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THE INEQUALITIES OF RANDOMLY WEIGHTED SUMS OF PAIRWISE NQD SEQUENCES AND ITS APPLICATION TO LIMIT THEORY

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Abstract. By using some inequalities of randomly weighted sums of pairwise NQD random variables, we investigate the single-indexed randomly weighted and double-indexed randomly weighted sums of these dependence structure. Some almost sure convergence and complete convergence results are obtained, which extend the corresponding results for the nonweighted and constant weighted cases to the case of randomly weighted. Last, some simulations are also illustrated in this paper.

1. Introduction

DEFINITION 1.1. Two random variables X and Y are said to be negative quadrant dependent (NQD) if for all real numbers x and y,

$$P(X \leqslant x, Y \leqslant y) \leqslant P(X \leqslant x)P(Y \leqslant y).$$

A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be pairwise NQD if X_i and X_j are NQD for any $i,\ j \in N^+$ and $i \neq j$.

An array of random variables $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is called rowwise pairwise NQD random variables if for every $n \geqslant 1$, $\{X_{ni}, 1 \leqslant i \leqslant n\}$ are pairwise NQD if X_{nj} and X_{nk} are NQD for any $1 \leqslant j,k \leqslant n$ and $j \neq k$.

The concept of pairwise NQD was introduced in Lehmann [13]. Pairwise NQD random variables are weak dependent random variables. The related concepts to pairwise NQD are negatively associated (NA), negatively superadditive dependent (NSD) and negatively orthant dependent (NOD). It can be found that NA and NSD random variables are NOD random variables, but the converse statement cannot always be true. For the counter-examples, one can refer to Joag-Dev and Proschan [12] and Wu [22]. Meanwhile, associated concept is closely related to negatively associated. For the examples and limit theorems of this random fields and related systems, one can refer to

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Bulinski and Shaskin [3]. It can be seen that NA, NSD and NOD sequences are pairwise NQD sequences. Thus, it is important to investigate the limit theory of pairwise NQD sequences. For more results of pairwise NQD sequences and related dependent sequences, we can refer to the references [4–7, 10, 11, 14–19, 21, 23–28] and so on.

Let C be some positive constant. Recall that a sequence $\{X_n, n \geqslant 1\}$ is stochastically dominated by a random variable Y if

$$\sup_{n\geqslant 1} P(|X_n| > x) \leqslant CP(|Y| > x), \ \ \forall\, x \geqslant 0. \tag{1.1}$$

Similarly, an array of random variables $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is stochastically dominated by a random variable Y if

$$\sup_{1\leqslant i\leqslant n, n\geqslant 1} P(|X_{ni}| > x) \leqslant CP(|Y| > x), \ \ \forall\, x \geqslant 0. \tag{1.2}$$

An array of random variables $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is stochastically dominated by random variable $\{Y_i, i \geqslant 1\}$ if

$$\sup_{n\geqslant i} P(|X_{ni}| > x) \leqslant CP(|Y_i| > x), \ \ \forall\, x \geqslant 0, \ \ \forall\, i \geqslant 1. \tag{1.3}$$

For more details of stochastically dominated, one can refer to Adler and Rosalsky [1], Adler et al. [2], Ghosal and Chandra [8], Hanson et al. [9], Wright [20], etc. As far as we know, there is no result of randomly weighted sums of pairwise NQD sequences. In this paper, by using some inequalities of randomly weighted sums of pairwise NQD sequences, we investigate the limit theorems of these dependent sequences, including single-indexed randomly weighted and double-indexed randomly weighted, and obtain the results of almost sure convergence and complete convergence. For the details, please see our results in Section 2. We extend the results of Hu et al. [10], Wang et al. [18] and Wu and Guo [26] for nonweighted and constant weighted cases to the case of randomly weighted. Some simulations are also illustrated in Section 2. The proofs of main results are presented in Section 3. Through out the paper, let $C, C_1, C_2, C_3, \cdots$, denote some positive constants not depending on n, which may be different in various places, $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$ and $\log x = \ln\max(x, e)$.

2. Limit theorems of randomly weighted sums of pairwise NQD sequences

First, we investigate the almost sure convergence of single-indexed randomly weighted sums of pairwise NQD sequences.

THEOREM 2.1. *For some $1 \leqslant r < 2$ and $\alpha > 3r/2$, let $\{X_n, n \geqslant 1\}$ be a mean zero sequence of pairwise NQD random variables, which is stochastically dominated by a random variable X with $E(|X|^r \log^\alpha |X|) < \infty$. Suppose that $\{A_n, n \geqslant 1\}$ is a sequence of independent random variables, which is also independent of $\{X_n, n \geqslant 1\}$. Let*

$$\sum_{i=1}^{n} EA_i^2 = O(n). \tag{2.1}$$

Then

$$\frac{1}{n^{1/r}}\sum_{i=1}^{n} A_i X_i \to 0, \ \ \textit{almost sure, as} \ \ n \to \infty. \tag{2.2}$$

Taking $A_n \equiv 1$, $n \geqslant 1$ in Theorem 2.1, we have the following result.

