

## TWO-WEIGHT ENTROPY BOUNDEDNESS OF MULTILINEAR FRACTIONAL TYPE OPERATORS

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(Communicated by L. Liu)

*Abstract.* This paper will be devoted to study the two-weight norm inequalities of the multilinear fractional maximal operator  $\mathcal{M}_\alpha$  and the multilinear fractional integral operator  $\mathcal{I}_\alpha$ . The entropy conditions in the multilinear setting will be introduced and the entropy bounds for  $\mathcal{M}_\alpha$  and  $\mathcal{I}_\alpha$  will be given.

### 1. Introduction

#### 1.1. Background

Let  $M_\alpha$  and  $I_\alpha$  be the fractional maximal operator and fractional integral operator defined by

$$M_\alpha f(x) := \sup_Q |Q|^{\frac{\alpha}{n}} \langle f \rangle_Q \cdot \mathbf{1}_Q(x), \quad I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 \leq \alpha < n.$$

In 1982, Sawyer [22] first showed that  $M_\alpha(\cdot\sigma) : L^p(\sigma) \rightarrow L^q(w)$  holds if and only if  $(w, \sigma)$  satisfies the following testing condition

$$[w, \sigma]_{S_{(p,q)}} := \sup_Q \sigma(Q)^{-\frac{1}{p}} \|\mathbf{1}_Q M_\alpha(\mathbf{1}_Q \sigma)\|_{L^q(w)} < \infty.$$

Subsequently, using the similar testing conditions, Sawyer [23, 24] gave some characterizations of two-weight weak and strong type inequalities of  $I_\alpha$ .

After the works of Sawyer, many works have been done in the characterizations of two weighed boundedness of continuous operators. Among such achievements are the celebrated works of Hytönen [7], Lacey [10, 11], Lacey et al [15], which demonstrated the characterizations of the two weighted  $L^2$  inequality of Hilbert transform in terms of Sawyer type testing conditions and two-weight  $A_2$  condition. Recently, Lacey and Li [12] gave a characterization of two-weight norm inequalities for the classical

*Mathematics subject classification* (2010): 26D10.

*Keywords and phrases:* Two-weight, entropy conditions, Carleson embedding theorem, multilinear fractional integral operator.

The second author was supported partly by NSFC (No. 11471041 and No. 11671039), the Fundamental Research Funds for the Central Universities (No. 2014KJJC10) and NCET-13-0065.

Littlewood-Paley  $g$ -function. Still more recently, Cao, Li and Xue [1] obtained the characterization of two weighted inequalities for the  $g_\lambda^*$ -function with more general fractional type of Poisson kernels. As for the discrete operators, on the one hand, two-weight characterizations of martingale transforms and dyadic shifts were presented by Nazarov et al [20] and Hytönen [8]. The two weighted  $L^p(\sigma) \rightarrow L^q(w)$ -type inequalities of positive dyadic operators were established by Nazarov et al [20] with  $p = q = 2$ , Lacey et al [14] with  $p < q$  and Hytönen [6] with  $p, q \in (1, \infty)$ . On the other hand, in order to study the sufficient conditions for the two weight inequalities of the singular integral operators, Treil and Volberg [26] introduced the entropy conditions. Later on, the entropy conditions were used to obtain the two-weight norm inequalities of intrinsic square functions and fractional maximal and integral operators by Lacey, Li [13] and Rahm, Spencer [21], respectively.

In the multilinear setting, several works also have already been done for the multilinear fractional maximal operator  $\mathcal{M}_\alpha$  and fractional integral operators  $\mathcal{I}_\alpha$  ( $0 \leq \alpha < mn$ ), which are defined by

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_Q |Q|^{\frac{\alpha}{n}} \prod_{i=1}^m \langle |f_i| \rangle_Q \cdot \mathbf{1}_Q(x), \quad \mathcal{I}_\alpha(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^m f_i(x - y_i)}{|(y_1, \dots, y_m)|^{mn-\alpha}} d\vec{y}.$$

In 2013, Chen and Damián [4] first gave some sufficient conditions for the two-weight inequalities of the multilinear maximal operator  $\mathcal{M}_0$ . In 2015, Li and Sun [18] considered the problem of two weighted inequalities of multilinear fractional maximal operator  $\mathcal{M}_\alpha$ . But it is worth pointing out that their method is not valid for the case  $0 \leq \alpha < n(1/p - 1/\max\{p_i\})$ . In 2016, Cao and Xue [2] extended the ranges of exponents to  $0 \leq \alpha < mn$  by applying the atomic decomposition of tent space. Moreover, Cao, Xue and Yabuta [3] defined and studied the multilinear fractional strong maximal operator and the corresponding multiple weights associated with rectangles. Under the dyadic reverse doubling condition, a necessary and sufficient condition for two-weight inequalities of the multilinear fractional strong maximal operator was given.

