

Hayman [10, 11] himself confirmed it for $n \geqslant 3$ and for $n \geqslant 2$ in the case of entire f. Further, it was proved by Mues [17] when $n \geqslant 2$; Clunie [6] when $n \geqslant 1$ and f is entire; Bergweiler and Eremenko [2] verified the case when $n = 1$ and f is of finite order, and finally by Chen and Fang [5] for the case $n = 1$.

Correspondingly, there is a conjecture of Hayman [11] related to above problem concerning the normality of $\mathscr{F}$ (see [1]).

Conjecture B If each $f \in \mathscr{F}$ satisfies $f^n f' \neq a$ for a positive integer n and a finite non-zero complex number a, then $\mathscr{F}$ is normal.

Concerning this conjecture, there are many significant results have been obtained by Yang and Zhang [30]; Gu [8]; Oshkin [18]; Li and Xie [15]; Pang [19]; Zalcman [31]. Chen and Fang [5] verified the conjecture B completely. Schick([24]) was the first author to draw a connection between value shared by functions in $\mathscr{F}$ and the normality of the family $\mathscr{F}$. Moreover, many scholars had studied normality criterions such as Meng [3]; Lei and Fang [14]; Li and Gu [16]; Pang and Zalcman [20]; Xia and Xu [25].

In 2004, Fang and Zalcman [7] proved:

Theorem A Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let $n \geqslant 1$ be a positive integer, and b be a finite non-zero complex number. If, for each $f, g \in \mathscr{F}$, f and g share 0 IM; $f^n f'$ and $g^n g'$ share b IM in D, then $\mathscr{F}$ is normal in D.

Lately, Zhang [32] obtained:

Theorem B Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, let $n \geqslant 1$ be a positive integer, and b be a finite complex number. If, for each $f, g \in \mathscr{F}$, $f^n(f-1)f'$ and $g^n(g-1)g'$ share b IM, then $\mathscr{F}$ is normal in D.

In 2008, Zhang [33] improved the condition of Theorem A, and obtained:

Theorem C Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let $n \geqslant 2$ be a positive integer, and b be a finite non-zero complex number. If, for each $f, g \in \mathscr{F}$, $f^n f'$ and $g^n g'$ share b IM, then $\mathscr{F}$ is normal in D.

There are examples showing that this result is not true if $n = 1$. Recently, Lei and Fang [13] extended Theorems A,B,C and obtained as follows:

Theorem D. Let $\mathscr{F}$ be a family of meromorphic functions in the plane domain D, let P be a polynomial with either $\deg(P) \geqslant 3$ or $\deg(P) = 2$ and P having only one distinct zero. If, for each $f, g \in \mathscr{F}$, $P(f)f'$ and $P(g)g'$ share a nonzero constant b IM in D, then $\mathscr{F}$ is normal in D.

Theorem E. Let $\mathscr{F}$ be a family of meromorphic functions in the domain D, all of whose poles are multiple, and let P be a polynomial with two distinct zeros. If, for each $f, g \in \mathscr{F}$, $P(f)f'$ and $P(g)g'$ share complex number b IM in D, then $\mathscr{F}$ is normal in D.

In 2012, Qi, Ding and Yang[22] extended the Theorem D and Theorem E and obtained as follows:

Theorem D1. Let k be a positive integer and let $\mathscr{F}$ be a family of meromorphic functions in the plane domain D all of whose zeros with multiplicity at least k. Let $P = a_p z^p + \cdots + a_2 z^2 + z$ be a polynomial, $a_p, a_2 \neq 0$ and $p = \deg(P) \geqslant k+2$. If, for each $f, g \in F$, $P(f)G(f)$ and $P(g)G(g)$ share a non-zero constant b IM in D, where $G(f) = f^{(k)} + H(f)$ be a differential polynomial of f satisfying $\frac{w}{\deg}|H \leqslant \frac{k}{l+1} + 1$ or $w(H) - deg(H) < k$, then $\mathscr{F}$ is normal in D.

