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SOME COEFFICIENT INEQUALITIES RELATED TO THE HANKEL DETERMINANT FOR STRONGLY STARLIKE FUNCTIONS OF ORDER ALPHA

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Abstract. In the present paper, the estimate of the Hankel determinant

$$H_{3,1}(f) := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

over the class $\mathscr{S}^*_\alpha$, $0 < \alpha \leqslant 1$, of analytic functions f with $a_n := f^{(n)}(0)/n!$, $n \in \mathbb{N} \cup \{0\}$, such that $|\arg(zf'(z)/f(z))| < \alpha\pi/2$ for $z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, is examined.

1. Introduction

Let $\mathscr{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathscr{A}$ be its subclass of f normalized by $f(0) := 0$ and $f'(0) := 1$, so of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Given $n, q \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of a function $f \in \mathscr{A}$ of the form (1.1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix},$$

where $a_1 := 1$. To find the growth of the Hankel determinant $H_{q,n}(f)$ dependent on q and n for the whole class $\mathscr{S} \subset \mathscr{A}$ of univalent functions as well as for its subclasses is one of the main problem to study. For the class $\mathscr{S}$ some important result was shown by Pommerenke [16]. For fixed q and n the growth problem is reduced to find the bound of the Hankel determinant over selected compact subclasses of $\mathscr{A}$. Recently many authors examined the Hankel determinant $H_{2,2}(f)$ of order 2 as well as the Hankel determinant

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$H_{3,1}(f)$ of order 3 (see e.g., [8], [15], [10], [3]). Note that $H_{2,1}(f) = |a_3 - a_2^2|$. Thus the Hankel determinant $H_{2,1}(f)$ of order 2 reduces to the well known coefficient functional which for $\mathcal{S}$ was estimated in 1916 by Bieberbach (see e.g., [7, Vol. I, p. 35]).

Given $\alpha \in (0,1]$, by $\mathcal{S}_\alpha^*$ we denote a subclass of $\mathcal{A}$ of functions f such that

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \alpha\frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{1.2}$$

called *strongly starlike of order* α. The class $\mathcal{S}_\alpha^*$ was independently introduced by Brannan and Kirwan [5] and Stankiewicz [17], [18] (see also [7, Vol. I, pp. 138–139]). Clearly, $\mathcal{S}^* := \mathcal{S}_1^*$ is the class of *starlike functions.*

In this paper we estimate the Hankel determinant $H_{3,1}(f)$ over the class $\mathcal{S}_\alpha^*$.

Let $\mathcal{P}$ be the class of Carathéodory functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{1.3}$$

having a positive real part in $\mathbb{D}$. The results below for the class $\mathcal{P}$ will be used in further considerations.

LEMMA 1.1. [6, p. 41] *If $p \in \mathcal{P}$ is of the form* (1.3)*, then*

$$|c_n| \leqslant 2, \quad n \in \mathbb{D}. \tag{1.4}$$

The inequality (1.4) *is sharp and the equality holds for for the function*

$$p(z) = \frac{1+z}{1-z} =: L(z), \quad z \in \mathbb{D}. \tag{1.5}$$

LEMMA 1.2. ([11],[12]) *If $p \in \mathcal{P}$ is of the form* (1.3) *with $c_1 > 0$, then*

$$2c_2 = c_1^2 + \zeta(4 - c_1^2) \tag{1.6}$$

and

$$4c_3 = c_1^3 + 2c_1(4-c_1^2)\zeta - c_1(4-c_1^2)\zeta^2 + 2(4-c_1^2)(1-|\zeta|^2)\eta \tag{1.7}$$

for some ζ and η such that $|\zeta| \leqslant 1$ and $|\eta| \leqslant 1$.

LEMMA 1.3. ([13]) *If $p \in \mathcal{P}$ is of the form* (1.3)*, then*

$$|c_2 - \lambda c_1^2| \leqslant 2, \quad 0 \leqslant \lambda \leqslant 1. \tag{1.8}$$

The inequality (1.8) *is sharp and the equality holds for the function* $p(z) := L(z^2)$, $z \in \mathbb{D}$.

2. Main results

Since for $f \in \mathscr{A}$,

$$|H_{3,1}(f)| \leqslant |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|, \tag{2.1}$$

we will estimate each part on the right side of (2.1).