COROLLARY 2.1. *For some $1 \leqslant r < 2$ and $\alpha > 3r/2$, let $\{X_n, n \geqslant 1\}$ be a mean zero sequence of pairwise NQD random variables, which is stochastically dominated by a random variable X with $E(|X|^r \log^{\alpha}|X|) < \infty$. Then*

$$\frac{1}{n^{1/r}}\sum_{i=1}^{n} X_i \to 0, \ \ \textit{almost sure, as}\ n \to \infty. \tag{2.3}$$

Second, we investigate the complete convergence of double-indexed randomly weighted sums of rowwise pairwise NQD sequences.

THEOREM 2.2. *Let $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be a mean zero array of rowwise pairwise NQD random variables. For each $n \geqslant 1$, we assume that $\{A_{ni}, 1 \leqslant i \leqslant n\}$ are independent random variables, which is also independent of $\{X_{ni}, 1 \leqslant i \leqslant n\}$. Let $\{b_n, n \geqslant 1\}$ be a sequence of positive numbers satisfying*

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{b_n^2} \sum_{i=1}^{n} EA_{ni}^2 EX_{ni}^2 < \infty. \tag{2.4}$$

Then,

$$\frac{1}{b_n} \max_{1 \leqslant k \leqslant n} \Big| \sum_{i=1}^{k} A_{ni} X_{ni} \Big| \to 0, \ \textit{completely, as}\ n \to \infty, \tag{2.5}$$

which yields

$$\frac{1}{b_n}\sum_{i=1}^{n} A_{ni} X_{ni} \to 0, \quad \textit{completely}, \ \ \textit{as}\ n \to \infty. \tag{2.6}$$

As applications of Theorem 2.2, we have the following results.

COROLLARY 2.2. *Let $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be a mean zero array of rowwise pairwise NQD random variables. For every $n \geqslant 1$, we assume that $\{A_{ni}, 1 \leqslant i \leqslant n\}$ are independent random variables, which is also independent of the sequence $\{X_{ni}, 1 \leqslant i \leqslant n\}$. Suppose that $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is stochastically dominated by a random variable A with $EA^2 < \infty$. For some $r > 0$, let*

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2r}} \sum_{i=1}^{n} EX_{ni}^2 < \infty. \tag{2.7}$$

Then,

$$\frac{1}{n^r} \max_{1 \leqslant k \leqslant n} \Big| \sum_{i=1}^{k} A_{ni} X_{ni} \Big| \to 0, \ \textit{completely, as}\ n \to \infty, \tag{2.8}$$

which yields

$$\frac{1}{n^r}\sum_{i=1}^{n} A_{ni} X_{ni} \to 0, \quad \textit{completely}, \ \ \textit{as}\ n \to \infty. \tag{2.9}$$

COROLLARY 2.3. *Assume that* $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *is a mean zero array of rowwise pairwise NQD random variables. For every* $n \geqslant 1$*, let* $\{A_{ni}, 1 \leqslant i \leqslant n\}$ *be independent random variables, which is also independent of the sequence* $\{X_{ni}, 1 \leqslant i \leqslant n\}$*. Let* $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be stochastically dominated by a sequence of random variable* $\{B_i, i \geqslant 1\}$*,* $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be stochastically dominated by a sequence of random variable* $\{Y_i, i \geqslant 1\}$*. For some* $r > \frac{1}{2}$*, suppose that*

$$\sum_{n=1}^{\infty} \frac{\log^2 n EB_n^2 EY_n^2}{n^{2r-1}} < \infty. \tag{2.10}$$

Then, it has (2.8), which implies (2.9).

COROLLARY 2.4. *Assume that* $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *is a mean zero array of rowwise pairwise NQD random variables, which is stochastically dominated by a sequence of random variable* X *with* $EX^2 < \infty$*. For every* $n \geqslant 1$*, let* $\{A_{ni}, 1 \leqslant i \leqslant n\}$ *be independent random variables, which is also independent of the sequence* $\{X_n, 1 \leqslant i \leqslant n\}$*. For some* $\delta > 0$*, suppose that*