Theorem E1. Let k be a positive integer, suppose that $\mathscr{F}$ be a family of meromorphic functions in the plane domain D all of whose zeros and poles with multiplicity at least k and 2 respectively. Let P be a polynomial with two distinct zeros at least. If, for each $f, g \in \mathscr{F}$, $P(f)G(f)$ and $P(g)G(g)$ share a constant b IM in D, where $G(f) = f^{(k)} + H(f)$ be a differential polynomial of f with $w(H) - deg(H) < k$, then $\mathscr{F}$ is normal in D.

In 2009, Hu and Meng[12] obtained the following results.

Theorem F. Take positive integers n and k with $n, k \geqslant 2$ and take a non-zero complex number a. Let $\mathscr{F}$ be a family of meromorphic functions in the plane domain D such that each $f \in \mathscr{F}$ has only zeros of multiplicity at least k. For each pair $(f, g) \in \mathscr{F}$, if $f(f^{(k)})^n$ and $g(g^k)^n$ share a IM, then $\mathscr{F}$ is normal in D.

Also in the final paper of Hu and Meng[12], Hu and Meng pointed that if the form $f(f^{(k)})^n$ (resp. $g(g^k)^n$) in Theorem 1.1 was replaced by the form $f^l(f^{(k)})^n$ (resp. $g^l(g^k)^n$) for an integer $l \geqslant 2$, the conclusion also holds.

Based on the ideas of Theorem D and Theorem E, it is natural to ask whether $f(f^{(k)})^n$, in Theorem F can be replaced by $P(f)G(f)^n$, where $P(f)$ is polynomial about f, $G(f)$ is a differential polynomial about f. The main purpose of this paper is to investigate this problem. We prove the following results.

THEOREM 1. (Main) *Let m, k be two positive integers. Let $\mathscr{F}$ be a family of meromorphic functions in the domain D all of whose zeros with multiplicity at least k. Let P be a polynomial and have at least one simple zero, and $p = \deg(P) \geqslant k+2$. If, for each pair $f, g \in \mathscr{F}$, $P(f)G^m(f)$ and $P(g)G^m(g)$ share a non-zero constant b IM in D, where $G(f) = P(f^{(k)}) + H(f)$ be a differential polynomial of f satisfying*

$\frac{w}{\deg}|_H \leqslant \frac{kmq}{l+mq}+1$ *or* $w(H)-\deg(H)<qk$, *and* $q>l\geqslant k+1$ *is a positive integer, then* $\mathscr{F}$ *is normal in D.*

REMARK 1. If the polynomial $P(z)$ has only one zero, the Theorem 1 is established for $\deg(P)\geqslant k+1$.

COROLLARY 1. *Let k be two positive integers and* $\mathscr{F}$ *be a family of meromorphic functions in the plane domain D all of whose zeros with multiplicity at least* k. *Let P be a polynomial as theorem 1. If, for each* $f,g\in\mathscr{F}$, $P(f)(f^{(k)})^m$ *and* $P(g)(g^{(k)})^m$ *share a non-zero constant b IM in D, then* $\mathscr{F}$ *is normal in D.*

THEOREM 2. (Main) *Let* k *be a positive integer, suppose that* $\mathscr{F}$ *be a family of meromorphic functions in the domain* D *all of whose zeros and poles with multiplicity at least* k *and* 2 *respectively. Let* P *be a polynomial with two distinct zeros at least. If, for each* $f,g\in\mathscr{F}$, $P(f)G^m(f)$ *and* $P(g)G^m(g)$ *share a constant b IM in D, where* $G(f)=P(f^{(k)})+H(f)$ *be a differential polynomial of* f *with* $w(H)-\deg(H)<qk$, *then* $\mathscr{F}$ *is normal in* D.

COROLLARY 2. *Let* k *be a positive integer, suppose that* $\mathscr{F}$ *be a family of meromorphic functions in the plane domain D all of whose zeros and poles with multiplicity at least k and 2 respectively. Let P be a polynomial as theorem 2. If, for each* $f,g\in\mathscr{F}$, $P(f)(f^{(k)})^m$ *and* $P(g)(g^{(k)})^m$ *share* b *IM in* D, *then* $\mathscr{F}$ *is normal in* D.