THEOREM 2.1. *Let* $\alpha \in (0,1]$. *If* $f \in \mathscr{S}_\alpha^*$ *is the form* (1.1)*, then*

$$|a_2a_3 - a_4| \tag{2.2}$$

$$\leqslant \begin{cases} \dfrac{2}{3}\alpha, & 0 < \alpha \leqslant \alpha_0, \\ \dfrac{2}{9}\alpha(2\alpha+1)\sqrt{\dfrac{2(2\alpha+1)}{(1-\alpha)(5\alpha+2)}}, & \alpha_0 \leqslant \alpha \leqslant \dfrac{1}{10}\left(2+\sqrt{34}\right), \\ \dfrac{2}{9}\alpha(10\alpha^2-1), & \dfrac{1}{10}\left(2+\sqrt{34}\right) \leqslant \alpha \leqslant 1, \end{cases}$$

where $\alpha_0 = 0.559376\ldots$ *is the unique root in* $(0,1)$ *of the equation*

$$16\alpha^3 + 69\alpha^2 - 15\alpha - 16 = 0. \tag{2.3}$$

The inequality (2.2) *is sharp and the equality holds: when* $\alpha \in (0, \alpha_0]$ *for the function*

$$f_1(z) := z\exp\left[\int_0^z \frac{(L(u^3))^\alpha - 1}{u}\mathrm{d}u\right], \quad z \in \mathbb{D}; \tag{2.4}$$

when $\alpha \in \left[\left(2+\sqrt{34}\right)/10, 1\right]$ *for the function*

$$f_2(z) := z\exp\left[\int_0^z \frac{(L(u))^\alpha - 1}{u}\mathrm{d}u\right], \quad z \in \mathbb{D}; \tag{2.5}$$

when $\alpha \in \left[\alpha_0, \left(2+\sqrt{34}\right)/10\right]$ *for the function*

$$f_3(z) := z\exp\left[\int_0^z \frac{(K(u))^\alpha - 1}{u}\mathrm{d}u\right], \quad z \in \mathbb{D}, \tag{2.6}$$

where the function L *is defined by* (1.5) *and*

$$K(z) := \frac{1-z^2}{1-t_0z+z^2}, \quad z \in \mathbb{D}, \tag{2.7}$$

with

$$t_0 := \sqrt{\frac{2(2\alpha+1)}{(1-\alpha)(5\alpha+2)}}. \tag{2.8}$$

Proof. Fix $\alpha \in (0,1]$ and let $f \in \mathscr{S}_\alpha^*$ be of the form (1.1). Then by (1.2) we have

$$\frac{zf'(z)}{f(z)} = (p(z))^\alpha, \quad z \in \mathbb{D}, \tag{2.9}$$

for some function $p \in \mathscr{P}$ of the form (1.3). Putting the series (1.1) and (1.3) into (2.9) by equating the coefficients we get

$$a_2 = \alpha c_1, \quad a_3 = \frac{\alpha}{2}\left(c_2 - \frac{1-3\alpha}{2}c_1^2\right) \tag{2.10}$$

and

$$a_4 = \frac{\alpha}{3}\left(c_3 + \frac{5\alpha-2}{2}c_1c_2 + \frac{17\alpha^2-15\alpha+4}{12}c_1^3\right). \tag{2.11}$$

Hence

$$a_2a_3 - a_4 = \frac{1}{36}\alpha\left[12(1-\alpha)c_1c_2 + 2(\alpha+1)(5\alpha-2)c_1^3 - 12c_3\right]. \tag{2.12}$$

Now by using the equalities (1.6) and (1.7) we have

$$|a_2a_3 - a_4| = \frac{1}{36}\alpha\left|(10\alpha^2-1)c_1^3 + (4-c_1^2)(-6\alpha c_1\zeta + 3c_1\zeta^2 - 6(1-|\zeta|^2)\eta)\right|, \tag{2.13}$$

where $|\zeta| \leqslant 1$ and $|\eta| \leqslant 1$. Since the class $\mathscr{S}_\alpha^*$ is invariant under the rotations, by (1.4) we may assume that $c_1 =: t \in [0,2]$. Thus applying the triangle inequality in the right hand side of (2.13) with $x := |\zeta|$ and $y := |\eta|$ we obtain

$$|a_2a_3 - a_4| \leqslant \frac{1}{36}\alpha T(t,x,y), \tag{2.14}$$

where

$$T(t,x,y) := |10\alpha^2-1|t^3 + (4-t^2)(6\alpha tx + 3tx^2 + 6(1-x^2)y)$$

for $(t,x,y) \in [0,2]\times[0,1]\times[0,1]$. But

$$T(t,x,y) \leqslant T(t,x,1) =: F(t,x), \quad (t,x) \in \Delta := [0,2]\times[0,1],$$

so we will find the maximum of the function F on Δ.

