Journal of Mathematical Inequalities
Volume 11, Number 2 (2017), 301–322 doi:10.7153/jmi-11-27

IMPROVED JENSEN–TYPE INEQUALITIES VIA LINEAR INTERPOLATION AND APPLICATIONS

DAESHIK CHOI, MARIO KRNIĆ AND JOSIP PEČARIĆ

(*Communicated by T. Burić*)

Abstract. In this paper we develop a general method for improving Jensen-type inequalities for convex and, even more generally, for piecewise convex functions. Our main result relies on the linear interpolation of a convex function. As a consequence, we obtain improvements of some recently established Young-type inequalities. Finally, our method is also applied to matrix case. In such a way we obtain improvements of some important matrix inequalities known from the literature.

1. Introduction

The classical Young inequality, or the arithmetic-geometric mean inequality, states that

$$(1-v)a+vb \geqslant a^{1-v}b^v, \tag{1.1}$$

where $a,b>0$ and $0 \leqslant v \leqslant 1$. Refining this inequality and its reverse has taken the attention of numerous researchers. Kittaneh and Manasrah [8], improved (1.1) to

$$(1-v)a+vb \geqslant a^{1-v}b^v + r_0(v)(\sqrt{a}-\sqrt{b})^2, \tag{1.2}$$

where $r_0(v)=\min\{v,1-v\}$. Moreover, Zhao and Wu [16], established even more accurate improvement:

$$\begin{aligned}(1-v)a+vb \geqslant{}& a^{1-v}b^v + r_0(v)(\sqrt{a}-\sqrt{b})^2 \\ &+r_1(v)\left[\left(\sqrt{a}-\sqrt[4]{ab}\right)^2\chi_{(0,\frac{1}{2})}(v)+\left(\sqrt[4]{ab}-\sqrt{b}\right)^2\chi_{(\frac{1}{2},1)}(v)\right],\end{aligned} \tag{1.3}$$

where $r_1(v)=\min\{2r_0(v),1-2r_0(v)\}$ and $\chi_I(v)$ stands for the characteristic function of an interval I, defined by $\chi_I(v)=\begin{cases}1, & v\in I\\ 0, & v\notin I\end{cases}$.

On the other hand, the reverses of inequalities (1.2) and (1.3) read as follows [9, 16]:

$$(1-v)a+vb \leqslant a^{1-v}b^v + R_0(v)(\sqrt{a}-\sqrt{b})^2 \tag{1.4}$$

Mathematics subject classification (2010): 26D15, 26A51, 15A45.
Keywords and phrases: Convex function, Jensen inequality, Young inequality, Kantorovich constant, Specht ratio, arithmetic mean, geometric mean, Heinz mean.

Paper JMI-11-27

and

$$(1-v)a+vb \leqslant a^{1-v}b^v + R_0(v)(\sqrt{a}-\sqrt{b})^2 \qquad (1.5)$$
$$-r_1(v)\left[\left(\sqrt{b}-\sqrt[4]{ab}\right)^2\chi_{(0,\frac{1}{2})}(v)+\left(\sqrt[4]{ab}-\sqrt{a}\right)^2\chi_{(\frac{1}{2},1)}(v)\right],$$

where $R_0(v) = 1 - r_0(v)$.

Other types of improvements of the Young inequality have been studied in numerous recent papers. For example, Wu and Zhao [14], showed a pair of relations

$$(1-v)a+vb \geqslant K_1(a,b)^{r_1(v)}a^{1-v}b^v + r_0(v)(\sqrt{a}-\sqrt{b})^2, \qquad (1.6)$$
$$(1-v)a+vb \leqslant K_1(a,b)^{-r_1(v)}a^{1-v}b^v + R_0(v)(\sqrt{a}-\sqrt{b})^2,$$

where $K_1(a,b) = \frac{(\sqrt{a}+\sqrt{b})^2}{4\sqrt{ab}}$. Recently, Liao and Wu [10], have proven the inequalities

$$(1-v)a+vb \geqslant K_2(a,b)^{r_2(v)}a^{1-v}b^v + r_0(v)(\sqrt{a}-\sqrt{b})^2 \qquad (1.7)$$
$$+r_1(v)\left[(\sqrt[4]{ab}-\sqrt{a})^2\chi_{(0,\frac{1}{2})}(v)+(\sqrt{b}-\sqrt[4]{ab})^2\chi_{(\frac{1}{2},1)}(v)\right],$$
$$(1-v)a+vb \leqslant K_2(a,b)^{-r_2(v)}a^{1-v}b^v + R_0(v)(\sqrt{a}-\sqrt{b})^2$$
$$-r_1(v)\left[(\sqrt[4]{ab}-\sqrt{b})^2\chi_{(0,\frac{1}{2})}(v)+(\sqrt{a}-\sqrt[4]{ab})^2\chi_{(\frac{1}{2},1)}(v)\right],$$

where $r_2(v) = \min\{2r_1(v), 1-2r_1(v)\}$ and $K_2(a,b) = \frac{\left(\sqrt[4]{a}+\sqrt[4]{b}\right)^2}{4\sqrt[4]{ab}}$. The constants of the form $\frac{(M+m)^2}{4Mm}$ are called Kantorovich constants.

Further, Dragomir [3], showed the following pair of inequalities that hold for any $a,b > 0$ and $0 \leqslant v \leqslant 1$:

$$(1-v)a+vb \geqslant a^{1-v}b^v + \frac{1}{2}v(1-v)\left(\ln\frac{b}{a}\right)^2\min\{a,b\}, \qquad (1.8)$$
$$(1-v)a+vb \leqslant a^{1-v}b^v + \frac{1}{2}v(1-v)\left(\ln\frac{b}{a}\right)^2\max\{a,b\}.$$

Meanwhile, assuming $a,b \geqslant 1$ and $0 \leqslant v \leqslant 1$, Minculete [11], proved that

$$(1-v)a+vb \geqslant a^{1-v}b^v + r_0(v)(\sqrt{a}-\sqrt{b})^2 + \alpha(v)\left(\ln\frac{b}{a}\right)^2, \qquad (1.9)$$
$$(1-v)a+vb \leqslant a^{1-v}b^v + R_0(v)(\sqrt{a}-\sqrt{b})^2 + \alpha(v)\left(\ln\frac{b}{a}\right)^2,$$

where

$$\alpha(v) = \frac{1}{2}v(1-v) - \frac{1}{4}r_0(v) = \frac{1}{4}r_0(v)|2v-1|.$$

Finally, utilizing the Specht ratio

$$S(t) = \frac{t^{1/(t-1)}}{e\ln t^{1/(t-1)}},$$

Furuichi and Tominaga [4, 13], showed that the series of inequalities

$$S\left(c^{r_0(v)}\right)a^{1-v}b^v \leqslant (1-v)a+vb \leqslant S(c)a^{1-v}b^v, \tag{1.10}$$

where $c=a^{-1}b$, holds for any $a,b>0$ and $0\leqslant v\leqslant 1$.

Basically, the Young inequality (1.1) is a consequence of the famous Jensen inequality

$$f((1-v)a+vb)\leqslant (1-v)f(a)+vf(b), \tag{1.11}$$

where f is a convex function defined on the interval I, $a,b\in I$, and $0\leqslant v\leqslant 1$. Clearly, the Young inequality (1.1) follows from (1.11) by putting $f(x)=-\ln x$, where ln stands for a natural logarithm.

The main objective of this paper is to provide a unified treatment of Young-type inequalities presented in this Introduction. More precisely, we will present a general improvement of a Jensen-type inequality related to piecewise convex functions and use it to refine some well-known classical inequalities. As an application, we will also derive improved versions of some important matrix inequalities known from the literature. It should be noticed here that the operator or matrix inequalities related to the scalar inequalities introduced in this section can be found in some recent papers including [3, 4, 8, 10, 11, 14, 16].

2. The main result related to convex and piecewise convex functions

In this section we give an improved version of the Jensen inequality that will, in some way, gather the relations presented in the previous section. Our main result will rely on the linear interpolation of a convex function.