It is well known that it is difficult to give a two-weight characterization of  $\mathcal{M}_\alpha$  and  $\mathcal{I}_\alpha$  with respect to Sawyer-type testing condition. Even if we make it, it is generally very hard to verify Sawyer-type testing condition in practice. This leads us to quest some sufficient conditions for two-weight norm inequalities of  $\mathcal{M}_\alpha$  and  $\mathcal{I}_\alpha$ . This kind of conditions should mainly concerned with  $A_p$  like conditions.

In this paper, we are mainly concerned with  $A_p$  like conditions that are sufficient for two-weight norm inequalities of  $\mathcal{M}_\alpha$  and  $\mathcal{I}_\alpha$ . We will work with the multiple version of entropy conditions and try to obtain the entropy bounds of  $\mathcal{M}_\alpha$  and  $\mathcal{I}_\alpha$ . For simplicity, we only give the results and the proofs in the case  $m=2$ , although our results still hold for general  $m \geq 2$ .

### 1.2. Main results

First, we give one definition related to multiple weights.

DEFINITION 1.1. (Multiple weights class) Let  $0 \leq \alpha < mn$ ,  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$  with  $1 < p_1, \dots, p_m < \infty$ , and  $0 < p \leq q < \infty$ . Let  $w, \sigma_i$  ( $i = 1, \dots, m$ ) be nonnegative

and locally integrable functions on  $\mathbb{R}^n$ , and  $v_{\vec{\sigma}} = \prod_{i=1}^m \sigma_i^{p/p_i}$ . We define

$$\begin{aligned} [w, \vec{\sigma}]_{A_{(\vec{p},q)}} &:= \sup_Q \mathcal{A}_{\vec{p},q}(w, \vec{\sigma}; Q) < \infty, \\ [w, \vec{\sigma}]_{A_{(\vec{p},q)A_\infty^{\text{exp}}}} &:= \sup_Q \mathcal{A}_{\vec{p},q}(w, \vec{\sigma}; Q) A_\infty^{\text{exp}}(v_{\vec{\sigma}}; Q)^{\frac{1}{p}} < \infty, \\ [w, \vec{\sigma}]_{A_{(\vec{p},q)H_\vec{p}^\infty}} &:= \sup_Q \mathcal{A}_{\vec{p},q}(w, \vec{\sigma}; Q) \prod_{i=1}^m A_\infty^{\text{exp}}(\sigma_i; Q)^{\frac{1}{p_i}} < \infty, \end{aligned}$$

where

$$A_\infty^{\text{exp}}(w; Q) := \langle w \rangle_Q \exp(\langle \log w^{-1} \rangle_Q), \quad \mathcal{A}_{\vec{p},q}(w, \vec{\sigma}; Q) := |Q|^{\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}} \langle w \rangle_Q^{\frac{1}{q}} \prod_{i=1}^m \langle \sigma_i \rangle_Q^{\frac{1}{p_i}}.$$

REMARK 1.2. If we denote

$$[\vec{\sigma}]_{H_\vec{p}^\infty} := \sup_Q \prod_{i=1}^m A_\infty^{\text{exp}}(\sigma_i; Q)^{\frac{p}{p_i}}, \quad [\vec{\sigma}]_{RH_\vec{p}} := \sup_Q v_{\vec{\sigma}}(Q)^{-1} \prod_{i=1}^m \sigma_i(Q)^{\frac{p}{p_i}},$$

it is easy to check that

$$\begin{aligned} [\vec{\sigma}]_{H_\vec{p}^\infty} &\leq [\vec{\sigma}]_{RH_\vec{p}} [v_{\vec{\sigma}}]_{A_\infty^{\text{exp}}}, \quad [w, \vec{\sigma}]_{A_{(\vec{p},q)A_\infty^{\text{exp}}}} \leq [w, \vec{\sigma}]_{A_{(\vec{p},q)}} [v_{\vec{\sigma}}]_{A_\infty^{\text{exp}}}^{1/p}, \\ [w, \vec{\sigma}]_{A_{(\vec{p},q)A_\infty^{\text{exp}}}} &\leq [w, \vec{\sigma}]_{A_{(\vec{p},q)H_\vec{p}^\infty}} \leq [\vec{\sigma}]_{RH_\vec{p}}^{1/p} [w, \vec{\sigma}]_{A_{(\vec{p},q)A_\infty^{\text{exp}}}}. \end{aligned}$$

Now, we give the definition of multilinear version of entropy conditions.