Example 1. Let $D=\{|z|<1\}$ and take a non-zero complex number a. Fix three positive integers $n\geqslant 2$, k, m.

$$\mathscr{F}=\{f_d(z)=dz^{k-1}|d=1,2...\}.$$

Obviously, for distinct positive integers, i, j, we have $f_i^n(f_i^{(k)})^m$ and $f_j^n(f_j^{(k)})^m$ share a IM. However, the family $\mathscr{F}$ is not normal at $z=0$.

Example 2. Set

$$\mathscr{F}=\{f_d(z)=dz-\frac{d}{3}+\frac{a}{d^m}|d=1,2,..\}.$$

For distinct positive integers, i, j, we have $f_i(f_i')^m$ and $g_i(g_i')^m$ share a IM. However, the family $\mathscr{F}$ is not normal at $z=\frac{1}{3}$.

Example 3. Set

$$\mathscr{F}=\{e^{dz}|d=1,2,...\}.$$

Obviously, any $f_d\in mathcalF$ has only zeros of multiplicity at least k. For distinct positive integers i, j, we have $f_i^n(f_i^{(k)})^m$ and $f_j^n(f_j^{(k)})^m$ share 0 IM. However, the families are not normal at $z=0$.

REMARK 2. Example 1 shows that the condition that f has only zeros of multiplicity at least k is sharp in Theorem 1 and 2. Example 2 shows the conditions $p = \deg(P) \geqslant k+2$ in Theorem 1 and P is a polynomial with two distinct zeros at least in Theorem 2 seems not be omitted. Example 3 shows the condition $a \neq 0$ in Theorem 1 is necessary.

REMARK 3. Obviously, Theorem 1 and Theorem 2 both extend the Theorem D1 and Theorem D2.

2. Some lemmas

In order to prove our theorem, we need the following Lemmas:

Lemma 2.1 [Pang-Zalcman's lemma](see [4], [21], [31]) Let k be a positive integer, let $\mathscr{F}$ be a family of meromorphic functions in the unit disc $\triangle$ with the property that for each $f \in \mathscr{F}$, all zeros of multiplicity at least k. Suppose that there exists a number $A \geqslant 1$ such that $|f^{(k)}(z)| \leqslant A$ whenever $f = 0$. Suppose that $\mathscr{F}$ is not normal at z_0, then for $0 \leqslant \alpha \leqslant k$, there exist

a). points $z_n \in \triangle$, $z_n \to z_0$;

b). functions $f_n \in \mathscr{F}$; and

c). positive numbers $\rho_n \to 0^+$;

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^\sharp(\xi) \leqslant g^\sharp(0) = kA+1$. In particular, g has order at most 2.

Lemma 2.2([28]) Let $n \geqslant 2$, k be a positive integer and n_k be a positive integer. If f is a transcendental meromorphic function, then $f^n(f^{(k)})^{n_k}$ assume every finite non-zero complex value infinitely often.

Lemma 2.3 ([26]) Let f be a non-constant rational function and n, m be two positive integers. let k be a positive integer, and let b be a non-zero finite complex number. then $f^n(f^{(k)})^m - b$ has one zero at least.

Lemma 2.4 Let f be a non-constant rational function in the plane $\mathbb{C}$ and $n \geqslant 2, m$ be two positive integers. let k be a positive integer, and let b be a non-zero finite complex number. then $f^n(f^{(k)})^m - b$ has two distinct zeros at least.

Proof. By Lemma 2.3, $f^n(f^{(k)})^m - b$ has one zero at least. On the contrary that $f^n(f^{(k)})^m - b$ has one zero at most.