$$\sum_{i=1}^{n} EA_{ni}^2 = O(n^{\delta}). \tag{2.11}$$

Then for all $r > \frac{1+\delta}{2}$*, it has (2.8), which yields (2.9).*

REMARK 2.1. For some $1 \leqslant r < 2$ and $\alpha > 1+r$, by the moment condition such as $E(|X|^r \log^{\alpha}|X|) < \infty$, Hu et al. [10] obtained the almost sure convergence (2.3) for the nonweighted sums of pairwise NQD sequences. For some $1 \leqslant r < 2$, by the moment condition $E(|X|^r \log^2|X|) < \infty$, Wu and Guo [26] obtained the result (2.3) for the non-weighted case too. In our Theorem 2.1, by the moment condition $E(|X|^r \log^{\alpha}|X|) < \infty$ with $1 \leqslant r < 2$ and $\alpha > 3r/2$, we obtain the almost sure convergence (2.2) for the randomly weighted sums of pairwise NQD sequences, which yields the result of (2.3) in Corollary 2.1. On the one hand, in view of $1 \leqslant r < 2$, it has $1+r > 3/2r$, which implies that our condition $E(|X|^r \log^{\alpha}|X|) < \infty$ is weaker than the one of Hu et al. [10]. On the other hand, it can can be checked that $3r/2 < 2$ if $r \in [1, 4/3)$, and $3r/2 \geqslant 2$ if $r \in [4/3, 2)$. So we improve the result of Hu et al. [10] and extend the result of Wu and Guo [26] to the randomly weighted case. Moreover, Wang et al. [18] investigated the complete convergence for double-indexed constant weighted sums of END random variables, and obtained some results such as $\frac{1}{b_n}\sum_{i=1}^{n} a_{ni}X_{ni} \to 0$, completely, as $n \to \infty$ (see Theorem 4.1 of Wang et al. [18]). Inspired by Wang et al. [18], in this paper, we studied the double-indexed and randomly weighted sums of pairwise NQD sequences and get some similar results such as $\frac{1}{b_n}\max_{1 \leqslant k \leqslant n}|\sum_{i=1}^{k} A_{ni}X_{ni}| \to 0$, completely, as $n \to \infty$, in Theorem 2.2. With the method of stochastically dominated, we obtain some complete convergence results such as $\frac{1}{n^r}\max_{1 \leqslant k \leqslant n}|\sum_{i=1}^{k} A_{ni}X_{ni}| \to 0$ and $\frac{1}{n^r}\sum_{i=1}^{n} A_{ni}X_{ni} \to 0$, completely, as $n \to \infty$, in Corollaries 2.2-2.4. So we extend the result of Wang et al. [18] for constant weighted sums of END random variables to the case of randomly weighted sums of pairwise NQD random variables. Since that pairwise NQD sequences contain many dependent sequences such as NA sequences, NSD sequences and NOD sequences, the results obtained in this paper also hold true for these dependent sequences.

SIMULATION 2.1. In the following, we do some simulations for the convergence of (2.2) in Theorem 2.1. Let $(X_1,X_2,\ldots,X_n)$ be a normal random vector such as $(X_1,X_2,\ldots,X_n) \sim N_n(0,\Sigma)$, where 0 is zero vector,

$$\Sigma = \begin{bmatrix} 1+\rho^2 & -\rho & -\rho^2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & -\rho^2 & \cdots & 0 & 0 & 0 & 0 \\ -\rho^2 & -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 & 0 \\ 0 & -\rho^2 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & -\rho^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho & -\rho^2 \\ 0 & 0 & 0 & 0 & \cdots & -\rho^2 & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\rho^2 & -\rho & 1+\rho^2 \end{bmatrix}_{n\times n},$$

and $0<\rho<1$. By Joag-Dev and Proschan [12], it can be seen that $(X_1,X_2,\ldots,X_n)$ is a NA vector. So, $(X_1,X_2,\ldots,X_n)$ is also a pairwise NQD vector. Let $\{A_n, n\geqslant 1\}$ be a *i.i.d.* random variables $A_1 \sim U(-a,b)$ with $a>0$ and $b>0$ (or $A_1 \sim \Gamma(d,\lambda)$ with $d>0$ and $\lambda>0$), which is also independent of $\{X_n, n\geqslant 1\}$. Then we use MATLAB software to plot the Box plot to illustrate

$$\frac{1}{n^{1/r}}\sum_{i=1}^{n} A_i X_i \to 0. \tag{2.12}$$

For $r=1.5$ (or $r=1$), $\rho=0.2$ (or $\rho=0.3$), the distribution $A_1 \sim U(-1,1)$ (or $A_1 \sim \Gamma(3,1)$) and sample size $n=100,200,\ldots,1000$, we repeat the experiments 10000 times and obtain the Box plots such as Fig 1 and Fig 2.