DEFINITION 1.3. (Multilinear version of entropy conditions) Let  $0 \leq \alpha < mn$ ,  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$  with  $1 < p_1, \dots, p_m < \infty$ , and  $0 < p \leq q < \infty$ . Let  $w, \sigma_i$  ( $i = 1, \dots, m$ ) be nonnegative and locally integrable functions on  $\mathbb{R}^n$ . We define

$$\begin{aligned} [w, \vec{\sigma}]_{\vec{p},q,\varepsilon} &:= \sup_Q \mathcal{A}_{\vec{p},q}(w, \vec{\sigma}; Q) \rho_{v_{\vec{\sigma}}}(Q)^{\frac{1}{p}} \varepsilon(\rho_{v_{\vec{\sigma}}}(Q)) \\ [w, \vec{\sigma}]_{\vec{p},q,\varepsilon,\eta} &:= \sup_Q \mathcal{A}_{\vec{p},q}(w, \vec{\sigma}; Q) \rho_{w,\eta}(Q)^{\frac{1}{q}} \prod_{i=1}^m \rho_{\sigma_i,\varepsilon_i}(Q)^{\frac{1}{p_i}} \\ [[\vec{\sigma}]]_{(i,j,k),\varepsilon_i} &:= \sup_Q \left( |Q|^{\frac{\alpha}{n}} \prod_{i=1}^2 \langle \sigma_i \rangle_Q \right)^{\frac{p'_k}{p_{ij}}} \cdot \langle \sigma_3 \rangle_Q \gamma_{(i,j,k)}(Q) \varepsilon_i(\gamma_{(i,j,k)}(Q)). \end{aligned}$$

where  $\varepsilon, \eta, \varepsilon_i$  are monotonic increasing functions on  $(1, \infty)$ , and

$$\begin{aligned} \rho_w(Q) &:= w(Q)^{-1} \int_Q M(\mathbf{1}_Q w)(x) dx \quad \text{and} \quad \rho_{w,\varepsilon}(Q) = \rho_w(Q) \varepsilon(\rho_w(Q)), \\ \gamma_{(i,j,k)}(Q) &:= \frac{\int_Q \mathcal{M}_\alpha(\mathbf{1}_Q \sigma_i, \mathbf{1}_Q \sigma_j)(x)^{\frac{p'_k}{p_{ij}}} dx}{\left( \int_Q \sigma_i^{\frac{p_{ij}}{p_i}} \sigma_j^{\frac{p_{ij}}{p_j}} dx \right)^{\frac{p'_k}{p_{ij}}}} \quad \text{and} \quad \frac{1}{p_{ij}} = \frac{1}{p_i} + \frac{1}{p_j}. \end{aligned}$$

We now state the main results of this paper as follows.

**THEOREM 1.1.** *Let  $0 \leq \alpha < 2n$ ,  $0 < p \leq q < \infty$  and  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$  with  $1 < p_1, p_2 < \infty$ . Suppose that  $\sigma_1, \sigma_2, w$  are weights on  $\mathbb{R}^n$ . Let  $\varepsilon$  be a monotonic increasing function on  $(1, \infty)$  that satisfies  $\int_1^\infty \frac{dt}{t\varepsilon(t)^q} < \infty$ . Then the following inequality holds*

$$\|\mathcal{M}_\alpha(f_1\sigma_1, f_2\sigma_2)\|_{L^q(w)} \lesssim \lceil w, \vec{\sigma} \rceil_{\vec{p}, q, \varepsilon} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}.$$

**THEOREM 1.2.** *Let  $0 \leq \alpha < 2n$ ,  $0 < p \leq q < \infty$  and  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$  with  $1 < p_1, p_2, q < \infty$ . Suppose that  $\sigma_1, \sigma_2, w$  are weights on  $\mathbb{R}^n$ . Let  $\varepsilon_1, \varepsilon_2, \eta$  be monotonic increasing functions on  $(1, \infty)$  that satisfy  $\int_1^\infty \frac{dt}{t\varepsilon_i(t)^{p_i}} < \infty$  and  $\int_1^\infty \frac{dt}{t\eta(t)^q} < \infty$ . Then there holds that*

$$\|\mathcal{I}_\alpha(f_1\sigma_1, f_2\sigma_2)\|_{L^q(w)} \lesssim \lfloor w, \vec{\sigma} \rfloor_{\vec{p}, q, \varepsilon, \eta} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}.$$