Suppose that f is a non-constant polynomial, we have

$$f^n(f^{(k)})^m(z) = A(z-z_0)^l + b, \tag{2.1}$$

where A is a non-zero constant and $l \geqslant 2$ is a positive integer. The right hand side of (2.1) has only simple zeros, but the left has multiple zeros, a contradiction. Thus f is a non-polynomial rational function.
Set

$$f^n(f^{(k)})^m(z) = A\frac{(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}} = \frac{P}{Q}, \tag{2.2}$$

where A is a non-zero constant, P and Q are polynomials of degree M and N respectively. Also P and Q have no common factor. By the zeros of f are at least k, we obtain $m_i \geqslant 2 (i = 1,2,\cdots,s)$, $n_j \geqslant 2 (j = 1,2,\cdots,t)$. Hence

$$m_1 + m_2 + \cdots + m_s = M \geqslant 2s, \tag{2.3}$$

$$n_1 + n_2 + \cdots + n_t = N \geqslant 2t. \tag{2.4}$$

From (2.2), we obtain

$$(f^n(f^{(k)})^m(z))' = \frac{(z-\alpha_1)^{m_1-1}(z-\alpha_2)^{m_2-1}\cdots(z-\alpha_s)^{m_s-1}g_1(z)}{(z-\beta_1)^{n_1+1}(z-\beta_2)^{n_2+1}\cdots(z-\beta_t)^{n_t+1}}, \tag{2.5}$$

where g_1 is a polynomial of degree at most $s+t-1$.

Since $f^n(f^{(k)})^m - b$ has only one zero. So we may set

$$f^n(f^{(k)})^m(z) = b + \frac{B(z-z_0)^l}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}} = \frac{P}{Q}. \tag{2.6}$$

Note that $b \neq 0$, we obtain $z_0 \neq \alpha_i (i = 1,\cdots,s)$, where B is a non-zero constant, and $l \geqslant 2$ is a positive integer.

From (2.6), we obtain

$$[f^n(f^{(k)})^m(z)]' = \frac{(z-z_0)^{l-1}g_2(z)}{(z-\beta_1)^{n_1+1}\cdots+(z-\beta_t)^{n_t+1}}. \tag{2.7}$$

Where $g_2(z) = B(l-N)z^t + B_1z^{t-1}\cdots + B_t$ is a polynomial ($B_1,\cdots,B_t$ are constants).

Case 1. If $l \neq N$, by (2.6), then we obtain the $\deg(P) \geqslant \deg(Q)$. So $M \geqslant N$. By (2.5) and (2.7), we obtain $\sum_{i=1}^{s}(m_i - 1) \leqslant \deg(g_2) = t$. So $M - s - \deg(g_1) \leqslant t$, and $M \leqslant s + t + \deg(g_1) \leqslant 2(s+t) - 1 < 2(s+t)$. By (2.3) and (2.4), we obtain

$$M < 2(s+t) \leqslant 2[\frac{M}{2} + \frac{N}{2}] \leqslant M.$$

So we deduce that $M < M$. This is impossible.

Case 2. If $l = N$, then we consider two subcases.

Case 2.1. If $M \geqslant N$, by (2.5) and (2.7), we obtain $\sum_{i=1}^{s}(M_i - 1) \leqslant \deg(g_2) = t$. So $M - s - \deg(g_1) \leqslant t$, and $M \leqslant s + t + \deg(g_1) \leqslant 2(s+t) - k < 2(s+t)$, by the same reasoning mentioned in the case 1. This is impossible.

Case 2.2. If $M < N$, by (2.5) and (2.7), we obtain $l - 1 \leqslant \deg g_1 \leqslant s + t - 1$, then

$$\begin{aligned} N = l \leqslant \deg(g_1) + 1 \leqslant s + t - 1 < s + t \\ \leqslant 2[\frac{M}{2} + \frac{N}{2}] \leqslant N. \end{aligned}$$

So we deduce that $N < N$. It is impossible. $\square$

Lemma 2.5([27]) Let $f(z)$ be a non-constant rational function, then $f(z)$ has only one deficient value.

3. Proof of Theorem 2

Proof. Without loss of generality, we assume that $P(z) = Q(z)z(z-1)$, where $Q(z) \not\equiv 0$ is a polynomial. Suppose that $\mathscr{F}$ is not normal in D. Then there exists at least one z_0 such that $\mathscr{F}$ is not normal at z_0, we assume that $z_0 = 0$. By Lemma 2.1, there exist points $z_j \to 0$; a sequence $\rho_j \to 0^+$ and functions $f_j \in \mathscr{F}$ such that

$$g_j(\xi) = f_j(z_j + \rho_j \xi) \to g(\xi), \tag{3.1}$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function in $\mathbb{C}$, all of whose zeros and poles are of multiplicity at least k and 2 respectively.