Throughout the paper, we will use the functions $r_n(v)$ defined recursively by

$$\begin{aligned} r_0(v) &= \min\{v,1-v\},\\ r_n(v) &= \min\{2r_{n-1}(v),1-2r_{n-1}(v)\}, \end{aligned}$$

for $0\leqslant v\leqslant 1$. Note that $r_0(v)$ and $r_1(v)$ can be rewritten as

$$r_0(v)=\begin{cases} v, & 0\leqslant v\leqslant \frac{1}{2}\\ 1-v, & \frac{1}{2}<v\leqslant 1 \end{cases}, \quad r_1(v)=\begin{cases} 2v, & 0\leqslant v\leqslant \frac{1}{4}\\ 1-2v, & \frac{1}{4}<v\leqslant \frac{1}{2}\\ 2v-1, & \frac{1}{2}<v\leqslant \frac{3}{4}\\ 2-2v, & \frac{3}{4}<v\leqslant 1 \end{cases}.$$

Generally, $r_n(v)$ can be expressed as multipart functions.

LEMMA 1. [2] *Let n be a nonnegative integer and $0\leqslant v\leqslant 1$. If $\frac{k-1}{2^n}\leqslant v\leqslant \frac{k}{2^n}$ for $k=1,\ldots,2^n$, then*

$$r_n(v)=\begin{cases} 2^n v-k+1, & \frac{k-1}{2^n}\leqslant v\leqslant \frac{2k-1}{2^{n+1}}\\ k-2^n v, & \frac{2k-1}{2^{n+1}}<v\leqslant \frac{k}{2^n} \end{cases}.$$

Proof. We prove it by induction on n. The case $n=0$ is obvious. Assume $\frac{k-1}{2^{n+1}} \leqslant v \leqslant \frac{k-1}{2^{n+1}} + \frac{1}{2^{n+2}}$. If $k=2m-1$ is odd, then $\frac{m-1}{2^n} \leqslant v \leqslant \frac{m-1}{2^n} + \frac{1}{2^{n+2}} < \frac{2m-1}{2^{n+1}}$ and $r_n(v) = 2^n v - m + 1$ by induction. Since $v \leqslant \frac{2k-1}{2^{n+2}}$,

$$r_n(v) = 2^n v - \frac{k-1}{2} \leqslant \frac{2k-1}{4} - \frac{k-1}{2} = \frac{1}{4}$$

and

$$r_{n+1}(v) = \min\{2r_n(v), 1-2r_n(v)\} = 2r_n(v) = 2^{n+1}v - k + 1.$$

If $k=2m$ is even, then $\frac{2m-1}{2^{n+1}} \leqslant v \leqslant \frac{m}{2^n} - \frac{1}{2^{n+2}} < \frac{m}{2^n}$ and $r_n(v) = m - 2^n v$ by induction. Since $v \leqslant \frac{2k-1}{2^{n+2}}$,

$$r_n(v) = \frac{k}{2} - 2^n v \geqslant \frac{k}{2} - \frac{2k-1}{4} = \frac{1}{4}$$

and

$$r_{n+1}(v) = 1 - 2r_n(v) = 2^{n+1}v - k + 1.$$

Using the same argument, we can show that if $\frac{k-1}{2^{n+1}} + \frac{1}{2^{n+2}} < v \leqslant \frac{k}{2^{n+1}}$, then $r_{n+1}(v) = k - 2^{n+1}v$. We omit the detailed proof. □

The functions r_n can be used for linear interpolation as follows.

LEMMA 2. *Let f be a function defined on $[0,1]$. For a nonnegative integer N, define $\varphi_N(v)$ by*

$$\varphi_N(v) = (1-v)f(0) + vf(1) - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v),$$

where

$$\Delta_f(n,k) = f\left(\frac{k-1}{2^n}\right) + f\left(\frac{k}{2^n}\right) - 2f\left(\frac{2k-1}{2^{n+1}}\right)$$

and the summation is assumed to be zero if $N=0$. Then, $\varphi_N(v)$ is the linear interpolation of $f(v)$ at $v = k/2^N$, $k = 0,1,\ldots,2^N$.

Proof. First we note that since $r_n(\frac{k}{2^n}) = 0$ for $0 \leqslant k \leqslant 2^n$, the interval of the characteristic function may contain boundary points. For example, $\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}$ can be replaced by $\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right]}$ or $\chi_{\left[\frac{k-1}{2^n},\frac{k}{2^n}\right]}$. We will show that

$$\varphi_N(v) = (k-2^N v)f\left(\frac{k-1}{2^N}\right) + (2^N v - k + 1)f\left(\frac{k}{2^N}\right) \tag{2.1}$$

for $\frac{k-1}{2^N} \leqslant v \leqslant \frac{k}{2^N}$ and $k = 1,\ldots,2^N$ by induction on N. It is obvious for $N=0$. Now, assume that (2.1) holds and let $\frac{m-1}{2^{N+1}} \leqslant v \leqslant \frac{m}{2^{N+1}}$ for $m = 1,\ldots,2^{N+1}$. If $m = 2k-1$, then $\frac{k-1}{2^N} \leqslant v \leqslant \frac{2k-1}{2^{N+1}} < \frac{k}{2^N}$ and

$$\begin{aligned}
\varphi_{N+1}(v) &= \varphi_N(v) - r_N(v)\Delta_f(N,k) \\
&= (k-2^N v) f\left(\frac{k-1}{2^N}\right) + (2^N v - k + 1) f\left(\frac{k}{2^N}\right) - (2^N v - k + 1)\Delta_f(N,k) \\
&= (2k - 2^{N+1} v - 1) f\left(\frac{k-1}{2^N}\right) + (2^{N+1} v - 2k + 2) f\left(\frac{2k-1}{2^{N+1}}\right) \\
&= (m - 2^{N+1} v) f\left(\frac{m-1}{2^{N+1}}\right) + (2^{N+1} v - m + 1) f\left(\frac{m}{2^{N+1}}\right)
\end{aligned}$$

by Lemma 1. Similarly, if $m = 2k$, then $\frac{k-1}{2^N} < \frac{2k-1}{2^{N+1}} \leqslant v \leqslant \frac{k}{2^N}$ and

$$\begin{aligned}
\varphi_{N+1}(v) &= \varphi_N(v) - r_N(v)\Delta_f(N,k) \\
&= (k-2^N v) f\left(\frac{k-1}{2^N}\right) + (2^N v - k + 1) f\left(\frac{k}{2^N}\right) - (k - 2^N v)\Delta_f(N,k) \\
&= (2k - 2^{N+1} v) f\left(\frac{2k-1}{2^{N+1}}\right) + (2^{N+1} v - 2k + 1) f\left(\frac{k}{2^N}\right) \\
&= (m - 2^{N+1} v) f\left(\frac{m-1}{2^{N+1}}\right) + (2^{N+1} v - m + 1) f\left(\frac{m}{2^{N+1}}\right). \quad \square
\end{aligned}$$

From now on, any summation having $\sum_{n=0}^{N-1}$ will be assumed to be zero for $N = 0$ and $\Delta_f(n,k)$ defined in Lemma 2 will be used throughout the paper.

Now, we are ready to state and prove our main result. The following theorem is based on a fact that a convex function can be estimated by using the linear interpolations $\varphi_N(v)$ in Lemma 2. In fact, such estimation provides a refinement of the Jensen inequality for a convex function defined on the interval $[0,1]$.

THEOREM 3. *Let N be a nonnegative integer. If $f(v)$ is convex on $[0,1]$, then*

$$(1-v)f(0) + vf(1) \geqslant f(v) + \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)$$

and

$$\begin{aligned}
(1-v)f(0) + vf(1) \leqslant\ & f(0) + f(1) - f(1-v) \\
& - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(n, 2^n - k + 1)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v).
\end{aligned} \tag{2.2}$$

Proof. By Lemma 2, we have $\varphi_N(v) \geqslant f(v)$ which represents (2.2). Replacing v by $1-v$ in (2.2) and noting that $r_n(v) = r_n(1-v)$, we have

$$(1-v)f(0) + vf(1) \leqslant f(0) + f(1) - f(1-v) - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(1-v).$$

Now, replacing k by 2^n-k+1 in the inner summation and noting that

$$\frac{k-1}{2^n}<1-v<\frac{k}{2^n} \iff 1-\frac{k}{2^n}<v<1-\frac{k-1}{2^n},$$

we obtain (2.2) and the proof is completed. □

It should be noticed here that $\Delta_f \geqslant 0$ in the previous theorem since f is convex. Therefore the inequality (2.2) represents the refinement of the Jensen inequality for a convex function defined on the interval $[0,1]$.