Fig 1: Box plots with r=1.5, ρ=0.2 and U(−1,1)

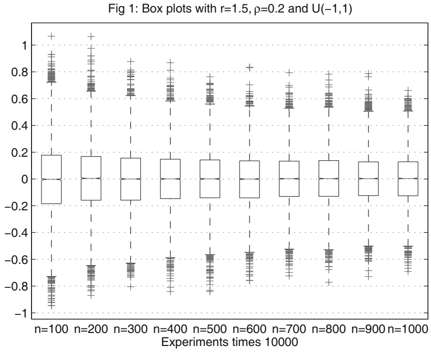
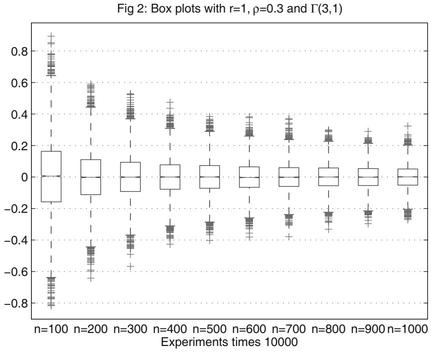


Fig 2: Box plots with r=1, ρ=0.3 and Γ(3,1)



In Fig 1 and Fig 2, the label of y-axis is the value of (2.12) and the label of x-axis is the number of sample n, by repeatting the experiments 10000 times. In Fig 1, for $r=1.5$, $\rho=0.2$ and $A_1 \sim U(-1,1)$, it can be seen that the median of (2.12) is close to 0 and the variation range becomes smaller as the sample n increases by $100,200,\ldots,1000$.

Likewise, in Fig 2, with $r = 1$, $\rho = 0.3$ and $A_1 \sim \Gamma(3,1)$, the median of (2.12) is close to 0 and the variation range becomes smaller too as the sample n increases. For the different ρ and distribution A_1, we also obtain some similar Box plots and omit them in this paper.

3. Some lemmas and the proofs of main results

LEMMA 3.1. (Lehmann [13]) *If random variables X and Y are NQD, then*
(i) $EXY \leqslant EXEY$;
(ii) $P(X > x, Y > y) \leqslant P(X > x)P(Y > y)$, $\forall\, x,\ y \in R$;
(iii) If f and g are both nondecreasing (or nonincreasing) functions, then $f(X)$ and $g(Y)$ are NQD.

REMARK 3.1. Let $\{X_n, n \geqslant 1\}$ be a pairwise NQD sequence and $\{Y_n, n \geqslant 1\}$ be a sequence of nonnegative and independent random variables, which is also independent of $\{X_n, n \geqslant 1\}$. Let $Z_n = X_n Y_n$. Then, for all $i \neq j$ and all real numbers x and y, we have

$$\begin{aligned} P(Z_i \leqslant x, Z_j \leqslant y) &= P(X_iY_i \leqslant x, X_jY_j \leqslant y) \\ &= \int_0^\infty \int_0^\infty P(X_iu \leqslant x, X_jv \leqslant y)dF_{Y_i}(u)dF_{Y_j}(v) \\ &\leqslant \int_0^\infty \int_0^\infty P(X_iu \leqslant x)P(X_jv \leqslant y)dF_{Y_i}(u)dF_{Y_j}(v) \\ &= P(X_iY_i \leqslant x)P(X_jY_j \leqslant y) \\ &= P(Z_i \leqslant x)P(Z_j \leqslant y) \end{aligned}$$

which yields that $\{Z_n, n \geqslant 1\}$ is also a pairwise NQD sequence.

LEMMA 3.2. (Wu [21, Lemma 2]) *Let $\{X_n, n \geqslant 1\}$ be a pairwise NQD sequence with $EX_n = 0$ and $EX_n^2 < \infty$ for all $n \geqslant 1$. Then for all $n \geqslant 1$, it has*

$$E\Big(\max_{1 \leqslant k \leqslant n}\Big(\sum_{i=1}^{k} X_i\Big)^2\Big) \leqslant C\log^2 n \sum_{i=1}^{n} EX_i^2,$$

where C is a positive constant not dependent on n.

LEMMA 3.3. (Adler and Rosalsky [1, Lemma 1] and Adler et al. [2, Lemma 3]) *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X. Then, for any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$\begin{aligned} E[|X_n|^\alpha I(|X_n| \leqslant b)] &\leqslant C_1\{E[|X|^\alpha I(|X| \leqslant b)] + b^\alpha P(|X| > b)\}, \\ E[|X_n|^\alpha I(|X_n| > b)] &\leqslant C_2 E[|X|^\alpha I(|X| > b)]. \end{aligned}$$

Consequently, it has $E[|X_n|^\alpha] \leqslant C_3 E|X|^\alpha$ for all $n \geqslant 1$. Here C_1, C_2 and C_3 are positive constants not depending on n.

LEMMA 3.4. *For every positive constant* $\alpha > 0$ *and integer* $m \geqslant 1$, *it has that*

$$\sum_{n=m}^{\infty} \frac{n(n+1)}{2^{\alpha n}} \leqslant C \frac{m^2}{2^{\alpha m}},$$

where C *is a positive constant not depending on* m.