(i) On the vertices of Δ we have

$$F(0,0) = 24, \quad F(0,1) = 0, \quad F(2,0) = F(2,1) = 8|10\alpha^2-1|.$$

(ii) On the side $x = 0$ the function F becomes

$$G(t) := |10\alpha^2-1|t^3 + 6(4-t^2), \quad t \in (0,2).$$

For $0 < \alpha \leqslant \sqrt{3/10}$ we have

$$G'(t) = 3t(|10\alpha^2-1|t-4) \leqslant 0, \quad t \in (0,2).$$

Therefore the function G is decreasing and consequently

$$G(t) \leqslant 24, \quad t \in (0,2).$$

For $\sqrt{3/10} < \alpha \leqslant 1$ the function G has a unique critical point in $(0,2)$, namely, a minimum at $t = 4/(10\alpha^2-1)$ since $G''(4/(10\alpha^2-1)) = 12 > 0$. Because $8(10\alpha^2-1) \geqslant 24$ for $2/\sqrt{10} \leqslant \alpha \leqslant 1$ and $8(10\alpha^2-1) \leqslant 24$ for $0 < \alpha \leqslant 2/\sqrt{10}$, we see that for all $t \in (0,2)$,

$$G(t) \leqslant \begin{cases} 24, & 0 < \alpha \leqslant 2/\sqrt{10}, \\ 8(10\alpha^2-1), & 2/\sqrt{10} \leqslant \alpha \leqslant 1. \end{cases} \tag{2.15}$$

(iii) On the side $x = 1$ the function F becomes

$$H(t) := |10\alpha^2-1|t^3 + t(4-t^2)(6\alpha+3), \quad t \in (0,2). \tag{2.16}$$

For $(2+\sqrt{34})/10 \leqslant \alpha \leqslant 1$ the function H is increasing since

$$H'(t) = 6(\alpha-1)(5\alpha+2)t^2 + 12(2\alpha+1) \geqslant 0, \quad t \in (0,2).$$

Thus $H(t) \leqslant 8(10\alpha^2-1)$ for $t \in (0,2)$.

For $1/\sqrt{10} < \alpha < (2+\sqrt{34})/10$ the function H has a unique critical point in $(0,2)$, namely, a maximum at

$$t = \sqrt{\frac{2(2\alpha+1)}{(1-\alpha)(5\alpha+2)}} =: t_0$$

because

$$H''(t_0) = 12(\alpha-1)(5\alpha+2)t_0 < 0.$$

Thus for all $0 < t < 2$,

$$H(t) \leqslant H(t_0) = 8(2\alpha+1)\sqrt{\frac{2(2\alpha+1)}{(1-\alpha)(5\alpha+2)}}.$$

Note now that

$$H(t_0) \leqslant 24, \quad 1/\sqrt{10} < \alpha \leqslant \alpha_0,$$

and

$$H(t_0) > 24, \quad \alpha_0 < \alpha \leqslant (2+\sqrt{34})/10,$$

where α_0 is the unique root in $(0,1)$ of the equation (2.3).

For $0 < \alpha < 1/\sqrt{10}$ the function H has a unique critical point in $(0,2)$, namely, a maximum at

$$t = \sqrt{\frac{2(2\alpha+1)}{5\alpha^2+3\alpha+1}} =: t_1$$

because

$$H''(t_1) = -12(5\alpha^2+3\alpha+1)t_1 < 0.$$

Thus for all $0 < t < 2$,

$$H(t) \leqslant H(t_1) = 8(2\alpha+1)\sqrt{\frac{2(2\alpha+1)}{5\alpha^2+3\alpha+1}}.$$

Note that the inequality

$$H(t_1) \leqslant 24, \quad 0 < \alpha \leqslant 1/\sqrt{10},$$

is equivalent to the inequality

$$16\alpha^3 - 21\alpha^2 - 15\alpha - 7 \leqslant 0, \quad 0 \leqslant \alpha < 1/\sqrt{10},$$

which as easy to see is true.