REMARK 1. It is important to emphasize that Theorem 3 can also be applied to piecewise convex functions. For example, if $f(v)$ is convex on intervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, and $f(\frac{1}{2}) \leqslant \frac{1}{2}(f(0)+f(1))$, then $f(v)$ fulfills the inequalities as in the theorem. More generally, if $f(v)$ is convex on intervals of the form $[\frac{m-1}{2^{N+1}},\frac{m}{2^{N+1}}]$, $1 \leqslant m \leqslant 2^{N+1}$, and $\Delta_f(N,k) \geqslant 0$ for $1 \leqslant k \leqslant 2^N$, then the inequalities (2.2) and (2.2) are still valid for f. To see this, note that $\varphi_N(v)$, $v \in [\frac{k-1}{2^N},\frac{k}{2^N}]$ is the line segment joining the two points of $f(v)$ at $v=\frac{k-1}{2^N}$ and $v=\frac{k}{2^N}$. Thus, if $\Delta_f(N,k) \geqslant 0$ and f is convex on the intervals of the form $[\frac{m-1}{2^{N+1}},\frac{m}{2^{N+1}}]$, then the graph of $f(v)$ is still below $\varphi_N(v)$ for $v \in [\frac{k-1}{2^N},\frac{k}{2^N}]$, $1 \leqslant k \leqslant 2^N$, that is, we have $\varphi_N(v) \geqslant f(v)$, as in the proof of Theorem 3.

3. Improved versions of Young-type inequalities

In this section, we will see how the Jensen-type inequalities from Theorem 3 can be used to improve Young-type inequalities. The most general forms of (1.2), (1.3), (1.4), and (1.5) have been proved recently.

THEOREM 4. [2, 12] *Let $a,b>0$, $0 \leqslant v \leqslant 1$, and N be a nonnegative integer. Then,*

$$\begin{aligned}(1-v)a+vb &\geqslant a^{1-v}b^v+\sum_{n=0}^{N-1} r_n(v)\sum_{k=1}^{2^n} g_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v), \\ &= a^{1-v}b^v+r_0(v)(\sqrt{a}-\sqrt{b})^2+\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^n} g_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}(1-v)a+vb &\leqslant a+b-a^vb^{1-v}-\sum_{n=0}^{N-1} r_n(v)\sum_{k=1}^{2^n} g_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v) \\ &= 2\sqrt{ab}-a^vb^{1-v}+R_0(v)(\sqrt{a}-\sqrt{b})^2-\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^n} g_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v) \\ &\leqslant a^{1-v}b^v+R_0(v)(\sqrt{a}-\sqrt{b})^2-\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^n} g_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),\end{aligned} \tag{3.2}$$

where $g_{n,k}(a,b) = \Delta_f(n,k)$ *with* $f(v) = a^{1-v}b^v$, *i.e.,*

$$g_{n,k}(a,b) = a^{1-(k-1)/2^n}b^{(k-1)/2^n} + a^{1-k/2^n}b^{k/2^n} - 2a^{1-(2k-1)/2^{n+1}}b^{(2k-1)/2^{n+1}}$$
$$= \left(\sqrt{a^{1-(k-1)/2^n}b^{(k-1)/2^n}} - \sqrt{a^{1-k/2^n}b^{k/2^n}}\right)^2.$$

Note that the inequalities (1.2), (1.3), (1.4), and (1.5) follow directly from Theorem 4 for $N=1$ and $N=2$. The original proof of Theorem 4 was rather lengthy, here we give a simple and elegant proof based on our Theorem 3.

Proof. Since $f(v) = a^{1-v}b^v$ is convex on $[0,1]$, the inequality (3.1) follows from (2.2), where we note that if $n=0$, then

$$r_n(v)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v) = r_0(v)(\sqrt{a}-\sqrt{b})^2.$$

Further, utilizing (2.2) we have

$$(1-v)a+vb \leqslant a+b-a^vb^{1-v} - \sum_{n=0}^{N-1}r_n(v)\sum_{k=1}^{2^n}g_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)$$
$$= 2\sqrt{ab}-a^vb^{1-v}+R_0(v)(\sqrt{a}-\sqrt{b})^2-\sum_{n=1}^{N-1}r_n(v)\sum_{k=1}^{2^n}g_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v).$$

Now, the second inequality in (3.2) follows by the arithmetic-geometric mean inequality $2\sqrt{ab} \leqslant a^vb^{1-v} + a^{1-v}b^v$. □

The inequalities (1.6) and (1.7) involving Kantorovich constants can also be generalized in the following way.

THEOREM 5. [2] *Let* $a,b>0$, $0\leqslant v\leqslant 1$, *and* N *be a nonnegative integer. Then*

$$(1-v)a+vb \geqslant K_N(a,b)^{r_N(v)}a^{1-v}b^v + \sum_{n=0}^{N-1}r_n(v)\sum_{k=1}^{2^n}g_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v) \tag{3.3}$$
$$= K_N(a,b)^{r_N(v)}a^{1-v}b^v + r_0(v)(\sqrt{a}-\sqrt{b})^2 + \sum_{n=1}^{N-1}r_n(v)\sum_{k=1}^{2^n}g_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)$$

and

$$(1-v)a+vb \leqslant a+b-K_N(a,b)^{r_N(v)}a^vb^{1-v} - \sum_{n=0}^{N-1}r_n(v)\sum_{k=1}^{2^n}g_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v) \tag{3.4}$$
$$= 2\sqrt{ab}-K_N(a,b)^{r_N(v)}a^vb^{1-v}+R_0(v)(\sqrt{a}-\sqrt{b})^2 - \sum_{n=1}^{N-1}r_n(v)\sum_{k=1}^{2^n}g_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)$$

$$\leqslant K_N(a,b)^{-r_N(v)}a^{1-v}b^v + R_0(v)(\sqrt{a}-\sqrt{b})^2 \\ - \sum_{n=1}^{N-1} r_n(v) \sum_{k=1}^{2^n} g_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),$$

where

$$K_N(a,b) = \frac{\left(a^{1/2^N}+b^{1/2^N}\right)^2}{4(ab)^{1/2^N}}$$

and $g_{n,k}$ *is defined in Theorem 4.*

The original proof of the above theorem can also be simplified by virtue of Theorem 3.

Proof. Let $f(v) = K_N(a,b)^{r_N(v)}a^{1-v}b^v$. Since $K_N(a,b)$ does not depend on variable v and $r_N(v)$ is a line segment on each interval $I_m = [\frac{m-1}{2^{N+1}}, \frac{m}{2^{N+1}}]$ for $1 \leqslant m \leqslant 2^{N+1}$, $f(v)$ is of the form $\alpha\beta^v$ on I_m for some $\alpha, \beta > 0$. Thus f is convex on I_m for $1 \leqslant m \leqslant 2^{N+1}$. Moreover, since $r_N(\frac{k}{2^N}) = 0$ for $0 \leqslant k \leqslant 2^N$, a direct computation shows that

$$\Delta_f(n,k) = \begin{cases} g_{n,k}(a,b), & 0 \leqslant n < N \\ 0, & n = N \end{cases}.$$

Although the function f is not convex on $[0,1]$, it is convex on intervals I_m. Moreover, since $\Delta_f(N,k) = 0$, Theorem 3 can be applied to function f. This yields the inequality (3.3) and the first inequality in (3.4). Finally, the second inequality in (3.4) follows simply from the arithmetic-geometric mean inequality:

$$2\sqrt{ab} \leqslant K_N(a,b)^{r_N(v)}a^v b^{1-v} + K_N(a,b)^{-r_N(v)}a^{1-v}b^v. \quad \square$$

It should be noticed here that the inequalities (1.6) and (1.7) are the special cases of Theorem 5 with $N = 1$ and $N = 2$. In order to conclude our discussion regarding the previous theorem, we show that the Kantorovich constants $K_N(a,b)$ appearing in Theorem 5 are the best possible.

PROPOSITION 6. *Let N be a nonnegative integer. If $\xi(a,b)$ is a nonnegative function such that*

$$(1-v)a + vb \geqslant \xi(a,b)^{r_N(v)}a^{1-v}b^v + \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} g_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v) \tag{3.5}$$

for $a, b > 0$ and $0 \leqslant v \leqslant 1$, then $\xi(a,b) \leqslant K_N(a,b)$.