**THEOREM 1.3.** *Let  $0 \leq \alpha < 2n$ , and  $1 < p_i < \infty$  ( $i = 1, 2, 3$ ) with  $\frac{1}{p_i} + \frac{1}{p_j} \geq 1$  for  $i \neq j$ . Suppose that  $\sigma_1, \sigma_2, \sigma_3$  are weights on  $\mathbb{R}^n$ . Let  $\varepsilon_i$  be a monotonic increasing function on  $(1, \infty)$  such that  $\int_1^\infty \frac{dt}{t\varepsilon_i(t)^{1/p_i}} < \infty$ ,  $i = 1, 2, 3$ . Then the following inequalities hold*

$$\|\mathcal{I}_\alpha(f_1\sigma_1, f_2\sigma_2)\|_{L^{p'_3}(\sigma_3)} \lesssim \sum_{(i,j,k) \in \Omega} [[\vec{\sigma}]]_{(i,j,k), \varepsilon_i}^{1/p'_k} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}; \tag{1.1}$$

$$\|\mathcal{I}_\alpha(f_1\sigma_1, f_2\sigma_2)\|_{L^{p'_3, \infty}(\sigma_3)} \lesssim \sum_{\substack{i \neq 3 \\ (i,j,k) \in \Omega}} [[\vec{\sigma}]]_{(i,j,k), \varepsilon_i}^{1/p'_k} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}; \tag{1.2}$$

where  $\Omega$  is the set of all permutations of  $(1, 2, 3)$ .

The article is organized as follows: In Section 2, some notations and lemmas will be given. In Section 3, we will demonstrate Theorem 1.1 and Theorem 1.2. Section 4 will be devoted to complete the proofs of Theorem 1.3.

### 2. Preliminaries

First, we present some definitions and lemmas, which will be used later.

**DEFINITION 2.1.** A collection,  $\mathcal{D}$  of cubes is said to be a dyadic grid if it satisfies

- (1) The side length of every  $Q \in \mathcal{D}$  equals  $2^k$  for some  $k \in \mathbb{Z}$ .
- (2) For any  $Q, R \in \mathcal{D}$ ,  $Q \cap R = \{Q, R, \emptyset\}$ .

$$(3) \mathbb{R}^n = \bigcup_{Q \in \mathcal{D}_k} Q, \mathcal{D}_k = \{Q \in \mathcal{D}; \ell(Q) = 2^k\} \text{ for any } k \in \mathbb{Z}.$$

DEFINITION 2.2. A subset  $\mathcal{S}$  of a dyadic grid is said to be sparse, if for every  $Q \in \mathcal{S}$  there holds that

$$\left| \bigcup_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q}} Q' \right| \leq \frac{1}{2}|Q|.$$

Equivalently, if  $E(Q) = Q \setminus \bigcup_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q}} Q'$ , then the sets  $\{E(Q)\}_{Q \in \mathcal{S}}$  are pairwise disjoint and  $|Q| \leq 2|E(Q)|$ .

DEFINITION 2.3. Let  $0 \leq \alpha < mn$  and  $\mathcal{D}, \mathcal{S}$  be a given dyadic grid and a sparse set. The dyadic versions of multilinear fractional maximal and fractional integral operators are defined by

$$\begin{aligned} \mathcal{M}_\alpha^\mathcal{D}(\vec{f})(x) &:= \sup_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}} \prod_{i=1}^m \langle |f_i| \rangle_Q \cdot \mathbf{1}_Q(x), \\ \mathcal{I}_\alpha^\mathcal{D}(\vec{f})(x) &:= \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}} \prod_{i=1}^m \langle f_i \rangle_Q \cdot \mathbf{1}_Q(x), \\ \mathcal{I}_{\mathcal{S}, \alpha}(\vec{f})(x) &:= \sum_{Q \in \mathcal{S}} |Q|^{\frac{\alpha}{n}} \prod_{i=1}^m \langle f_i \rangle_Q \cdot \mathbf{1}_{E(Q)}(x). \end{aligned}$$

We will need the following lemma given by Hytönen and Pérez in [9].

LEMMA 2.1. *There are  $2^n$  dyadic grids  $\mathcal{D}_t$ ,  $t \in \{0, 1/3\}^n$  such that for any cube  $Q \subset \mathbb{R}^n$  there exists a cube  $Q_t \in \mathcal{D}_t$  satisfying  $Q \subset Q_t$  and  $\ell(Q_t) \leq 6\ell(Q)$ , where the dyadic grid  $\mathcal{D}_t$  is defined by*

$$\mathcal{D}_t := \{2^{-k}([0, 1]^n + m + (-1)^k t) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad t \in \{0, 1/3\}^n.$$

We also need the following lemma.

LEMMA 2.2. *Let  $\mathcal{D}$  be a dyadic grid. For any non-negative integrable  $f_i$  ( $i = 1, \dots, m$ ), there exist sparse families  $\mathcal{S} \subset \mathcal{D}$  such that for all  $x \in \mathbb{R}^n$ , it holds that*

$$\mathcal{M}_\alpha(\vec{f})(x) \simeq \sup_{t \in \{0, 1/3\}^n} \mathcal{M}_{\alpha}^{\mathcal{D}_t}(\vec{f})(x), \quad \mathcal{M}_\alpha^\mathcal{D}(\vec{f})(x) \simeq \mathcal{I}_{\mathcal{S}, \alpha}(\vec{f})(x); \tag{2.1}$$

$$\mathcal{I}_\alpha(\vec{f})(x) \simeq \sup_{t \in \{0, 1/3\}^n} \mathcal{I}_\alpha^{\mathcal{D}_t}(\vec{f})(x), \quad \mathcal{I}_\alpha^\mathcal{D}(\vec{f})(x) \simeq \mathcal{I}_{\mathcal{S}, \alpha}(\vec{f})(x). \tag{2.2}$$

The proof of (2.1) can be found in [17] and (2.2) was shown in [19].