If $Q(g)g(g-1)(g^{(k)})^{qm} \equiv 0$, then g is a constant, a contradiction.

If $Q(g)g(g-1)(g^{(k)})^{qm} \neq 0$. Because of the zeros of g have multiplicity at least k, we obtain $g \neq 0, 1$.

We can claim that g is not a transcendental meromorphic function. In fact, if it is not true, we have

$$\begin{aligned} T(r,g) &\leqslant \overline{N}(r,g) + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{g-1}) + S(r,g) \\ &\leqslant \frac{1}{2}N(r,g) + S(r,g) \\ &\leqslant \frac{1}{2}T(r,g) + S(r,g) \quad . \end{aligned}$$

Thus $T(r,g) = S(r,g)$, a contradiction. So g is a rational function. Since $g \neq 0, 1$, then g is a constant, a contradiction.

Thus $Q(g)g(g-1)(g^{(k)})^{qm}$ is a non-constant meromorphic function and has one zero at least.

Next we will prove that $Q(g)g(g-1)(g^{(k)})^{qm}$ has just a single zero. In fact, let ξ_0 and ξ_0^* be two distinct solutions of $Q(g)g(g-1)(g^{(k)})^{qm}$. We choose a positive number δ small enough such that g and g_j are holomorphic in $D(\xi_0,\delta_1)D(\xi_0^*,\delta_1)$ and $D(\xi_0,\delta_1)\cap D(\xi_0^*,\delta_1)=\emptyset$.

From (3.1), we have

$$\begin{aligned}&\rho_j^k[Q(f_j(z_j+\rho_j\xi)f_j(z_j+\rho_j\xi)(f_j(z_j+\rho_j\xi)-1)\cdot G^m(f_j(z_j+\rho_j\xi)-b]\\=&\rho_j^{mqk}[Q(g_j(\xi)g_j(\xi)(g_j(\xi)-1)\cdot(\rho_j^{-qk}(g_j^{(k)}(\xi))^q+\cdots+\rho_j^{-k}b_1(z_j+\rho_j\xi)g_j^{(k)}(\xi)\\&+\sum_{j=1}^{n}a(z_j+\rho_j\xi)\rho_j^{\deg(M_j)-w(M_j)}M_i(g,g',\cdots,g^{(k)}))-b]^m\\&\qquad\to Q(g(\xi))g(\xi)(g(\xi)-1)(g^{(k)}(\xi))^{qm}.\end{aligned}\tag{3.2}$$

By Hurwitz's theorem, there exist points $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ such that for sufficiently large j

$$Q(f_j(z_j+\rho_j\xi_j))f_j(z_j+\rho_j\xi_j)(f_j(z_j+\rho_j\xi_j)-1)G^m(f_j(z_j+\rho_j\xi_j))=b,$$

$$Q(f_j(z_j+\rho_j\xi_j^*))f_j(z_j+\rho_j\xi_j^*)(f_j(z_j+\rho_j\xi_j^*)-1)G^m(f_j(z_j+\rho_j\xi_j^*))=b.$$

By the hypothesis that for each pair of functions f and g in $\mathscr{F}$, $P(f)G^m(f)$ and $P(g)G^m(g)$ share 0 in D, we know that for any positive integer d

$$Q(f_d(z_j+\rho_j\xi_j))f_d(z_j+\rho_j\xi_j)(f_d(z_j+\rho_j\xi_j)-1)G^m(f_d(z_j+\rho_j\xi_j)=b,$$

$$Q(f_d(z_j+\rho_j\xi_j^*))f_d(z_j+\rho_j\xi_j^*)(f_d(z_j+\rho_j\xi_j^*)-1)G^m(f_d(z_j+\rho_0^*\xi_j)=b.$$

Fix d, take $j\to\infty$, and note $z_j+\rho_j\xi_j\to 0$, $z_j+\rho_j\xi_j^*\to 0$, then

$$Q(f_d(0))f_d(0)(f_d(0)-1)G^m(f_d(0))=b.$$

Since the zeros of $P(f_d)G^m(f_d)-b$ has no accumulation point, so $z_j+\rho_j\xi_j=0$, $z_j+\rho_j\xi_j^*=0$.