Proof. Let $f(v) = \xi(a,b)^{r_N(v)}a^{1-v}b^v$. Similarly to the proof of Theorem 5, we can show that $\Delta_f(n,k) = g_{n,k}(a,b)$ and that f is convex on $I_m = [\frac{m-1}{2^{N+1}}, \frac{m}{2^{N+1}}]$, for $1 \leqslant m \leqslant 2^{N+1}$. Since

$$(1-v)a + vb - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} g_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)$$

is the linear interpolation of $f(v)$ at $v = k/2^N$, for $0 \leqslant k \leqslant 2^N$, by Lemma 2, the inequality (3.5) holds *if and only if* $\Delta_f(N,k) \geqslant 0$, for $0 \leqslant k \leqslant 2^N$.

Now, let $v_k = \frac{k}{2^N}$. Since $r_N(v_{k-1}) = r_N(v_k) = 0$ and $r_N(\frac{v_{k-1}+v_k}{2}) = \frac{1}{2}$, the condition $\Delta_f(N,k) \geqslant 0$ is equivalent to

$$\xi(a,b)^{1/2} a^{1-(v_{k-1}+v_k)/2} b^{(v_{k-1}+v_k)/2} \leqslant \frac{1}{2}\left(a^{1-v_{k-1}} b^{v_{k-1}} + a^{1-v_k} b^{v_k}\right),$$

that is,

$$\xi(a,b) \leqslant \frac{\left(a^{1/2^N} + b^{1/2^N}\right)^2}{4(ab)^{1/2^N}}.$$

Therefore we have $\xi(a,b) \leqslant K_N(a,b)$, $a,b > 0$. □

The inequalities in (1.8), due to Dragomir, can also be improved by virtue of Theorem 3.

THEOREM 7. *Let $a,b > 0$, $0 \leqslant v \leqslant 1$, and let N be a nonnegative integer. Then,*

$$(1-v)a + vb \geqslant a^{1-v}b^v + \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} g_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v) + \left(\frac{v(1-v)}{2} - \sum_{n=0}^{N-1} \frac{r_n(v)}{2^{n+2}}\right)\left(\ln\frac{b}{a}\right)^2 \min\{a,b\} \tag{3.6}$$

and

$$(1-v)a + vb \leqslant a + b - a^v b^{1-v} - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} g_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v) - \left(\frac{v(1-v)}{2} - \sum_{n=0}^{N-1} \frac{r_n(v)}{2^{n+2}}\right)\left(\ln\frac{b}{a}\right)^2 \min\{a,b\}, \tag{3.7}$$

where the function $g_{n,k}$ is defined in Theorem 4.

Proof. Putting $f(v) = a^{1-v}b^v + \frac{1}{2}v(1-v)\left(\ln\frac{b}{a}\right)^2 \min\{a,b\}$, we have

$$f''(v) = (\ln\frac{b}{a})^2\left(a^{1-v}b^v - \min\{a,b\}\right) \geqslant 0,$$

so the inequalities (3.6) and (3.7) follows directly from Theorem 3, since

$$\Delta_f(n,k) = g_{n,k}(a,b) - \frac{1}{2^{2n+2}}\left(\ln\frac{b}{a}\right)^2 \min\{a,b\}. \quad \square$$

Note that the inequality (1.8) follows from the above theorem for $N = 0$. It is very interesting to compare relations (3.1) and (3.6). It can be shown that if $N \geqslant 2$, then

$$\frac{v(1-v)}{2} \leqslant \sum_{n=0}^{N-1} \frac{r_n(v)}{2^{n+2}},$$

for $0 \leqslant v \leqslant 1$. Thus, the inequality (3.6) is weaker than (3.1) for $N \geqslant 2$. On the other hand, in the case when $N = 1$, the relation (3.6) is stronger than (3.1), since

$$\frac{1}{2}v(1-v) - \frac{1}{4}r_0(v) = \frac{1}{4}r_0(v)|1-2v| \geqslant 0.$$

Similarly, the inequality (3.7) is stronger than the first inequality in (3.2) when $N = 1$, and we have the following result.

COROLLARY 8. *Let $a,b > 0$ and $0 \leqslant v \leqslant 1$. Then,*

$$(1-v)a + vb \geqslant a^{1-v}b^v + r_0(v)(\sqrt{a}-\sqrt{b})^2 + \alpha(v)\zeta(a,b) \tag{3.8}$$

and

$$\begin{aligned}(1-v)a + vb &\leqslant a+b-a^v b^{1-v} - r_0(v)(\sqrt{a}-\sqrt{b})^2 - \alpha(v)\zeta(a,b) \\ &\leqslant a^{1-v}b^v + R_0(v)(\sqrt{a}-\sqrt{b})^2 - \alpha(v)\zeta(a,b),\end{aligned} \tag{3.9}$$

where

$$\alpha(v) = \frac{1}{2}v(1-v) - \frac{1}{4}r_0(v) = \frac{1}{4}r_0(v)|1-2v|,$$
$$\zeta(a,b) = \left(\ln\frac{b}{a}\right)^2 \min\{a,b\}.$$

Moreover, (3.8) and the first inequality in (3.9) are stronger than the corresponding ones in (1.9) for $a,b \geqslant 1$.

Proof. The relations (3.8) and (3.9) follow from (3.6) and (3.7) with $N = 1$ respectively, where the second inequality in (3.9) follows from the arithmetic-geometric mean inequality $2\sqrt{ab} \leqslant a^v b^{1-v} + a^{1-v}b^v$.

Now, assume that $a,b \geqslant 1$. Since $\min\{a,b\} \geqslant 1$, it is obvious that (3.8) is stronger than the first inequality in (1.9). Moreover, from (3.9) we have

$$\begin{aligned}(1-v)a + vb &\leqslant a+b-a^v b^{1-v} - r_0(v)(\sqrt{a}-\sqrt{b})^2 - \alpha(v)\left(\ln\frac{b}{a}\right)^2 \\ &= 2\sqrt{ab} + R_0(v)(\sqrt{a}-\sqrt{b})^2 - a^v b^{1-v} - \alpha(v)\left(\ln\frac{b}{a}\right)^2 \\ &= a^{1-v}b^v + R_0(v)(\sqrt{a}-\sqrt{b})^2 + \alpha(v)\left(\ln\frac{b}{a}\right)^2 \\ &\quad + 2\sqrt{ab} - a^v b^{1-v} - a^{1-v}b^v - \left(v(1-v)-\frac{1}{4}\right)\left(\ln\frac{b}{a}\right)^2.\end{aligned}$$

Thus, it suffices to show the relation

$$2\sqrt{ab} \leqslant a^{1-v}b^v + a^v b^{1-v} + \left(v(1-v)-\frac{1}{4}\right)\left(\ln\frac{b}{a}\right)^2$$

for $a,b \geqslant 1$ and $0 \leqslant v \leqslant 1$. Denoting the right-hand side of the above inequality by $f(v)$, we have

$$f''(v) = \left(\ln\frac{b}{a}\right)^2 \left(a^{1-v}b^v + a^v b^{1-v} - 2\right).$$

Since $a^{1-v}b^v + a^v b^{1-v} \geqslant 2\sqrt{ab} \geqslant 2$, it follows that f is convex. Moreover, since $f(v) = f(1-v)$, f attains its minimum value at $v = \frac{1}{2}$, that is, $f(v) \geqslant f(\frac{1}{2}) = 2\sqrt{ab}$. $\square$

Now, our aim is to improve the series of inequalities in (1.10) which includes the Specht ratio. Note that the Specht ratio $S(t) = t^{1/(t-1)}/(e\ln t^{1/(t-1)})$ has the following properties (see e.g. [13]):

- $S(1) = \lim_{t\to 1} S(t) = 1$ and $S(t) = S(t^{-1})$ for $t > 0$.
- $S'(t) < 0$ for $0 < t < 1$ and $S'(t) > 0$ for $t > 1$.

Before the corresponding improvement, we first give an auxiliary result regarding the Specht ratio.