We will apply the following multilinear version of Carleson embedding theorem [25] at certain key points in the proofs of our results.

LEMMA 2.3. (Carleson embedding theorem) *Let  $0 < p \leq q < \infty$  and  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$  satisfying  $1 < p_1, \dots, p_m < \infty$ . Suppose that the nonnegative numbers  $\{c_Q\}_Q$  satisfy*

$$\sum_{Q \subset Q'} c_Q \leq A v_{\vec{\sigma}}(Q')^{q/p}, \text{ for any } Q' \in \mathcal{D},$$

where  $\sigma_i$  ( $i = 1, \dots, m$ ) are weights and  $v_{\vec{\sigma}} = \prod_{i=1}^m \sigma_i^{p/p_i}$ . Then for all nonnegative functions  $f_i \in L^{p_i}(\sigma_i)$ , we have

$$\sum_{Q \in \mathcal{D}} c_Q \prod_{i=1}^m (\langle f_i \rangle_Q^{\sigma_i})^q \lesssim A \|\mathcal{M}_{\vec{\sigma}}^d(\vec{f})\|_{L^{p,q}(v_{\vec{\sigma}})}^q \lesssim A \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}^q,$$

where  $L^{p,q}(w)$  is the Lorentz space defined by

$$\|f\|_{L^{p,q}(w)} = \left[ \int_0^\infty \left( \lambda w(\{x \in \mathbb{R}^n; |f(x)| > \lambda\})^{1/p} \right)^q \frac{d\lambda}{\lambda} \right]^{1/q} < \infty.$$

### 3. Proofs of Theorems 1.1–1.2

In this section, our aim is to demonstrate Theorem 1.1 and Theorem 1.2 by making use of dyadic techniques (see for examples, [8] and [16]).

#### 3.1. Proof of Theorem 1.1

Let  $\mathcal{S}$  be any sparse set of  $\mathcal{D}$ . By Lemma 2.2, it suffices to show that

$$\|\mathcal{T}_{\mathcal{S},\alpha}(f_1 \sigma_1, f_2 \sigma_2)\|_{L^q(w)} \lesssim [w, \vec{\sigma}]_{\vec{p},q,\varepsilon} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}. \tag{3.1}$$

We may assume that each  $f_i$  is a non-negative function for  $i = 1, 2$ . Denote

$$\mathcal{S}_k := \left\{ Q \in \mathcal{S}; 2^{-k} [w, \vec{\sigma}]_{\vec{p},q,\varepsilon} \leq \Gamma(Q) \leq 2^{-k+1} [w, \vec{\sigma}]_{\vec{p},q,\varepsilon} \right\},$$

where  $\Gamma(Q) := |Q|^{\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}} \langle w \rangle_Q^{\frac{1}{q}} \prod_{i=1}^2 \langle \sigma_i \rangle_Q^{\frac{1}{p_i}} \cdot \rho_{v_{\vec{\sigma}}}(Q)^{\frac{1}{p}} \varepsilon(\rho_{v_{\vec{\sigma}}}(Q))$ . Using the pairwise disjointness of the sets  $\{E(Q)\}_{Q \in \mathcal{S}}$ , we deduce that

$$\|\mathcal{T}_{\mathcal{S},\alpha}(f_1 \sigma_1, f_2 \sigma_2)\|_{L^q(w)}^q = \sum_{k=1}^\infty \sum_{Q \in \mathcal{S}_k} \left( |Q|^{\frac{\alpha}{n}} \prod_{i=1}^2 \langle f_i \sigma_i \rangle_Q \right)^q w(E(Q)) := \sum_{k=1}^\infty \Delta_k.$$

To obtain the bound of  $\Delta_k$ , we need to introduce the notion

$$c_Q = (|Q|^{\alpha/n} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q)^q w(E(Q)).$$

Then, it is easy to see that

$$\Delta_k = \sum_{Q \in \mathcal{S}_k} c_Q \left( \langle f_1 \rangle_Q^{\sigma_1} \langle f_2 \rangle_Q^{\sigma_2} \right)^q.$$