Proof. With the techniques of mathematical analysis, it is easy to establish the result of Lemma 3.4. □

Proof of Theorem 2.1. Combining Lemma 3.1 with Remark 3.1, for all fixed n, we obtain that $\{A_i^+ X_i, 1 \leqslant i \leqslant n\}$, $\{A_i^- X_i, 1 \leqslant i \leqslant n\}$ are also pairwise NQD random variables. In view of $A_i X_i = A_i^+ X_i - A_i^- X_i$, without loss of generality, we assume that $A_i \geqslant 0$ in the proof. Denote $S_n = \sum_{i=1}^n A_i X_i$, $n \geqslant 1$. For any integer n, there exists some integer $k = k(n)$ such that $2^k \leqslant n < 2^{k+1}$. Therefore, it follows

$$\frac{1}{n^{1/r}} |S_n| \leqslant \max_{2^k \leqslant n < 2^{k+1}} \frac{1}{2^{k/r}} |S_n|.$$

Consequently, to prove (2.2), it is suffices to show that

$$\lim_{k \to \infty} \max_{2^k \leqslant n < 2^{k+1}} \frac{1}{2^{k/r}} |S_n| = 0, \quad a.s. \tag{3.1}$$

Take $r < \mu < 3r/2$. Denote $a_k = 2^{\frac{k+1}{r}} / (k+1)^{\frac{\mu}{r}}$ and

$$\begin{aligned}
X_i^{(k)} &= -a_k I(X_i < -a_k) + X_i I(|X_i| \leqslant a_k) + a_k I(X_i > a_k), \\
\tilde{X}_i^{(k)} &= X_i - X_i^{(k)} = a_k I(X_i < -a_k) + X_i I(|X_i| > a_k) - a_k I(X_i > a_k), \\
S_n^{(k)} &= \sum_{i=1}^n A_i X_i^{(k)}, \quad \tilde{S}_n^{(k)} = \sum_{i=1}^n A_i \tilde{X}_i^{(k)}, \quad k \geqslant 1,\ n \geqslant 1.
\end{aligned}$$

Making use of Hölder inequality and (2.1), one has

$$\sum_{i=1}^n E|A_i| \leqslant (\sum_{i=1}^n EA_i^2)^{1/2} (\sum_{i=1}^n 1)^{1/2} = O(n). \tag{3.2}$$

It is easy to see that $E(A_i X_i) = EA_i EX_i = 0$, $i \geqslant 1$. Then, for the k lager enough such that $((k+1)\log 2 - \mu \ln(k+1))^\alpha > 0$, we have by (1.1), Lemma 3.3, $E(|X|^r \log^\alpha |X|) < \infty$ and (3.2) that

$$\begin{aligned}
& 2^{-k/r} \max_{2^k \leqslant n < 2^{k+1}} \left| \sum_{i=1}^n E[A_i X_i I(|X_i| \leqslant a_k)] \right| = 2^{-k/r} \max_{2^k \leqslant n < 2^{k+1}} \left| \sum_{i=1}^n E[A_i X_i I(|X_i| > a_k)] \right| \\
\leqslant\ & 2^{-k/r} \sum_{i=1}^{2^{k+1}} E|A_i| E[|X_i| I(|X_i| > a_k)] \leqslant C_1 2^{-k/r} 2^{k+1} E\left[|X| I\left(|X| > \frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}} \right) \right] \\
\leqslant\ & C_2 \frac{2^{k+1} (k+1)^{\mu(r-1)/r} E(|X|^r \log^\alpha |X|)}{2^{k/r} 2^{(k+1)(r-1)/r} ((k+1)\log 2 - \mu \log(1+k))^\alpha} \\
\leqslant\ & C_3 \frac{1}{k^{\alpha - \mu + \mu/r}} \to 0, \quad as\ k \to \infty.
\end{aligned} \tag{3.3}$$

In view of (3.3), it has

$$2^{-k/r}\max_{2^k\leqslant n<2^{k+1}}\Big|\sum_{i=1}^{n}E[A_i(-a_k)I(X_i<-a_k)]\Big|$$
$$\leqslant 2^{-k/r}\sum_{i=1}^{2^{k+1}}E|A_i|E[|X_i|I(|X_i|>a_k)]\leqslant C\frac{1}{k^{\alpha-\mu+\mu/r}}\to 0,\quad as\ k\to\infty \tag{3.4}$$

and

$$2^{-k/r}\max_{2^k\leqslant n<2^{k+1}}\Big|\sum_{i=1}^{n}E[A_ia_kI(X_i>a_k)]\Big|$$
$$\leqslant 2^{-k/r}\sum_{i=1}^{2^{k+1}}E|A_i|E[|X_i|I(|X_i|>a_k)]\leqslant C\frac{1}{k^{\alpha-\mu+\mu/r}}\to 0,\quad as\ k\to\infty. \tag{3.5}$$