LEMMA 9. *Let $S(t)$ be the Specht ratio and define $D(t)$ by*

$$D(t) = \frac{1}{2}(t + t^{-1}),$$

for $t > 0$. Then,

1. *$S(t) \leqslant D(t)$ for $t > 0$,*
2. *For any $c > 0$, $f(v) = D(c^{r_0(v)})c^v$ is convex on $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. Moreover, $f(\frac{1}{2}) = \frac{1}{2}(f(0) + f(1))$.*

Proof. Since $S(t^{-1}) = S(t)$ and $D(t^{-1}) = D(t)$, for $t > 0$, we will show $S(t) \leqslant D(t)$ for $t > 1$. Taking a natural logarithm, we can show that $S(t) \leqslant D(t) \iff \psi(t) \geqslant 0$, where

$$\psi(t) = \ln(t^2+1) - \ln(2t) - \ln(t-1) - \frac{\ln t}{t-1} + 1 + \ln\ln t.$$

A direct computation yields

$$\begin{aligned}\psi'(t) &= \frac{\ln t}{(t-1)^2} + \frac{1}{t\ln t} - \frac{2(t+1)}{(t^2+1)(t-1)} \\ &\geqslant \frac{2}{\sqrt{t}(t-1)} - \frac{2(t+1)}{(t^2+1)(t-1)} \\ &= 2\frac{t\sqrt{t}-1}{(t+\sqrt{t})(t^2+1)} > 0,\end{aligned}$$

for $t > 1$. Since $\lim_{t\to 1+} \psi(t) = 0$, it follows that $\psi(t) \geqslant 0$ for $t \geqslant 1$.

The convexity of f is obvious, since

$$f(v)=\begin{cases}\frac{1}{2}(c^{2v}+1), & 0\leqslant v\leqslant \frac{1}{2}\\ \frac{1}{2}(c+c^{2v-1}), & \frac{1}{2}<v\leqslant 1\end{cases}.$$

Finally, $f(\frac{1}{2})=D(\sqrt{c})\sqrt{c}=\frac{1}{2}(1+c)=\frac{1}{2}(f(0)+f(1))$. □

Now, the following improvement of the series of inequalities in (1.10) is also based on our Theorem 3.

THEOREM 10. *Let $a,b>0$ and $0\leqslant v\leqslant 1$. If N is a nonnegative integer, then*

$$(1-v)a+vb\geqslant S(c^{r_0(v)})a^{1-v}b^v+\sum_{n=0}^{N-1}r_n(v)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v) \tag{3.10}$$

and

$$\begin{aligned}(1-v)a+vb\leqslant a+b-a^vb^{1-v}S(c^{r_0(v)})\\ -\sum_{n=0}^{N-1}r_n(v)\sum_{k=1}^{2^n}\Delta_f(n,2^n-k+1)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),\end{aligned} \tag{3.11}$$

where $c=a^{-1}b$ and $f(v)=S(c^{r_0(v)})a^{1-v}b^v$.

Proof. For $c>0$, let $f_c(v)=c^vS(c^{r_0(v)})$. We will show that f_c is convex on $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. Since

$$f_{-c}(v)=c^{-v}S(c^{r_0(v)})=c^{-1}c^{1-v}S(c^{r_0(1-v)})=c^{-1}f_c(1-v),$$

we may assume $c>1$ and show that $g(v)\equiv e(\ln c)f_c(v)$ is convex on $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. From now on, we will write any function $\alpha(v)$ simply as α, for a convenience. Letting $x=r_0/(c^{r_0}-1)$, $h=c^x/x$, and $g=c^vh$, a straightforward computation yields

$$\begin{aligned}h'&=x'\left(\ln c-\frac{1}{x}\right)h,\\ h''&=x''\left(\ln c-\frac{1}{x}\right)h+\left(\frac{x'}{x}\right)^2h+x'\left(\ln c-\frac{1}{x}\right)h'\\ &=h\left[x''\left(\ln c-\frac{1}{x}\right)+\left(\frac{x'}{x}\right)^2+(x')^2\left(\ln c-\frac{1}{x}\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}g'&=c^v\left(h\ln c+h'\right),\\ g''&=c^v\left(h(\ln c)^2+2h'\ln c+h''\right)\\ &=c^vh\left(\left[\ln c+x'\left(\ln c-\frac{1}{x}\right)\right]^2+\frac{1}{x^2}\left[xx''\left(x\ln c-1\right)+(x')^2\right]\right).\end{aligned}$$

Thus, it suffices to show that

$$xx''\,(x\ln c-1)+(x')^2\geqslant 0. \tag{3.12}$$

Since

$$\begin{aligned}
x' &= r_0'\left(\frac{1}{c^{r_0}-1}-\frac{r_0c^{r_0}\ln c}{(c^{r_0}-1)^2}\right)=\pm\frac{x}{r_0}\,(1-xc^{r_0}\ln c)\,,\\
x'' &= \pm\frac{\left[x'\,(1-2xc^{r_0}\ln c)-x^2c^{r_0}r_0'(\ln c)^2\right]r_0-x\,(1-xc^{r_0}\ln c)\,r_0'}{r_0^2}\\
&= \frac{x^2}{r_0^2}\,((2xc^{r_0}-r_0)\ln c-2)\,c^{r_0}\ln c,
\end{aligned}$$

the relation (3.12) can be rewritten as

$$((2xc^{r_0}-r_0)\ln c-2)\,(x\ln c-1)xc^{r_0}\ln c+(1-xc^{r_0}\ln c)^2\geqslant 0.$$

Replacing x by $r_0/(c^{r_0}-1)$ and denoting c^{r_0} by t, the above inequality reads

$$\left(\frac{t+1}{t-1}\ln t-2\right)\left(\frac{\ln t}{t-1}-1\right)\frac{t\ln t}{t-1}+\left(1-\frac{t\ln t}{t-1}\right)^2\geqslant 0$$

for $t>1$. Multiplying by $(t-1)^3$ and letting $s=\ln t$, the above expression becomes

$$((t+1)s-2(t-1))\,(s-t+1)ts+(t-1)\,(t-1-ts)^2\geqslant 0,$$

or equivalently,

$$\xi\,(t)\equiv t^3+\left(s^3-3s^2-3\right)t^2+(s^3+3s^2+3)t-1\geqslant 0$$

for $t>1$. A straightforward computation shows

$$\begin{aligned}
\xi_1=\xi' &= 3t^2+(2s^3-3s^2-6s-6)t+s^3+6s^2+6s+3,\\
\xi_2=t\xi_1' &= 6t^2+(2s^3+3s^2-12s-12)t+3s^2+12s+6,\\
\xi_3=t\xi_2' &= 12t^2+(2s^3+9s^2-6s-24)t+6s+12,\\
\xi_4=t\xi_3' &= 24t^2+(2s^3+15s^2+12s-30)t+6,\\
\xi_5=t\xi_4' &= 48t+2s^3+21s^2+42s-18.
\end{aligned}$$

Since $t>1$ and $s>0$, it follows that $\xi_5>0$. Thus, $\xi\,(t)\geqslant 0$ results from

$$\xi_4(1)=\cdots=\xi_1(1)=\xi\,(1)=0.$$

We have shown that f_c is convex on $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. Now, by Lemma 9 it follows that $\frac{1}{2}(\sqrt{c}+\sqrt{c}^{-1})\geqslant S(\sqrt{c})$ which is equivalent to $f(0)+f(1)\geqslant 2f(\frac{1}{2})$. Thus, f_c satisfies

$$1-v+vc\geqslant c^vS(c^{r_0(v)})-\sum_{n=0}^{N-1}r_n(v)\sum_{k=1}^{2^n}\Delta_{f_c}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v),$$

$$1-v+vc \leqslant 1+c-c^{1-v}S(c^{r_0(v)})-\sum_{n=0}^{N-1} r_n(v)\sum_{k=1}^{2^n}\Delta_{f_c}(n,2^n-k+1)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),$$

by Theorem 3. Finally, letting $c=a^{-1}b$, we obtain (3.10) and (3.11). □

In order to conclude this section, we give yet another improvement of the Young inequality, based on Theorem 3 and Lemma 9.