In order to apply the Carleson embedding theorem, we need to analyze  $\{c_Q\}_{Q \in \mathcal{S}_k}$ . Fix  $Q' \in \mathcal{S}_k$ . Since  $\Gamma(Q) \simeq 2^{-k} [w, \vec{\sigma}]_{\vec{p}, q, \varepsilon}$  for each  $Q \in \mathcal{S}_k$ , we get

$$\begin{aligned} \sum_{Q \in \mathcal{S}_k: Q \subset Q'} c_Q &\leq \sum_{Q \in \mathcal{S}_k: Q \subset Q'} \Gamma(Q)^q \left( \frac{\sigma_1(Q)^{\frac{1}{p_1}} \sigma_2(Q)^{\frac{1}{p_2}}}{\rho_{v_{\vec{\sigma}}}(Q)^{\frac{1}{p}} \varepsilon(\rho_{v_{\vec{\sigma}}}(Q))} \right)^q \\ &\lesssim 2^{-kq} [w, \vec{\sigma}]_{\vec{p}, q, \varepsilon}^q \sum_{Q \in \mathcal{S}_k: Q \subset Q'} \frac{v_{\vec{\sigma}}(Q)^{q/p}}{\rho_{v_{\vec{\sigma}}}(Q)^{q/p} \varepsilon(\rho_{v_{\vec{\sigma}}}(Q))^q} \\ &:= 2^{-kq} [w, \vec{\sigma}]_{\vec{p}, q, \varepsilon}^q \Delta'_k. \end{aligned}$$

Now, we tentatively claim that

$$\Delta'_k \lesssim v_{\vec{\sigma}}(Q')^{q/p}. \tag{3.2}$$

Therefore, if the above claim is true, we actually obtain that

$$\sum_{Q \in \mathcal{S}_k: Q \subset Q'} c_Q \lesssim 2^{-kq} [w, \vec{\sigma}]_{\vec{p}, q, \varepsilon}^q v_{\vec{\sigma}}(Q')^{q/p},$$

and

$$\Delta_k \lesssim 2^{-kq} [w, \vec{\sigma}]_{\vec{p}, q, \varepsilon}^q \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}^q,$$

where we have used Lemma 2.3. Consequently, it yields that

$$\|\mathcal{I}_{\mathcal{S}, \alpha}(\vec{f} \cdot \vec{\sigma})\|_{L^q(w)} = \sum_{k=1}^{\infty} \Delta_k \lesssim [w, \vec{\sigma}] \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}.$$

This shows that inequality (3.1) is true.

Now, we are in the position to demonstrate (3.2). Set

$$\mathcal{S}_{k,j} := \{Q \in \mathcal{S}_k; Q \subset Q', 2^{j-1} \leq \rho_{v_{\vec{\sigma}}}(Q) < 2^j\},$$

and  $\mathcal{S}_{k,j}^*$  is the collection of maximal elements in  $\mathcal{S}_{k,j}$ . Thereby, we have

$$\begin{aligned} \left( \sum_{Q \in \mathcal{S}_{k,j}} v_{\vec{\sigma}}(Q)^{q/p} \right)^{p/q} &\leq \sum_{Q^* \in \mathcal{S}_{k,j}^*} \sum_{Q \subset Q^*} v_{\vec{\sigma}}(Q) \\ &\lesssim \sum_{Q^* \in \mathcal{S}_{k,j}^*} \sum_{Q \subset Q^*} \int_{E(Q)} \langle \mathbf{1}_{Q^*} v_{\vec{\sigma}} \rangle_Q \mathbf{1}_Q(x) dx \\ &\lesssim \sum_{Q^* \in \mathcal{S}_{k,j}^*} \sum_{Q \subset Q^*} \int_{E(Q)} \sup_{P \in \mathcal{D}} \langle \mathbf{1}_{Q^*} v_{\vec{\sigma}} \rangle_P \mathbf{1}_P(x) dx \\ &\leq \sum_{Q^* \in \mathcal{S}_{k,j}^*} \int_{Q^*} \sup_{P \in \mathcal{D}} \langle \mathbf{1}_{Q^*} v_{\vec{\sigma}} \rangle_P \mathbf{1}_P(x) dx \end{aligned}$$

$$\begin{aligned} &\leq \sum_{Q^* \in \mathcal{S}_{k,j}^*} \int_{Q^*} M(\mathbf{1}_{Q^*} v_{\vec{\sigma}})(x) dx \\ &= \sum_{Q^* \in \mathcal{S}_{k,j}^*} v_{\vec{\sigma}}(Q^*) \rho_{v_{\vec{\sigma}}}(Q^*) \lesssim 2^j v_{\vec{\sigma}}(Q'). \end{aligned}$$