(iv) We can easily compute that

$$F(2,x) = 8|10\alpha^2-1|, \quad F(0,x) \leqslant 24.$$

(v) It remains to consider the interior of Δ. Solving the equations

$$\frac{\partial F}{\partial t} = 3|10\alpha^2-1|t^2 \tag{2.17}$$

$$-2t(6\alpha t x + (3t-6)x^2+6) + (4-t^2)(6\alpha x + 3x^2) = 0$$

and

$$\frac{\partial F}{\partial x} = 6(4-t^2)[\alpha t + (t-2)x] = 0, \tag{2.18}$$

we get for $\alpha \neq 1/\sqrt{5}$ a unique critical point (t_2, x_2), where

$$x_2 := \frac{2\alpha(1-\alpha^2)}{|10\alpha^2-1|+5\alpha^2-2}$$

and

$$t_2 := \frac{4(1-\alpha^2)}{|10\alpha^2-1|+3\alpha^2}$$

which possible lies in Δ. Since $t_2 \geqslant 2$ for $0 \leqslant \alpha < 1/\sqrt{5}$ and $x_2 < 0$ for $0 < \alpha < 1/2$, so $(t_2, x_2) \in \Delta$ when $\alpha \geqslant 1/2$. As by (2.17) with $t := t_2$ we have $x_2 = \alpha t_2/(2-t_2)$, so hence and from (2.16) we get

$$F(t_2,x_2) = 2(\alpha^2-1)t_2^2 + 24 \leqslant 24$$

for all $\alpha > 1/2$.

At the end observe that for $\alpha = 1/\sqrt{5}$ the system of equations (2.17) and (2.18) has no solution in Δ.

Summarizing all considered cases, we conclude that

$$F(t,x) \leqslant \begin{cases} 24, & 0 < \alpha \leqslant \alpha_0, \\ 8(2\alpha+1)\sqrt{\dfrac{2(2\alpha+1)}{(1-\alpha)(5\alpha+2)}}, & \alpha_0 < \alpha < \dfrac{1}{10}(2+\sqrt{34}), \\ 8(10\alpha^2-1), & \dfrac{1}{10}(2+\sqrt{34}) \leqslant \alpha \leqslant 1 \end{cases}$$

on Δ, which together with (2.14) proves the inequality (2.2).

To show the sharpness for the case $0 < \alpha \leqslant \alpha_0$, set $c_1 := 0$, $\zeta := 0$, $\eta := 1$ into (1.6) and (1.7) which yield $c_2 = 0$ and $c_3 = 2$. Hence and by (2.10) and (2.11), $a_2 = 0$, $a_3 = 0$ and $a_4 = 2\alpha/3$ which holds for the function (2.4) and makes the equality in (2.2). For the case $(2+\sqrt{34})/10 < \alpha \leqslant 1$, set $c_1 := 2$, $\zeta := 0$ and $\eta := 1$ into (1.6) and (1.7) which yield $c_2 = c_3 = 2$. Hence and by (2.10) and (2.11), $a_2 = 2\alpha$, $a_3 = 3\alpha$ and $a_4 = 2\alpha(17\alpha^2+1)/9$ which holds for the function (2.5) and makes the equality in (2.2). For the case $\alpha_0 \leqslant \alpha \leqslant (2+\sqrt{34})/10$ consider the function f_3 given by (2.6). Since the function K given by (2.7) is in $\mathscr{P}$ with $c_1 = t_0$, $c_2 = t_0^2 - 2$ and $c_3 = t_0^3 - 3t_0$, where t_0 is given by (2.8), from (2.12) it follows that

$$\begin{aligned} |a_2a_3 - a_4| &= \frac{1}{36}\alpha\left|12(1-\alpha)t_0(t_0^2-2) + 2(\alpha+1)(5\alpha-2)t_0^3 - 12t_0^3 + 36t_0\right| \\ &= \frac{2}{9}\alpha(2\alpha+1)\sqrt{\frac{2(2\alpha+1)}{(1-\alpha)(5\alpha+2)}}, \end{aligned}$$

which makes the equality in (2.2). □

REMARK 2.2. *For $\alpha := 1$, i.e., for the class $\mathscr{S}^*$ the above theorem reduces to Theorem 2.2 of* [3].