THEOREM 11. *Let $a,b>0$ and let N be a nonnegative integer. Define $g_{a,b}(v)$ by $g_{a,b}(v)=a^{1-2v}b^{2v}$, $0\leqslant v\leqslant 1$.*

1. If $0\leqslant v\leqslant \frac{1}{2}$, then

$$(1-v)a+vb \geqslant \frac{1}{2}(a^{1-2v}b^{2v}+a)+\frac{1}{2}\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^{n-1}}\Delta_{g_{a,b}}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),$$

$$(1-v)a+vb \leqslant a+\frac{1}{2}b-\frac{1}{2}a^{2v}b^{1-2v}-\frac{1}{2}\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^{n-1}}\Delta_{g_{b,a}}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v).$$

2. If $\frac{1}{2}<v\leqslant 1$, then

$$(1-v)a+vb \geqslant \frac{1}{2}(a^{2-2v}b^{2v-1}+b)+\frac{1}{2}\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^{n-1}}\Delta_{g_{a,b}}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\left(v-\frac{1}{2}\right),$$

$$(1-v)a+vb \leqslant \frac{1}{2}a+b-\frac{1}{2}a^{2v-1}b^{2-2v}-\frac{1}{2}\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^{n-1}}\Delta_{g_{b,a}}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\left(v-\frac{1}{2}\right).$$

Proof. Utilizing Theorem 3 and Lemma 9 with

$$\begin{aligned} f(v) &= D(c^{r_0(v)})a^{1-v}b^v \\ &= \frac{1}{2}(a^{-r_0(v)+1-v}b^{r_0(v)+v}+a^{r_0(v)+1-v}b^{-r_0(v)+v}) \\ &= \begin{cases} \frac{1}{2}(a^{1-2v}b^{2v}+a), & 0\leqslant v\leqslant \frac{1}{2} \\ \frac{1}{2}(a^{2-2v}b^{2v-1}+b), & \frac{1}{2}<v\leqslant 1 \end{cases}, \end{aligned}$$

we have

$$(1-v)a+vb \geqslant f(v)+\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v), \tag{3.13}$$

$$(1-v)a+vb \leqslant a+b-f(1-v)-\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^n}\Delta_f(n,2^n-k+1)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v).$$

Note that the outer summation starts at $n=1$, since $\Delta_f(0,1)=0$.

If $0 \leqslant v \leqslant \frac{1}{2}$, then $f(v) = \frac{1}{2}(a^{1-2v}b^{2v} + a)$ and $f(1-v) = \frac{1}{2}(a^{2v}b^{1-2v} + b)$. Further, taking into account (3.13), we have

$$(1-v)a + vb \geqslant \frac{1}{2}(a^{1-2v}b^{2v} + a) + \sum_{n=1}^{N-1} r_n(v) \sum_{k=1}^{2^{n-1}} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),$$

$$(1-v)a + vb \leqslant a + \frac{1}{2}b - \frac{1}{2}a^{2v}b^{1-2v} - \sum_{n=1}^{N-1} r_n(v) \sum_{k=1}^{2^{n-1}} \Delta_f(n,2^n-k+1)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v).$$

Finally, since $1 \leqslant k \leqslant 2^{n-1}$, it follows that

$$\begin{aligned}
\Delta_f(n,k) &= \frac{1}{2}\Delta_{g_{a,b}}(n,k),\\
\Delta_f(n,2^n-k+1) &= f\left(1-\frac{k}{2^n}\right) + f\left(1-\frac{k-1}{2^n}\right) - 2f\left(1-\frac{2k-1}{2^{n+1}}\right)\\
&= \frac{1}{2}\Delta_{g_{b,a}}(n,k).
\end{aligned}$$

On the other hand, if $\frac{1}{2} < v \leqslant 1$, then $f(v) = \frac{1}{2}(a^{2-2v}b^{2v-1} + b)$ and $f(1-v) = \frac{1}{2}(a^{2v-1}b^{2-2v} + a)$. Thus, utilizing (3.13) we have,

$$\begin{aligned}
(1-v)a + vb &\geqslant \frac{1}{2}(a^{2-2v}b^{2v-1} + b) + \sum_{n=1}^{N-1} r_n(v) \sum_{k=2^{n-1}+1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\\
&= \frac{1}{2}(a^{2-2v}b^{2v-1} + b) + \sum_{n=1}^{N-1} r_n(v) \sum_{k=1}^{2^{n-1}} \Delta_f(n,k+2^{n-1})\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\left(v-\frac{1}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
(1-v)a+vb &\leqslant \frac{1}{2}a+b-\frac{1}{2}a^{2v-1}b^{2-2v} - \sum_{n=1}^{N-1} r_n(v) \sum_{k=2^{n-1}+1}^{2^n} \Delta_f(n,2^n-k+1)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\\
&= \frac{1}{2}a+b-\frac{1}{2}a^{2v-1}b^{2-2v} - \sum_{n=1}^{N-1} r_n(v) \sum_{k=1}^{2^{n-1}} \Delta_f(n,2^{n-1}-k+1)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\left(v-\frac{1}{2}\right).
\end{aligned}$$

Finally, if $1 \leqslant k \leqslant 2^{n-1}$, we have

$$\begin{aligned}
\Delta_f(n,k+2^{n-1}) &= f\left(\frac{1}{2}+\frac{k-1}{2^n}\right) + f\left(\frac{1}{2}+\frac{k}{2^n}\right) - 2f\left(\frac{1}{2}+\frac{2k-1}{2^{n+1}}\right)\\
&= \frac{1}{2}\Delta_{g_{a,b}}(n,k)
\end{aligned}$$

and

$$\begin{aligned}
\Delta_f(n,2^{n-1}-k+1) &= f\left(\frac{1}{2}-\frac{k}{2^n}\right) + f\left(\frac{1}{2}-\frac{k-1}{2^n}\right) - 2f\left(\frac{1}{2}-\frac{2k-1}{2^{n+1}}\right)\\
&= \frac{1}{2}\Delta_{g_{b,a}}(n,k),
\end{aligned}$$

which completes the proof. □

4. Applications to some matrix inequalities

Our aim in this section is to discuss some matrix inequalities that correspond to scalar inequalities derived in the previous section.

Throughout this section, we will use M_n for the set of $n\times n$ complex matrices, M_n^+ for the subset of M_n consisting of positive definite matrices, and $|||\cdot|||$ for any unitarily invariant norm. For $A\in M_n$, $A>0$ $(A\geqslant 0)$ means that A is positive definite (semidefinite). For Hermitian matrices $A,B\in M_n$, $A<B$ $(A\leqslant B)$ implies that $B-A$ is positive definite (semidefinite). The absolute value of $A\in M_n$ will be defined by $|A|=(A^*A)^{1/2}$.

For $A,B\in M_n^+$ and $0\leqslant v\leqslant 1$, the v-weighted arithmetic mean and geometric mean of A and B are defined, respectively, by

$$A\nabla_v B=(1-v)A+vB,$$
$$A\sharp_v B=A^{1/2}(A^{-1/2}BA^{-1/2})^v A^{1/2}.$$

For convenience of notation, we use $A\nabla B$ for $A\nabla_{\frac{1}{2}}B$ and $A\sharp B$ for $A\sharp_{\frac{1}{2}}B$.

In order to obtain matrix inequalities from the corresponding scalar inequalities, we will use the operator monotonicity of continuous functions, that is, if f is a real valued continuous function defined on the spectrum of a self-adjoint operator A, then $f(t)\geqslant 0$ for every t in the spectrum of A implies that $f(A)$ is a positive operator.

Matrix inequalities that correspond to Theorems 4 and 5 have been already established in papers [2, 12]. Now, we are going to discuss matrix inequalities that correspond to Corollary 8, closely connected to some recent matrix inequalities due to Dragomir.

In order to do this, we will first generalize the definition of the geometric mean $A\sharp_v B=A^{1/2}(A^{-1/2}BA^{-1/2})^v A^{1/2}$. Let f be a continuous function defined on an interval I containing the spectrum of $A^{-1/2}BA^{-1/2}$. Then, using the functional calculus for continuous functions, we define $A\sharp_f B$ by

$$A\sharp_f B=A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}.$$

Utilizing the scalar relation (1.8), Dragomir [3], established the following series of inequalities

$$\frac{1}{2}v(1-v)A\sharp_{f_{\min}}B\leqslant A\nabla_v B-A\sharp_v B\leqslant \frac{1}{2}v(1-v)A\sharp_{f_{\max}}B, \tag{4.1}$$

where $A,B\in M_n^+$, $0\leqslant v\leqslant 1$, and

$$f_{\min}(x)=\min\{1,x\}(\ln x)^2,$$
$$f_{\max}(x)=\max\{1,x\}(\ln x)^2,$$

where $x>0$. Now, by virtue of our Corollary 8 we can obtain more accurate relations than those in (4.1).

THEOREM 12. *Let* $A,B\in M_n^+$ *and* $0\leqslant v\leqslant 1$. *Then,*

$$A\nabla_v B\geqslant A\sharp_v B+r_0(v)(A+B-2A\sharp B)+\alpha(v)A\sharp_{f_{\min}}B$$

and

$$\begin{aligned}A\nabla_v B&\leqslant A+B-A\sharp_{1-v}B-r_0(v)(A+B-2A\sharp B)-\alpha(v)A\sharp_{f_{\min}}B,\\&=A\sharp B-A\sharp_{1-v}B+R_0(v)(A+B-2A\sharp B)-\alpha(v)A\sharp_{f_{\min}}B,\end{aligned}$$

where $\alpha(v)=\frac{1}{2}v(1-v)-\frac{1}{4}r_0(v)$ *and* $f_{\min}(x)=\min\{1,x\}(\ln x)^2$.