Hence, there exists a k_0 such that for all $\varepsilon>0$,

$$2^{-k/r}\max_{2^k\leqslant n<2^{k+1}}\Big|\sum_{i=1}^{n}E[A_iX_i^{(k)}]\Big|<\frac{\varepsilon}{4},\ \ k\geqslant k_0.$$

Since $S_n=S_n^{(k)}+\tilde{S}_n^{(k)}$, $k\geqslant 1$, $n\geqslant 1$, it can be argued that for all $\varepsilon>0$,

$$\begin{aligned}
&\sum_{k=1}^{\infty}P\Big(\max_{2^k\leqslant n<2^{k+1}}|S_n|>\varepsilon 2^{k/r}\Big)\\
\leqslant\ &\sum_{k=1}^{\infty}P\Big(\max_{2^k\leqslant n<2^{k+1}}|S_n^{(k)}|>\frac{\varepsilon 2^{k/r}}{2}\Big)+\sum_{k=1}^{\infty}P\Big(\max_{2^k\leqslant n<2^{k+1}}|\tilde{S}_n^{(k)}|>\frac{\varepsilon 2^{k/r}}{2}\Big)\\
\leqslant\ &\sum_{k=1}^{\infty}P\Big(\max_{2^k\leqslant n<2^{k+1}}|S_n^{(k)}-ES_n^{(k)}|>\frac{\varepsilon 2^{k/r}}{4}\Big)+\sum_{k=1}^{\infty}P\Big(\max_{2^k\leqslant n<2^{k+1}}|\tilde{S}_n^{(k)}|>\frac{\varepsilon 2^{k/r}}{2}\Big)\\
&+C+\sum_{k=k_0}^{\infty}P\Big(\max_{2^k\leqslant n<2^{k+1}}|ES_n^{(k)}|>\frac{\varepsilon 2^{k/r}}{4}\Big)\\
\leqslant\ &C+\sum_{k=1}^{\infty}P\Big(\max_{2^k\leqslant n<2^{k+1}}|S_n^{(k)}-ES_n^{(k)}|>\frac{\varepsilon 2^{k/r}}{4}\Big)+\sum_{k=1}^{\infty}P\Big(\max_{2^k\leqslant n<2^{k+1}}|\tilde{S}_n^{(k)}|>\frac{\varepsilon 2^{k/r}}{2}\Big)\\
:=\ &C+I+J.
\end{aligned}\tag{3.6}$$

On the one hand, it follows from $r<\mu<\alpha$ that $\alpha-\mu+\mu/r>1$. Combining with the proofs of (3.3), (3.4), (3.5), we check by Markov inequality, (1.1), Lemma 3.3 and $E(|X|^r\log^{\alpha}|X|)<\infty$ that

$$\begin{aligned}
J&\leqslant\sum_{k=1}^{\infty}\frac{2}{\varepsilon 2^{k/r}}E\Big(\max_{2^k\leqslant n<2^{k+1}}|\tilde{S}_n^{(k)}|\Big)\leqslant\sum_{k=1}^{\infty}\frac{2}{\varepsilon 2^{k/r}}\sum_{i=1}^{2^{k+1}}E|A_i|E[|X_i|I(|X_i|>a_k)]\\
&\leqslant C_1+C_2\sum_{k=k_0}^{\infty}\frac{1}{k^{\alpha-\mu+\mu/r}}<\infty.
\end{aligned}\tag{3.7}$$

On the other hand, by Lemma 3.1, $\{X_i^{(k)}, i \geqslant 1\}$ is also a pairwise NQD sequence with

$$E(X_i^{(k)})^2 = E[X_i^2 I(|X_i| \leqslant a_k)] + a_k^2 E[I(|X_i| > a_k)],\ i \geqslant 1.$$

Combining with Lemma 3.1 and Remark 3.1, the sequence $\{A_i X_i^{(k)}, i \geqslant 1\}$ is also a pairwise NQD sequence. So, it follows from Markov inequality, (1.1), (2.1), Lemma 3.2 and Lemma 3.3 that