Accordingly, we deduce that

$$\begin{aligned} \Delta'_k &\leq \sum_{j=0}^{\infty} \frac{1}{2^{jq/p} \mathcal{E}(2^j)^q} \sum_{Q \in \mathcal{S}_{k,j}} v_{\vec{\sigma}}(Q)^{q/p} \\ &\lesssim v_{\vec{\sigma}}(Q')^{q/p} \sum_{j=0}^{\infty} \frac{1}{\mathcal{E}(2^j)^q} \\ &\lesssim v_{\vec{\sigma}}(Q')^{q/p} \int_1^{\infty} \frac{dt}{t \mathcal{E}(t)^q} \lesssim v_{\vec{\sigma}}(Q')^{q/p}. \end{aligned}$$

The proof of (3.2) is finished.  $\square$

### 3.2. Proof of Theorem 1.2

By duality, we have

$$\begin{aligned} \|\mathcal{T}_{\mathcal{S},\alpha}(\vec{f} \cdot \vec{\sigma})\|_{L^q(w)} &= \sup_{\|g\|_{L^{q'}(w)} \leq 1} \left| \sum_{Q \in \mathcal{S}} (|Q|^{\frac{\alpha}{n}} \langle f_1 \sigma_1 \rangle_Q \langle f_2 \sigma_2 \rangle_Q) \int_Q g(x) w dx \right| \\ &:= \sup_{\|g\|_{L^{q'}(w)} \leq 1} |\mathcal{V}(g)|. \end{aligned}$$

Denote

$$\mathcal{S}_k = \{Q \in \mathcal{S}; 2^k < \lambda_Q \leq 2^{k+1}\}, \quad \lambda_Q = \mathcal{A}_{\vec{p},q}(w, \vec{\sigma}; Q) \rho_{w,\eta}(Q)^{\frac{1}{q}} \prod_{i=1}^2 \rho_{\sigma_i, \varepsilon_i}(Q)^{\frac{1}{p_i}}.$$

Then, we have  $k \leq K_0 := \log_2 \lfloor w, \vec{\sigma} \rfloor_{\vec{p},q,\vec{\varepsilon},\eta}$ . Therefore, by the Hölder inequality, it now follows that

$$\begin{aligned} \mathcal{V}(g) &= \sum_{k=1}^{K_0} \sum_{Q \in \mathcal{S}_k} \lambda_Q \prod_{i=1}^2 \frac{\langle f_i \rangle_Q^{\sigma_i} \sigma_i(Q)^{\frac{1}{p_i}} \langle g \rangle_Q^w w(Q)^{\frac{1}{q}}}{\rho_{\sigma_i, \varepsilon_i}(Q)^{\frac{1}{p_i}} \rho_{w,\eta}(Q)^{\frac{1}{q}}} \\ &\lesssim \sum_{k=1}^{K_0} 2^k \left( \sum_{Q \in \mathcal{S}_k} \prod_{i=1}^2 \frac{(\langle f_i \rangle_Q^{\sigma_i})^q \sigma_i(Q)^{\frac{q}{p_i}}}{\rho_{\sigma_i, \varepsilon_i}(Q)^{\frac{q}{p_i}}} \right)^{\frac{1}{q}} \left( \sum_{Q \in \mathcal{S}_k} (\langle g \rangle_Q^w)^{q'} \frac{w(Q)}{\rho_{w,\eta}(Q)} \right)^{\frac{1}{q'}} \\ &\lesssim [w, \vec{\sigma}]_{\vec{p},q,\vec{\varepsilon},\eta} \prod_{i=1}^2 \left( \sum_{Q \in \mathcal{S}_k} (\langle f_i \rangle_Q^{\sigma_i})^{p_i} \frac{\sigma_i(Q)}{\rho_{\sigma_i, \varepsilon_i}(Q)} \right)^{\frac{1}{p_i}} \left( \sum_{Q \in \mathcal{S}_k} (\langle g \rangle_Q^w)^{q'} \frac{w(Q)}{\rho_{w,\eta}(Q)} \right)^{\frac{1}{q'}}. \end{aligned}$$

By the Carleson embedding theorem 2.3, it is enough to show that for each  $Q' \in \mathcal{S}_k$

$$\sum_{\substack{Q \in \mathcal{S}_k \\ Q \subset Q'}} \frac{\sigma_i(Q)}{\rho_{\sigma_i, \varepsilon_i}(Q)} \lesssim \sigma_i(Q'), \quad i = 1, 2, 3,$$



where  $\sigma_3 = w$  and  $\varepsilon_3 = \eta$ . A completely analogous calculation to that of the preceding subsection yields the desired result.  $\square$

### 4. Proof of Theorem 1.3

In this section, we shall give the proof of Theorem 1.3. We need the following two-weight characterization of  $\mathcal{I}_\alpha^\mathcal{S}$ , which was proved in [6] and [19].