Proof. By Corollary 8, we have

$$\begin{aligned}1-v+vc&\geqslant c^v+r_0(v)(c+1-2\sqrt{c})+\alpha(v)f_{\min}(c),\\1-v+vc&\leqslant 1+c-c^{1-v}-r_0(v)(c+1-2\sqrt{c})-\alpha(v)f_{\min}(c)\\&=2\sqrt{c}-c^{1-v}+R_0(v)(c+1-2\sqrt{c})-\alpha(v)f_{\min}(c)\end{aligned}$$

for $c>0$ and $0\leqslant v\leqslant 1$. Now, substituting c by $A^{-1/2}BA^{-1/2}$ and multiplying each inequality by $A^{1/2}$ both-sidedly, which preserves operator order, we obtain desired relations. □

Next, we give the matrix interpretation of Theorem 11.

THEOREM 13. *Let* $A,B\in M_n^+$ *and* $0\leqslant v\leqslant 1$. *Define* $G_{n,k}(A,B)$ *by*

$$G_{n,k}(A,B)=A\sharp_{(k-1)/2^{n-1}}B+A\sharp_{k/2^{n-1}}B-2A\sharp_{(2k-1)/2^n}B.$$

1. *If* $0\leqslant v\leqslant\frac{1}{2}$, *then*

$$(1-v)A+vB\geqslant\frac{1}{2}(A\sharp_{2v}B+A)+\frac{1}{2}\sum_{n=1}^{N-1}r_n(v)\sum_{k=1}^{2^{n-1}}G_{n,k}(A,B)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),$$

$$(1-v)A+vB\leqslant A+\frac{1}{2}B-\frac{1}{2}A\sharp_{1-2v}B-\frac{1}{2}\sum_{n=1}^{N-1}r_n(v)\sum_{k=1}^{2^{n-1}}G_{n,k}(B,A)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v).$$

2. *If* $\frac{1}{2}<v\leqslant 1$, *then*

$$(1-v)A+vB\geqslant\frac{1}{2}(A\sharp_{2v-1}B+B)+\frac{1}{2}\sum_{n=1}^{N-1}r_n(v)\sum_{k=1}^{2^{n-1}}G_{n,k}(A,B)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\left(v-\frac{1}{2}\right),$$

$$(1-v)A+vB\leqslant\frac{1}{2}A+B-\frac{1}{2}A\sharp_{2-2v}B-\frac{1}{2}\sum_{n=1}^{N-1}r_n(v)\sum_{k=1}^{2^{n-1}}G_{n,k}(B,A)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\left(v-\frac{1}{2}\right).$$

Proof. Let $c>0$. Taking into account Theorem 11 with $0\leqslant v\leqslant\frac{1}{2}$, we have

$$(1-v)+vc\geqslant\frac{1}{2}(c^{2v}+1)+\frac{1}{2}\sum_{n=1}^{N-1}r_n(v)\sum_{k=1}^{2^{n-1}}\Delta_{g_{1,c}}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),$$

$$(1-v)+vc \leqslant 1+\frac{1}{2}c-\frac{1}{2}c^{1-2v}-\frac{1}{2}\sum_{n=1}^{N-1} r_n(v)\sum_{k=1}^{2^{n-1}} \Delta_{g_{c,1}}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),$$

where

$$\Delta g_{1,c}(n,k) = c^{(k-1)/2^{n-1}}+c^{k/2^{n-1}}-2c^{(2k-1)/2^n},$$
$$\Delta g_{c,1}(n,k) = c^{1-(k-1)/2^{n-1}}+c^{1-k/2^{n-1}}-2c^{1-(2k-1)/2^n}.$$

Now, the desired inequalities follow by substituting c by $A^{-1/2}BA^{-1/2}$ and multiplying each inequality by $A^{1/2}$ both-sidedly. The same conclusion can be drawn for the case $\frac{1}{2}<v\leqslant 1$. We omit the detailed proof. □

The rest of this section will be dedicated to improving some important matrix inequalities known from the literature. First, we deal with Heinz-type inequalities. For $0\leqslant v\leqslant 1$, the Heinz mean in parameter v is defined by

$$H_v(a,b)=\frac{a^{1-v}b^v+a^vb^{1-v}}{2}, \qquad a,b>0.$$

The Heinz mean is convex on $[0,1]$, as a function of variable v and attains its minimum value at $v=1/2$. Thus, the Heinz mean interpolates between the geometric mean and the arithmetic mean, that is,

$$\sqrt{ab}\leqslant H_v(a,b)\leqslant \frac{a+b}{2}.$$

Similarly, it is easy to see that for any $A,B\in M_n^+$ holds relation

$$A\sharp B\leqslant H_v(A,B)\leqslant A\nabla B, \tag{4.2}$$

where

$$H_v(A,B)=\frac{A\sharp_v B+A\sharp_{1-v}B}{2}.$$

Now, by virtue of Theorem 3, we can improve the second inequality in (4.2).

THEOREM 14. *Let $A,B\in M_n^+$. If N is a nonnegative integer, then*

$$H_v(A,B)\leqslant A\nabla B-\sum_{n=0}^{N-1} r_n(v)\sum_{k=1}^{2^n}\left(H_{\frac{k-1}{2^n}}(A,B)+H_{\frac{k}{2^n}}(A,B)-2H_{\frac{2k-1}{2^{n+1}}}(A,B)\right)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v).$$

Proof. Let $c>0$. Since $f(v)=H_v(1,c)=(c^v+c^{1-v})/2$ is convex on $[0,1]$, we have

$$f(v)\leqslant f(0)-\sum_{n=0}^{N-1} r_n(v)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)$$

by Theorem 3. By the functional calculus, we can replace c by $A^{-1/2}BA^{-1/2}$. Then, multiplying the obtained inequality by $A^{1/2}$ both-sidedly, we obtain the desired inequality. □

Note that the second inequality in (4.2) follows from the above theorem with $N = 0$. Moreover, if $N = 1$, we have

$$H_v(A,B) \leqslant (1-2r_0(v))A\nabla B + 2r_0(v)A\sharp B$$

for all $0 \leqslant v \leqslant 1$, which was proved in [7].

Kittaneh [6], showed that if $A,B \in M_n^+$, $X \in M_n$, and $0 \leqslant v \leqslant 1$, then

$$|||A^{1-v}XB^v + A^vXB^{1-v}||| \leqslant 4r_0(v)|||A^{1/2}XB^{1/2}||| + (1-2r_0(v))|||AX+XB|||. \quad (4.3)$$

This Heinz-type inequality for unitarily invariant norms can be improved as follows.

THEOREM 15. *Let $A,B \in M_n^+$ and $X \in M_n$. If $0 \leqslant v \leqslant 1$, then*

$$|||A^{1-v}XB^v + A^vXB^{1-v}||| \leqslant |||AX+XB||| - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),$$

where $f(v) = |||A^{1-v}XB^v + A^vXB^{1-v}|||$.

Proof. It follows from Theorem 3 since the function $f(v) = |||A^{1-v}XB^v + A^vXB^{1-v}|||$ is convex on $[0,1]$ (for more details, see [1, Corollary IX.4.10]). □

Considering the above theorem for $N = 0$, we obtain the well-known Heinz inequality

$$|||A^{1-v}XB^v + A^vXB^{1-v}||| \leqslant |||AX+XB|||,$$

while for $N = 1$, we have

$$|||A^{1-v}XB^v + A^vXB^{1-v}||| \leqslant |||AX+XB||| - 2r_0(v)\big(|||AX+XB||| - 2|||A^{1/2}XB^{1/2}|||\big)$$

which is simply (4.3).

Now, consider the following relation that interpolates the matrix Cauchy-Schwarz inequality [15, Corollary 4.31]:

$$\begin{aligned} |||\,|A^{1/2}XB^{1/2}|^t\,|||^2 &\leqslant |||\,|A^{1-v}XB^v|^t\,||| \cdot |||\,|A^vXB^{1-v}|^t\,||| \\ &\leqslant |||\,|AX|^t\,||| \cdot |||\,|XB|^t\,|||, \end{aligned}$$

where $A,B \in M_n^+$, $X \in M_n$, and $t > 0$. This series of inequalities can be improved as follows.