$$\begin{aligned}
I &\leqslant \frac{4^2}{\varepsilon^2}\sum_{k=1}^{\infty} 2^{-2k/r} E\Big(\max_{2^k \leqslant n < 2^{k+1}} |S_n^{(k)} - ES_n^{(k)}|^2\Big) \\
&\leqslant \frac{4^2}{\varepsilon^2}\sum_{k=1}^{\infty} 2^{-2k/r} E\Big(\max_{1 \leqslant n \leqslant 2^{k+1}} \Big|\sum_{i=1}^{n}[A_i X_i^{(k)} - E(A_i X_i^{(k)})]\Big|^2\Big) \\
&\leqslant C_1 \sum_{k=1}^{\infty} \frac{(\log 2^{k+1})^2}{2^{2k/r}} \sum_{i=1}^{2^{k+1}} EA_i^2 E(X_i^{(k)})^2 \\
&\leqslant \sum_{k=1}^{\infty} \frac{C_2 k^2}{2^{2k/r}} \sum_{i=1}^{2^{k+1}} EA_i^2 \Big\{E[X_i^2 I(|X_i| \leqslant a_k)] + a_k^2 E[I(|X_i| > a_k)]\Big\} \\
&\leqslant C_3 \sum_{k=1}^{\infty} \frac{k^2}{2^{2k/r}} 2^{k+1} E\left[X^2 I\left(|X| \leqslant \frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}}\right)\right] \\
&\quad + C_4 \sum_{k=1}^{\infty} \frac{k^2}{2^{2k/r}} 2^{k+1} \frac{2^{2(k+1)/r}}{(k+1)^{2\mu/r}} E\left[I\left(|X| > \frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}}\right)\right] \\
&:= C_3 I_1 + C_4 I_2.
\end{aligned} \tag{3.8}$$

It can be argued the fact that there exists a $m_0 > 0$ such that $\frac{2^m}{m^\mu} < \frac{2^{m+1}}{(m+1)^\mu}$, $m > m_0$. Let $B_m := \{\frac{2^m}{m^\mu} < |X|^r \leqslant \frac{2^{m+1}}{(m+1)^\mu}\}$, $m \geqslant m_0 + 1$. Thus, by $1 \leqslant r < 2$, $r < \mu < 3r/2$ and $\alpha > 3r/2$, one makes use of Lemma 3.4 and establish that

$$\begin{aligned}
I_1 &= \sum_{k=1}^{m_0} k^2 2^{k+1-\frac{2k}{r}} E\left[X^2 I\left(|X| \leqslant \frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}}\right)\right] \\
&\quad + \sum_{k=m_0+1}^{\infty} k^2 2^{k+1-\frac{2k}{r}} \left(EX^2 I\left(|X|^r \leqslant \frac{2^{m_0+1}}{(m_0+1)^\mu}\right) + \sum_{m=m_0+1}^{k} E[X^2 I(B_m)]\right) \\
&\leqslant C_1 + \sum_{m=m_0+1}^{\infty} E[X^2 I(B_m)] \sum_{k=m}^{\infty} k^2 2^{k+1-\frac{2k}{r}} \\
&\leqslant C_1 + C_2 \sum_{m=m_0+1}^{\infty} m^2 2^{m-\frac{2m}{r}} E\left[|X|^r \log^\alpha |X| \frac{|X|^{2-r}}{\log^\alpha |X|} I(B_m)\right] \\
&\leqslant C_1 + C_3 \sum_{m=m_0+1}^{\infty} m^2 2^{m-\frac{2m}{r}} \frac{2^{(m+1)\frac{2-r}{r}}}{(m+1)^{\mu\frac{2-r}{r}}} \frac{1}{\log^\alpha \frac{2^m}{m^\mu}} E[|X|^r \log^\alpha |X| I(B_m)] \\
&\leqslant C_1 + C_4 \sum_{m=m_0+1}^{\infty} m^{2+\mu-\frac{2\mu}{r}-\alpha} E[|X|^r \log^\alpha |X| I(B_m)].
\end{aligned}$$

In view of $\alpha > r$, we take μ such as $\alpha > 2+\mu-\frac{2\mu}{r}$. Combining the above inequality with $E(|X|^r \log^\alpha |X|) < \infty$, we obtain

$$I_1 \leqslant C_1 + C_5 E(|X|^r \log^\alpha |X|) < \infty. \tag{3.9}$$

By $\mu < 3r/2 < \frac{3r}{2-r}$, it has $2+\mu-2\mu/r > -1$. Then, for I_2, we have by $\alpha > 3r/2$ and $E(|X|^r \log^\alpha |X|) < \infty$ that

$$\begin{aligned} I_2 &\leqslant C_1 \sum_{k=1}^{\infty} k^{2-\frac{2\mu}{r}} 2^{k+1} E\left[I\left(|X|^r > \frac{2^{k+1}}{(k+1)^\mu} \right) \right] \qquad (3.10)\\ &\leqslant C_2 \sum_{k=1}^{\infty} k^{2+\mu-\frac{2\mu}{r}} E\left[|X|^r I\left(|X|^r > \frac{2^{k+1}}{(k+1)^\mu} \right) \right] \\ &= C_2 \sum_{k=1}^{\infty} k^{2-\mu(\frac{2}{r}-1)} \sum_{m=k}^{\infty} E[|X|^r I(B_{m+1})] = C_2 \sum_{m=1}^{\infty} E[|X|^r I(B_{m+1})] \sum_{k=1}^{m} k^{2+\mu-\frac{2\mu}{r}} \\ &\leqslant C_3 \sum_{m=1}^{\infty} m^{3-\mu(\frac{2}{r}-1)} E[|X|^r I(B_{m+1})] \leqslant C_4 \sum_{m=1}^{\infty} m^{3r/2} E[|X|^r I(B_{m+1})] \\ &\leqslant C_5 + C_6 \sum_{m=m_0}^{\infty} \frac{m^{3r/2}}{((m+1)\log 2 - \mu \log(m+1))^\alpha} E[|X|^r \log^\alpha |X| I(B_{m+1})] \\ &\leqslant C_5 + C_6 \sum_{m=1}^{\infty} E[|X|^r \log^\alpha |X| I(B_{m+1})] \leqslant C_5 + C_6 E(|X|^r \log^\alpha |X|) < \infty. \qquad (3.11) \end{aligned}$$