Hence

$$\xi_j=-\frac{z_j}{\rho_j},\qquad \xi_j^*=-\frac{z_j}{\rho_j}.$$

This contradicts with $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ and $D(\xi_0,\delta)\cap D(\xi_0^*,\delta)=\emptyset$ So $Q(g)g(g-1)(g^{(k)})^{qm}$ has just a single zero, which can be denoted by ξ_0.

Suppose that g is a transcendental meromorphic function. Since $Q(g)g(g-1)$ $(g^{(k)})^{qm}$ has only one zero, so $g=0$ and $g=1$ has only finite zeros. As the above argument, we obtain $T(r,g)=S(r,g)$, a contradiction.

Thus g is a rational function which is not a polynomial. Because $Q(g)g(g-1)(g^{(k)})^{qm}$ has only one zero, we have $g\neq 0$ or $g\neq 1$.

If $g \neq 0$, then $g(\xi) = \frac{1}{H(\xi)}$ where $H(\xi)$ is a non-constant polynomial. Since $g(\xi) - 1 = \frac{1-H(\xi)}{H(\xi)}$ has just a single zero, so

$$1 - H(\xi) = A(\xi - B)^k, \tag{3.3}$$

where $A \neq 0, B$ are constant, $k \geqslant 2$ is a positive integer.

We claim $H(\xi)$ has only simple zeros. Suppose, on the contrary, that $H(z_0) = 0$ and z_0 is multiple. Form (3.3), we arrive at $0 = H'(z_0) = (1 - H(z_0))' = Ak(z_0 - B)^{(k-1)}$, a contradiction, since $z_0 \neq B$.

Thus $H(\xi)$ has just simple zeros, this contradicts that g has no simple pole. If $g \neq 1$, we can argue it in the same way. So $\mathscr{F}$ is normal on D. □

4. Proof of Theorem 1

Proof. We may assume that $D = \{|z| < 1\}$. Suppose that $\mathscr{F}$ is not normal in D. Without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exist a sequence z_j of complex numbers with $z_j \to 0$ $(j \to \infty)$; a sequence f_j of $\mathscr{F}$; and a sequence $\rho_j \to 0^+$ such that

$$g_j(\xi) = \rho_j^{-\frac{mkq}{l+mq}} f_j(z_j + \rho_j \xi) \tag{3.4}$$

converges uniformly to a non-constant mermorphic functions $g(\xi)$ in C with respect to the spherical metric. Moreover, $g(\xi)$ is of order at most 2. Where $q > l(\geqslant k+1)$ is a constant. By Hurwitz's theorem, the zeros of $g(\xi)$ have at least multiplicity k. Next we will distinguish two cases:

Case 1. When $P(z)$ has two distinct zeros, then we can denote $P(f) = f^l(f + 1)$ $(q > l \geqslant k+1)$.

If $g^l(g^{(k)})^{qm} \equiv b$, then g has no zeros. Of course, g also has no poles. Since g is a non-constant meromorphic function of order at morst 2, we obtain $g(\xi) = e^{d\xi^2 + h\xi + c}$ (where D, h, c are constants and $dh \neq 0$). At this moment $g^l(g^{(k)})^{qm} \not\equiv b$. Which is a contradiction.

If $g^l(g^{(k)})^{qm} \neq b$, then by Lemma 2.2, we obtain that g is a constant. This contradicts that g is a non-zero meromorphic function.

Thus $g^l(g^{(k)})^{qm} - b$ is a non-constant meromorphic function and has one zero at least.

Next we will prove that $g^l(g^{(k)})^{qm} - b$ has just a single zero. In fact, let ξ_0 and ξ_0^* be two distinct solutions of $g^l(g^{(k)})^{qm} - b$. We choose a positive number δ_1 small enough such that g and g_j are holomorphic in $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$.