The theorem below can be found in [10] as Corollary 1 Part 4 however the authors did not remark on the extremal function. To complete this paper we reprove it again.

THEOREM 2.3. *Let $\alpha \in (0,1]$. If $f \in \mathscr{S}^*_\alpha$ is the form (1.1), then*

$$|a_2a_4 - a_3^2| \leqslant \alpha^2. \tag{2.19}$$

The inequality (2.19) is sharp and the equality holds for the function

$$f_4(z) := z\exp\left[\int_0^z \frac{(L(u^2))^\alpha - 1}{u}\mathrm{d}u\right], \quad z \in \mathbb{D}, \tag{2.20}$$

where the function L is defined by (1.5).

Proof. Fix $\alpha \in (0,1]$ and let $f \in \mathscr{S}^*_\alpha$ be of the form (1.1). Then by (2.10) and (2.11) we have

$$a_2a_4 - a_3^2 = \frac{1}{144}\alpha^2[(-13\alpha^2+4)c_1^4 + (4-c_1^2)(6\alpha c_1^2\zeta - (36+3c_1^2)\zeta^2 + 24c_1(1-|\zeta|^2)\eta)] \tag{2.21}$$

for $|\zeta| \leqslant 1$ and $|\eta| \leqslant 1$. As in the proof of Theorem 2.1 setting $c_1 = t \in [0,2]$, $x := |\zeta|$ and $y := |\eta|$ from the above we obtain

$$|a_2a_3 - a_4| \leqslant \frac{1}{144}\alpha^2 T(t,x,y), \tag{2.22}$$

where

$$T(t,x,y) := |4-13\alpha^2|t^4 + (4-t^2)(6\alpha t^2 x + (36+3t^2)x^2 + 24t(1-x^2)y),$$

with $(t,x,y) \in [0,2]\times[0,1]\times[0,1]$. But

$$T(t,x,y) \leqslant T(t,x,1) =: F(t,x), \quad (t,x) \in \Delta := [0,2]\times[0,1],$$

so we will find the maximum of the function F on Δ.

(i) On the vertices of Δ we have

$$F(0,0) = 0, \quad F(0,1,1) = 144, \quad F(2,0) = F(2,1) = 16|4-13\alpha^2|.$$

(ii) On the side $x = 1$ the function F becomes

$$G(t) := |4-13\alpha^2|t^4 + (4-t^2)[(6\alpha+3)t^2+36], \quad t \in (0,2). \tag{2.23}$$

We will show that G is decreasing. For $0 < \alpha \leqslant (-3+\sqrt{22})/13$ we have $13\alpha^2 + 6\alpha - 1 < 0$ and $4 - 13\alpha^2 \geqslant 0$. Therefore

$$G'(t) = 4t((-13\alpha^2 - 6\alpha + 1)t^2 + 12(\alpha-1)) \leqslant -16t(13\alpha^2+3\alpha+2) \leqslant 0, \quad t \in (0,2).$$

For $(-3+\sqrt{22})/13 \leqslant \alpha \leqslant 2/\sqrt{13}$ we have $-13\alpha^2 - 3\alpha + 1 \leqslant 0$ and $4 - 13\alpha^2 \geqslant 0$. Therefore

$$G'(t) = 4t((-13\alpha^2 - 3\alpha + 1)t^2 + 12(\alpha-1)) \leqslant 0, \quad t \in (0,2).$$

For $2/\sqrt{13} \leqslant \alpha \leqslant 1$ we have $4 - 13\alpha^2 \leqslant 0$. Therefore

$$G'(t) = 4(\alpha-1)t((13\alpha+7)t^2+48) \leqslant 0, \quad t \in (0,2).$$

Summarizing, for each $\alpha \in (0,1]$,

$$G'(t) \leqslant 0, \quad t \in (0,2),$$

so G is decreasing and consequently

$$G(t) \leqslant 144, \quad t \in (0,2).$$

(iii) On the edge $x = 0$ the function F becomes

$$H(t) = |4-13\alpha^2|t^4 + 24t(4-t^2), \quad t \in (0,2). \tag{2.24}$$

We have

$$H'(t) = 0$$

if and only if

$$|4-13\alpha^2|t^4 = 18t^3 - 24t. \tag{2.25}$$

In case when the above equation has no solution in $(0,2)$ the function H is increasing. Then for $\alpha \in (0,1]$,