Consequently, (3.1) follows from (3.6)–(3.11). □

Proof of Theorem 2.2. The proof is inspired by the Theorem 4.1 of Wang et al. [18]. For every fixed n, by Lemma 3.1 and Remark 3.1, one has that $\{A_{ni}^+ X_{ni}, 1 \leqslant i \leqslant n\}$ and $\{A_{ni}^- X_{ni}, 1 \leqslant i \leqslant n\}$ are also pairwise NQD random variables. In view of $A_{ni}X_{ni} = A_{ni}^+ X_{ni} - A_{ni}^- X_{ni}$, we also assume that $A_{ni} \geqslant 0$ in the proof. Then, it follows from Lemma 3.2 that

$$E\left(\max_{1\leqslant k\leqslant n} \left| \sum_{i=1}^{k} A_{ni} X_{ni} \right| \right)^2 \leqslant C_1 \log^2(n) \sum_{i=1}^{n} EA_{ni}^2 EX_{ni}^2, \tag{3.12}$$

where C_1 is a positive constants. Therefore, by Markov inequality, (2.4) and (3.12), we have that for all $\varepsilon > 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\frac{1}{b_n} \max_{1\leqslant k\leqslant n} \left| \sum_{i=1}^{k} A_{ni} X_{ni} \right| > \varepsilon \right) &\leqslant \sum_{n=1}^{\infty} \frac{1}{b_n^2 \varepsilon^2} E\left(\max_{1\leqslant k\leqslant n} \left| \sum_{i=1}^{k} A_{ni} X_{ni} \right|^2 \right) \\ &\leqslant \sum_{n=1}^{\infty} \frac{C_1 \log^2 n}{b_n^2 \varepsilon^2} \sum_{i=1}^{n} EA_{ni}^2 EX_{ni}^2 < \infty. \end{aligned}$$

So (2.5) holds. Consequently, (2.6) follows from (2.5) immediately. □

Proof of Corollary 2.2. Combining (1.2) with Lemma 3.3, it can be checked that $EA_{ni}^2 \leqslant CEA^2 < \infty$ for all $n \geqslant 1$ and $1 \leqslant i \leqslant n$. By taking $b_n = n^{1/r}$ in (2.7), we apply Theorem 2.2 and obtain the results of (2.8) and (2.9) immediately. □

Proof of Corollary 2.3. In view of (1.3) and Lemma 3.3, we establish that for all $n \geqslant 1$,

$$EA_{ni}^2 \leqslant C_1 EB_i^2, \quad EX_{ni}^2 \leqslant C_2 EY_i^2, \quad 1 \leqslant i \leqslant n.$$

Consequently, by (2.10), it follows

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2/r}} \sum_{i=1}^{n} EA_{ni}^2 EX_{ni}^2 \leqslant C_1 \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2r}} \sum_{i=1}^{n} EB_i^2 EY_i^2 = C_1 \sum_{i=1}^{\infty} EB_i^2 EY_i^2 \sum_{n=i}^{\infty} \frac{\log^2 n}{n^{2r}}$$
$$\leqslant C_2 \sum_{i=1}^{\infty} \frac{EB_i^2 EY_i^2 \log^2 i}{i^{2r-1}} < \infty.$$

Therefore, by (2.10) and Theorem 2.2 with $b_n = n^r$, (2.8) and (2.9) hold true. □

Proof of Corollary 2.4. In view of (1.2) and Lemma 3.3, we establish that $EX_{ni}^2 \leqslant CEX^2 < \infty$ for all $n \geqslant 1$ and $1 \leqslant i \leqslant n$. Then, by (2.11), it can be argued that for some $0 < \delta < 1$ and $r > \frac{1+\delta}{2}$,

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2/r}} \sum_{i=1}^{n} EA_{ni}^2 EX_{ni}^2 \leqslant C_1 \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2r}} \sum_{i=1}^{n} EA_{ni}^2 \leqslant C_2 \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2r-\delta}} < \infty.$$

Thus, by Theorem 2.2 with $b_n = n^{1/r}$, one has (2.8) and (2.9) immediately. □

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