From (3.4), we have

$$[f_j^{l+1}(z_j+\rho_j\xi)+f_j^l(z_j+\rho_j\xi)]\cdot[P(f_j^{(k)}(z_j+\rho_j\xi))+H(f,f',\cdots,f^{(k)})]^m-b$$
$$=[\rho_j^{-\frac{qlk}{l+mq}}(g_j^{(k)}(\xi))^q+\cdots+\sum_{j=1}^{n}a_j^*\rho_j^{(\frac{kmq}{l+mq}+1)\deg(M_j)-w(M_j)}M_i(g,g',\cdots,g^{(k)})]^m,$$

$$[\rho_j^{\frac{kmq(l+1)}{l+mq}}g_j^{l+1}(\xi)+\rho_j^{\frac{mqlk}{l+mq}}g_j^l(\xi)]-b\to g^l(\xi)(g^{(k)}(\xi))^{qm}-b. \tag{3.5}$$

Choose δ_2 such that $D(\xi_0,\delta_2)\cap D(\xi_0^*,\delta_2)=\emptyset$ and such that $g^l(g^{(k)})^{qm}-b$ has no other zeros in $D(\xi_0,\delta)\cup D(\xi_0^*,\delta)$. By Hurwitz's theorem, there exist points $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ such that for sufficiently large j

$$[f_j^{l+1}(z_j+\rho_j\xi_j)+f_j^l(z_j+\rho_j\xi_j)]G^m(f_j(z_j+\rho_j\xi_j))-b=0,$$
$$[f_j^{l+1}(z_j+\rho_j\xi_j^*)+f_j^l(z_j+\rho_j\xi_j^*)]G^m(f_j(z_j+\rho_j\xi_j^*))-b=0.$$

By the hypothesis that for each pair of functions f and g in $\mathscr{F}$, $P(f)G^m(f^{(k)})$ and $P(g)G^m(g^{(k)})$ share b in D, we know that for any positive integer d

$$[f_d^{l+1}(z_j+\rho_j\xi_j)+f_d^l(z_j+\rho_j\xi_j)]G^m(f_d(z_j+\rho_j\xi_j))-b=0,$$
$$[f_d^{l+1}(z_j+\rho_j\xi_j^*)+f_d^l(z_j+\rho_j\xi_j^*)]G^m(f_d(z_j+\rho_j\xi_j^*))-b=0.$$

Fix d, take $j\to\infty$, and note $z_j+\rho_j\xi_j\to 0$, $z_j+\rho_j\xi_j^*\to 0$, then

$$[f_d^{l+1}(0)+f_d^l(0)]G^m(f_d(0))-b=0.$$

Since the zeros of $P(f)G^m(f)-b$ has no accumulation point, so $z_j+\rho_j\xi_j=0$, $z_j+\rho_j\xi_j^*=0$.

Hence

$$\xi_j=-\frac{z_j}{\rho_j},\qquad \xi_j^*=-\frac{z_j}{\rho_j}.$$

This contradicts with $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ and $D(\xi_0,\delta)\cap D(\xi_0^*,\delta)=\emptyset$. So $g^l(g^{(k)})^{qm}-b$ has just a single zero, which can be denoted by ξ_0.

From the above, we know $g^l(g^{(k)})^{qm}-b$ has just a unique zero. This contradicts Lemma 2.2 and Lemma 2.4.

Case 2. If $P(z)$ has more than three distinct zeros, we can denote $P(z)=Q(g)g(g-1)(g-a)$.

Without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0=0$. Then, by Lemma 2.1, there exist a sequence z_j of complex numbers with $z_j\to 0$ $(j\to\infty)$; a sequence f_j of $\mathscr{F}$; and a sequence $\rho_j\to 0^+$ such that

$$g_j(\xi)=f_j(z_j+\rho_j\xi) \tag{3.6}$$

converges uniformly to a non-constant mermorphic functions $g(\xi)$ in C with respect to the spherical metric.

Proceeding as in the proof of Theorem 2, we have $Q(g)g(g-1)(g-a)G^m(g)$ only one zero.

Obviously, g is not a transcendental meromorphic function from Picard Theorem. Thus g is a non-constant rational function and g doesn't assume two complex number of $\{0,1,a\}$, due to Lemma 2.5, a contradiction. So $\mathscr{F}$ is normal in z_0. □

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