$$H(t) \leqslant 16|4-13\alpha^2| \leqslant 144.$$

In case when there exists a solution of the equation (2.24), say $t_0 \in (0,2)$, by using the the equation (2.25) with $t := t_0$ we have

$$H(t_0) = 6t_0(12 - t_0^2) < 144.$$

In consequence

$$H(t) \leqslant 144, \quad t \in (0,2).$$

(iv) We can easily compute that

$$F(2,x) = 8|10\alpha^2 - 1|, \quad F(0,x) \leqslant 144.$$

(v) It remains to consider the interior of Δ. Solving the equations Since $\alpha t^2 + (t-2)(t-6)x > 0$ for $0 < t < 2$ and $0 < x < 1$, we have

$$\frac{\partial F}{\partial x} = 6(4-t^2)(\alpha t^2 + (t-2)(t-6)x) > 0.$$

Thus the function F has has no critical point.

Summarizing all considered cases we conclude that

$$F(t,x) \leqslant 144$$

on Δ, which together with (2.22) proves the inequality (2.19).

To show the sharpness, set $c_1 := 0$, $\zeta := 1$ and $\eta = 1$ into (1.6) and (1.7) which yields $c_2 = 2$ and $c_3 = 0$. Hence and by (2.10) and (2.11), $a_2 = 0$, $a_3 = \alpha$ and $a_4 = 0$, which holds for the function (2.20) and makes the equality in (2.19). □

REMARK 2.4. *For $\alpha := 1$, i.e., for the class $\mathscr{S}^*$ the above theorem reduces to Theorem 3.1 of* [9].

THEOREM 2.5. *Let $\alpha \in (0,1]$. If $f \in \mathscr{S}^*_\alpha$ is the form* (1.1)*, then*

$$|a_3 - a_2^2| \leqslant \alpha. \tag{2.26}$$

The inequality (2.26) *is sharp and the equality holds for the function* (2.20).

Proof. From (2.10) we have

$$|a_3 - a_2^2| \leqslant \frac{1}{2}\alpha \left| c_2 - \frac{1+\alpha}{2} c_1^2 \right|.$$

The inequality (2.26) follows by applying Lemma 1.3 with $\lambda := (1+\alpha)/2 \in (1/2, 1]$.

The sharpness of the inequality (2.26) follows from the sharpness of the inequality (1.8) and the equality holds for the function (2.20). □

REMARK 2.6. *For* $\alpha := 1$, *i.e., for the class* $\mathscr{S}^*$ *the above result is as in the class* $\mathscr{S}$, *so it reduces to the well known theorem due to Bieberbach (1916) (see e.g.,* [7, Vol. I, p. 35]*).*

REMARK 2.7. *Let* $\alpha \in (0,1]$ *and* $f \in \mathscr{S}^*_\alpha$ *be the form* (1.1). *For the second and third coefficients of* f *the sharp estimates were given in* [4]*, namely,*

$$|a_2| \leqslant 2\alpha, \quad |a_3| \leqslant \begin{cases} \alpha, & 0 < \alpha \leqslant 1/3, \\ 3\alpha^2, & 1/3 \leqslant \alpha \leqslant 1. \end{cases}$$

For the fourth coefficient of f *the sharp estimate was found in* [14]*, namely,*

$$|a_4| \leqslant \begin{cases} 2\alpha/3, & 0 < \alpha \leqslant \sqrt{2/17}, \\ 2\alpha(1+17\alpha^2)/9, & \sqrt{2/17} \leqslant \alpha \leqslant 1. \end{cases}$$

For the fifth coefficient of f *the sharp result, however not complete for all* $\alpha \in (0,1]$, *was obtained in* [1]*, namely,*

$$|a_5| \leqslant \begin{cases} \alpha/2, & 228\alpha^4 - 194\alpha^3 + 2\alpha^2 + 39\alpha - 9 \leqslant 0, \\ \alpha^2(7+38\alpha^2)/9, & 76\alpha^3 - 60\alpha^2 + 32\alpha - 9 \geqslant 0. \end{cases}$$

Now from (2.1) *by using the above coefficient estimates with* (2.2)*,* (2.19) *and* (2.26) *we can obtain the bound of the Hankel determinant* $H_{3,1}(f)$*. Clearly, this bound is incomplete and also not sharp.*

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