THEOREM 16. *Let $A,B \in M_n^+$, $X \in M_n$, and N be a nonnegative integer. If $t > 0$ and $0 \leqslant v \leqslant 1$, then*

$$\begin{aligned} |||\,|A^{1-v}XB^v|^t\,||| \cdot |||\,|A^vXB^{1-v}|^t\,||| &\leqslant |||\,|AX|^t\,||| \cdot |||\,|XB|^t\,||| \\ &\quad - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v), \end{aligned}$$

where $f(v) = |||\,|A^{1-v}XB^v|^t\,||| \cdot |||\,|A^vXB^{1-v}|^t\,|||$.

Proof. It follows from Theorem 3 since $f(v) = |||\,|A^{1-v}XB^v|^t\,||| \cdot |||\,|A^vXB^{1-v}|^t\,|||$ is convex on $[0,1]$ (see [15, Theorem 4.30]). □

In particular, if $N = 1$ the above theorem reduces to

$$|||\,|A^{1-v}XB^v|^t\,||| \cdot |||\,|A^vXB^{1-v}|^t\,||| \leqslant (1-2r_0(v))|||\,|AX|^t\,||| \cdot |||\,|XB|^t\,||| \\ +2r_0(v)|||\,|A^{1/2}XB^{1/2}|^t\,|||^2,$$

where $0 \leqslant v \leqslant 1$.

Similarly to the previous theorem, we can also utilize convexity of a function $f(v) = |||\,|A^vXB^v|^t\,||| \cdot |||\,|A^{-v}XB^{-v}|^t\,|||$ on the interval $[-1,1]$ (for more details, see [15, Corollary 4.32]).

THEOREM 17. *Let $A,B \in M_n^+$, $X \in M_n$, and N be a nonnegative integer. If $t > 0$ and $-1 \leqslant v \leqslant 1$, then*

$$|||\,|A^vXB^v|^t\,||| \cdot |||\,|A^{-v}XB^{-v}|^t\,||| \leqslant |||\,|AXB|^t\,||| \cdot |||\,|A^{-1}XB^{-1}|^t\,||| \\ -\sum_{n=0}^{N-1} s_n(v) \sum_{k=1-2^{n-1}}^{2^{n-1}} \Delta_f(n,k)\chi_{(\frac{k-1}{2^{n-1}},\frac{k}{2^{n-1}})}(v),$$

where $f(v) = |||\,|A^vXB^v|^t\,||| \cdot |||\,|A^{-v}XB^{-v}|^t\,|||$ and $s_n(v) = r_n(\frac{v+1}{2})$.

Proof. Applying Theorem 3 to $g(v) = f(2v-1)$, $0 \leqslant v \leqslant 1$, we have

$$f(2v-1) \leqslant (1-v)f(-1) + vf(1) - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_g(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v). \tag{4.4}$$

Now, since

$$\Delta_g(n,k) = f\left(\frac{k-1}{2^{n-1}} - 1\right) + f\left(\frac{k}{2^{n-1}} - 1\right) - 2f\left(\frac{2k-1}{2^n} - 1\right),$$

replacing v by $\frac{v+1}{2}$ and k by $2^{n-1} - k$ in (4.4), we obtain

$$f(v) \leqslant \frac{1-v}{2}f(-1) + \frac{1+v}{2}f(1) - \sum_{n=0}^{N-1} s_n(v) \sum_{k=1-2^{n-1}}^{2^{n-1}} \Delta_f(n,k)\chi_{(\frac{k-1}{2^{n-1}},\frac{k}{2^{n-1}})}(v),$$

which represents the desired inequality. □

In particular, if $N = 1$ the above result reduces to

$$|||\,|A^vXB^v|^t\,||| \cdot |||\,|A^{-v}XB^{-v}|^t\,||| \leqslant (1-2s_0(v))|||\,|A^{-1}XB^{-1}|^t\,||| \cdot |||\,|AXB|^t\,||| \\ +2s_0(v)|||\,|X|^t\,|||^2,$$

where $-1 \leqslant v \leqslant 1$.

To conclude the paper, we will improve the following inequality involving positive definite matrices and arithmetic mean (see [5, pp. 554–555]):

$$(A\nabla_v B)^{-1} \leqslant A^{-1}\nabla_v B^{-1}, \tag{4.5}$$

where $A,B \in M_n^+$ and $0 \leqslant v \leqslant 1$. This inequality can also be refined by virtue of Theorem 3.

THEOREM 18. *If $A,B \in M_n^+$ and $0 \leqslant v \leqslant 1$, then*

$$(A\nabla_v B)^{-1} \leqslant A^{-1}\nabla_v B^{-1} - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} F_{n,k}(A,B)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v),$$

where

$$F_{n,k}(A,B) = (A\nabla_{\frac{k-1}{2^n}} B)^{-1} + (A\nabla_{\frac{k}{2^n}} B)^{-1} - 2(A\nabla_{\frac{2k-1}{2^{n+1}}} B)^{-1}.$$

Proof. Let $c > 0$. Applying Theorem 3 to the convex function $f(v) = (1-v+vc)^{-1}$, we have

$$(1-v+vc)^{-1} \leqslant 1-v+vc^{-1} - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v).$$

Now, the result follows by the functional calculus as in Theorems 12, 13, and 14. □

If $N = 0$, the above theorem reduces to inequality (4.5), while for $N = 1$ we obtain relation

$$(A\nabla_v B)^{-1} \leqslant A^{-1}\nabla_v B^{-1} - 2r_0(v)\left(A^{-1}\nabla B^{-1} - (A\nabla B)^{-1}\right),$$

where $0 \leqslant v \leqslant 1$.

REFERENCES

[1] R. BHATIA, *Matrix Analysis*, Springer-Verlag, 1997.
[2] D. CHOI, *Multiple-term refinements of Young type inequalities*, preprint.
[3] S. DRAGOMIR, *On new refinements and reverses of Young's operator inequality*, http://arxiv.org/abs/1510.01314.
[4] S. FURUICHI, *Refined Young inequalities with Specht's ratio*, J. Egypt. Math. Soc. **20** (2012), 46–49.
[5] R. HORN AND C. JOHNSON, *Topics in Matrix Analysis*, Cambridge U. P., New York, 1985.
[6] F. KITTANEH, *On the convexity of the Heinz means*, Integr. Equ. Oper. Theory **68** (2010), 519–527.
[7] F. KITTANEH AND M. KRNIĆ, *Refined Heinz operator inequalities*, Linear and Multilinear Algebra **61** (2013), 1148–1157.
[8] F. KITTANEH AND Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361** (2010), 262–269.
[9] F. KITTANEH AND Y. MANASRAH, *Reverse Young and Heinz inequalities for matrices*, Linear and Multilinear Algebra **59** (2011), 1031–1037.
[10] W. LIAO AND J. WU, *Improved Young and Heinz inequalities with the Kantorovich constant*, J. Math. Ineq. **10** (2016), 559–570.
[11] N. MINCULETE, *A refinement of the Kittaneh–Manasrah inequality*, Creat. Math. Inform. **20** (2011), 157–162.

[12] M. SABABHEH AND D. CHOI, *A complete refinement of Young's inequality*, J. Math. Anal. Appl. **440** (2016), 379–393.
[13] M. TOMINAGA, *Specht's ratio in the Young inequality*, Sci. Math. Japon. **55** (2002), 583–588.
[14] J. WU AND J. ZHAO, *Operator inequalities and reverse inequalities related to the Kittaneh-Manasrah inequalities*, Linear and Multilinear Algebra. **62** (2014), 884–894.
[15] X. ZHAN, *Matrix inequalities*, Springer-Verlag, Berlin (2002).
[16] J. ZHAO AND J. WU, *Operator inequalities involving improved Young and its reverse inequalities*, J. Math. Anal. Appl. **421** (2015), 1779–1789.

(Received September 10, 2016)

Daeshik Choi
Southern Illinois University, Edwardsville
Department of Mathematics and Statistics
Box 1653, Edwardsville, IL 62026
e-mail: dchoi@siue.edu

Mario Krnić
University of Zagreb
Faculty of Electrical Engineering and Computing
Unska 3, 10000 Zagreb, Croatia
e-mail: mario.krnic@fer.hr

Josip Pečarić
University of Zagreb
Faculty of Textile Technology
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
e-mail: pecaric@element.hr