

EXTENSION OF DETERMINANTAL INEQUALITIES OF POSITIVE DEFINITE MATRICES

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Abstract. In this short note, we extend some known determinantal inequalities of positive definite matrices to a larger class of matrices, namely, matrices whose numerical range is contained in a sector.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. For $A \in \mathbb{M}_n(\mathbb{C})$, the conjugate transpose of A is denoted by A^* , and recall the Cartesian decomposition ([8, p. 6]) $A = \Re A + i\Im A$, where $\Re A = \frac{1}{2}(A + A^*)$ and $\Im A = \frac{1}{2i}(A - A^*)$. For two Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, $A \prec B$ ($A \preceq B$) means that $B - A$ is positive definite (semidefinite). In particular, a positive definite (positive semidefinite) matrix A can be expressed as $A \succ 0$ ($A \succeq 0$). We also consider $A \in \mathbb{M}_n(\mathbb{C})$ to be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (1.1)$$

where diagonal blocks are square matrices. $\text{diag}(D_1, \dots, D_k)$ denotes the block diagonal matrix whose diagonal blocks are D_1, \dots, D_k . In (1.1), if A_{11} is nonsingular, the Schur complement of A_{11} in A is defined by $A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

The numerical range of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{x^*Ax \mid x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, let S_α be the sector in the complex plane given by

$$S_\alpha = \{z \in \mathbb{C} \mid \Re z > 0, |\Im z| \preceq (\Re z) \tan \alpha\} = \{re^{i\theta} \mid r > 0, |\theta| \leq \alpha\}.$$

Clearly, if A is positive definite, $W(A) \subset S_0$.

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For fundamentals of numerical range, the readers can refer to [9, 10]. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, A is necessarily nonsingular.

Based on a recent result of Lin [3], Choi [1] proved the following determinantal inequality for positive definite matrices.

THEOREM 1.1. [1] *Let $A_i, i = 1, \dots, m$, be positive definite matrices whose diagonal blocks are n_j -square matrices $A_i^{(j)}$ for $j = 1, \dots, k$. Then*

$$\det\left(\sum_{i=1}^m A_i^{-1}\right) \geq \det\left(\sum_{i=1}^m (A_i^{(1)})^{-1}\right) \cdots \det\left(\sum_{i=1}^m (A_i^{(k)})^{-1}\right). \tag{1.2}$$

In [7], Haynsworth gave the result for the Schur complement of 2×2 block matrices.

THEOREM 1.2. [7] *Suppose $A, B \in \mathbb{M}_n(\mathbb{C})$ are Hermitian matrices, partitioned as in (1.1), $A = (A_{ij}), B = (B_{ij}), i, j = 1, 2$, where A_{11} and B_{11} are square of order m . If $A \succeq 0, B \succeq 0, A_{11} \succ 0, B_{11} \succ 0$, then*

$$\det((A + B)/(A_{11} + B_{11})) \geq \det A / \det A_{11} + \det B / \det B_{11}. \tag{1.3}$$

This result (1.3) has been extended in [4]. In this paper, we extend the results (1.2) and (1.3) to the case of matrices whose numerical ranges are contained in a sector.

2. Main results

The following lemmas are useful for proving the above theorems.

LEMMA 2.1. [2, p. 68] *Let $A \in \mathbb{M}_n(\mathbb{C})$. If $\Re A$ is positive definite, then*

$$\det(\Re A) \leq |\det(A)|. \tag{2.1}$$

LEMMA 2.2. [5, 6] *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subset S_\alpha$. Then A can be decomposed as $A = XZX^*$ for some invertible matrix $X \in \mathbb{M}_n(\mathbb{C})$ and $Z = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$ with $|\theta_j| \leq \alpha$ for all j .*

LEMMA 2.3. [4, Lemma 2.6] *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subset S_\alpha$. Then*

$$\sec^n(\alpha) \det(\Re A) \geq |\det(A)|. \tag{2.2}$$

Proof. By Lemma 2.2, we have $A = XZX^*$ for some invertible matrix $X \in \mathbb{M}_n(\mathbb{C})$ and $Z = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$ with $|\theta_j| \leq \alpha$ for all j . Thus, $|\det(A)| = |\det(XZX^*)|$. After dividing by $|\det X|^2$, the inequality holds: $|\det A|/|\det X|^2 = |\det Z| = 1$. Furthermore, $\sec^n \alpha \det(\Re Z) \geq 1$. Therefore, $\sec^n \alpha \det(\Re Z) \geq |\det A|/|\det X|^2$. \square

LEMMA 2.4. [4, Lemma 2.4] Let $A \in \mathbb{M}_n(\mathbb{C})$ with $\Re A$ positive definite. Then

$$(\Re A)^{-1} \succeq \Re(A^{-1}). \quad (2.3)$$

Now we present some generalizations of Theorem 1.1 and Theorem 1.2. First of all, Theorem 1.1 is extended to the class of matrices whose numerical ranges are contained in a sector as follows:

THEOREM 2.5. Let A_i , $i = 1, \dots, m$, be a sequence of $n \times n$ matrices whose diagonal blocks are n_j -square matrices $A_i^{(j)}$ for $j = 1, \dots, k$. Assume that $W(A_i) \subset S_\alpha$, $i = 1, \dots, m$, $\alpha \in [0, \frac{\pi}{2})$. Then

$$|\det(\sum_{i=1}^m A_i^{-1})| \geq \cos^{3n}(\alpha) |\det(\sum_{i=1}^m (A_i^{(1)})^{-1})| \cdots |\det(\sum_{i=1}^m (A_i^{(k)})^{-1})|. \quad (2.4)$$

Proof. Compute

$$\begin{aligned} |\det(\sum_{i=1}^m A_i^{-1})| &\geq \det(\Re(\sum_{i=1}^m A_i^{-1})) = \det(\Re(A_1^{-1} + \cdots + A_m^{-1})) && \text{(by (2.1))} \\ &= \det \Re((X_1^{-1})^* Z_1^{-1} X_1^{-1} + \cdots + (X_m^{-1})^* Z_m^{-1} X_m^{-1}) && \text{(by Lemma 2.2)} \\ &= \det((X_1^{-1})^* \Re(Z_1^{-1}) X_1^{-1} + \cdots + (X_m^{-1})^* \Re(Z_m^{-1}) X_m^{-1}) \\ &\geq \det(\cos^2(\alpha)(X_1^{-1})^* (\Re Z_1)^{-1} X_1^{-1} + \cdots + \cos^2(\alpha)(X_m^{-1})^* (\Re Z_m)^{-1} X_m^{-1}) \\ &\quad \text{(by } \Re(Z^{-1}) \succeq \cos^2(\alpha)(\Re Z)^{-1}) \\ &= \cos^{2n}(\alpha) \det((X_1^{-1})^* (\Re Z_1)^{-1} X_1^{-1} + \cdots + (X_m^{-1})^* (\Re Z_m)^{-1} X_m^{-1}) \\ &= \cos^{2n}(\alpha) \det((X_1 \Re Z_1 X_1^*)^{-1} + \cdots + (X_m \Re Z_m X_m^*)^{-1}) \\ &\geq \cos^{2n}(\alpha) \det(\sum_{i=1}^m \Re A_i^{(1)})^{-1} \cdots \det(\sum_{i=1}^m \Re A_i^{(k)})^{-1} && \text{(by (1.2))} \\ &\geq \cos^{2n}(\alpha) \det(\sum_{i=1}^m \Re((A_i^{(1)})^{-1})) \cdots \det(\sum_{i=1}^m \Re((A_i^{(k)})^{-1})) && \text{(by (2.3))} \\ &\geq \cos^{3n}(\alpha) |\det(\sum_{i=1}^m (A_i^{(1)})^{-1})| \cdots |\det(\sum_{i=1}^m (A_i^{(k)})^{-1})|, && \text{(by (2.2))} \end{aligned}$$

where X_i ($i = 1, \dots, m$) and Z_i ($i = 1, \dots, m$) correspond to the invertible matrices and diagonal matrices in Lemma 2.2, respectively. \square

REMARK 2. When $\alpha = 0$, our Theorem 2.5 reduces to Theorem 1.1.

Next, Theorem 1.2 is extended to the class of matrices whose numerical ranges are contained in a sector as follows.

THEOREM 2.6. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathbb{M}_n(\mathbb{C})$ with $A_{11}, B_{11} \in \mathbb{M}_m(\mathbb{C})$ and $W(A), W(B) \subset S_\alpha$, $\alpha \in [0, \frac{\pi}{2})$, then

$$\frac{|\det(A+B)|}{|\det(A_{11}+B_{11})|} \geq \cos^{3(n-m)} \alpha \left(\frac{|\det A|}{|\det A_{11}|} + \frac{|\det B|}{|\det B_{11}|} \right). \quad (2.5)$$

Proof. Compute

$$\begin{aligned} \frac{|\det(A+B)|}{|\det(A_{11}+B_{11})|} &\geq \frac{\det \Re(A+B)}{|\det(A_{11}+B_{11})|} && \text{(by (2.1))} \\ &\geq \cos^{2(n-m)} \alpha \det(\Re(A/A_{11}) + \Re(B/B_{11})) && \text{(by [4, Theorem 3.1])} \\ &\geq \cos^{2(n-m)} \alpha (\det(\Re(A/A_{11})) + \det(\Re(B/B_{11}))) \\ &\geq \cos^{3(n-m)} \alpha (|\det(A/A_{11})| + |\det(B/B_{11})|). && \text{(by (2.2)) } \square \end{aligned}$$

REMARK 3. When $\alpha = 0$, our Theorem 2.6 reduces to Haynsworth's result (Theorem 1.2). In Theorem 2.6, if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{M}_{k+1}(\mathbb{C})$ with $A_{11} \in \mathbb{M}_k(\mathbb{C})$, then

$$\sec^3 \alpha \left| \frac{\det(A_{k+1} + B_{k+1})}{\det(A_k + B_k)} \right| \geq \left| \frac{\det A_{k+1}}{\det A_k} \right| + \left| \frac{\det B_{k+1}}{\det B_k} \right|,$$

which is the result in [4, (4.1)]. Thus, our result is a generalization of (4.1) in [4].

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