LEMMA 4.1. *Let  $\mathcal{D}$  be a dyadic grid and  $\mathcal{S} \subset \mathcal{D}$  be a sparse family. Suppose that  $\sigma_1, \sigma_2$  and  $\sigma_3$  are positive Borel measures and  $1 < p_i < \infty$  ( $i = 1, 2, 3$ ) with  $\frac{1}{p_i} + \frac{1}{p_j} \geq 1$  for  $i \neq j$ . Then*

(1) *The strong type inequality*

$$\|\mathcal{I}_\alpha^\mathcal{S}(f_1\sigma_1, f_2\sigma_2)\|_{L^q(\sigma_3)} \leq \mathfrak{N} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}$$

holds if and only if the following test conditions hold for any triple  $(i, j, k) \in \Omega$ ,

$$\mathfrak{T}_{\mathcal{S},(i,j,k)} := \sup_{R \in \mathcal{S}} \frac{\left\| \sum_{\substack{Q \in \mathcal{S} \\ Q \subset R}} |Q|^{\frac{\alpha}{n}} \langle \sigma_j \rangle_Q \langle \sigma_k \rangle_Q \mathbf{1}_Q \right\|_{L^{p'_i}(\sigma_i)}}{\sigma_j(R)^{1/p_j} \sigma_k(R)^{1/p_k}} < \infty.$$

(2) *The weak type inequality*

$$\|\mathcal{I}_\alpha^\mathcal{S}(f_1\sigma_1, f_2\sigma_2)\|_{L^{q,\infty}(\sigma_3)} \leq \mathfrak{N}_{weak} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}$$

holds if and only if  $\mathfrak{T}_{\mathcal{S},(i,j,k)} < \infty$ , for any triple  $(i, j, k) \in \Omega$  and  $i \neq 3$ .

Moreover, the best constants satisfy

$$\mathfrak{N} \simeq \sum_{(i,j,k) \in \Omega} \mathfrak{T}_{\mathcal{S},(i,j,k)}, \quad \mathfrak{N}_{weak} \simeq \sum_{i \neq 3, (i,j,k) \in \Omega} \mathfrak{T}_{\mathcal{S},(i,j,k)}.$$

*Proof of Theorem 1.3.* By Lemma 4.1, it suffices to show

$$\mathfrak{T}_{\mathcal{S},(i,j,k)} \lesssim [[\vec{\sigma}]]_{(i,j,k),\varepsilon_i}, \quad \text{for each } (i, j, k) \in \Omega.$$

By symmetry, we only focus on estimating the case  $(i, j, k) = (1, 2, 3)$ . For convenience, we write  $q = p'_3$ ,  $p = p_{12}$  and  $\gamma = \gamma_{(1,2,3)}$ . From now on, we fix the cube  $R \in \mathcal{D}$  and introduce the notations

$$\begin{aligned} \mathcal{A}(R) &:= \left\| \sum_{Q \in \mathcal{S}} |Q|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \mathbf{1}_Q \right\|_{L^q(R,w)}, \\ \mathcal{B}(Q) &:= \left( |Q|^{\frac{\alpha}{n}} \prod_{i=1}^m \langle \sigma_i \rangle_Q \right)^{\frac{q}{p'}} \cdot \langle w \rangle_Q \gamma(Q) \varepsilon_i(\gamma(Q)). \end{aligned}$$

Then, we make a partition of  $\mathcal{S}$  by setting

$$\mathcal{S}_{a,b} := \{Q \in \mathcal{S}; Q \subset R, 2^a < \mathcal{B}(Q) \leq 2^{a+1}, 2^b < \gamma(Q) \leq 2^{b+1}\}.$$

Note that  $2^a \leq [[\bar{\sigma}]]_{(1,2,3),\varepsilon_1}$ . Now we construct the stopping cubes  $\mathcal{F}$ . Let  $\mathcal{F}$  be the minimal subset of  $\mathcal{S}_{a,b}$  containing the maximal cubes in  $\mathcal{S}_{a,b}$  such that whenever  $F \in \mathcal{F}$ , the maximal cubes  $Q \subset F$ ,  $Q \in \mathcal{S}_{a,b}$  with  $|Q|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q > 4|F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F$  are also in  $\mathcal{F}$ . Denote by  $\pi_{\mathcal{F}}(Q)$  the minimal cube in  $\mathcal{F}$  which contains  $Q$ . Denote

$$\mathcal{S}_{a,b}^k := \left\{ Q \in \mathcal{S}_{a,b}; |Q|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \simeq 2^{-k} |\pi_{\mathcal{F}}(Q)|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_{\pi_{\mathcal{F}}(Q)} \langle \sigma_2 \rangle_{\pi_{\mathcal{F}}(Q)} \right\}.$$

Then Minkowski inequality implies that

$$\mathcal{A}(R) \leq \sum_{a,b} \sum_{k=1}^{\infty} \left\| \sum_{Q \in \mathcal{S}_{a,b}^k} |Q|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \mathbf{1}_Q \right\|_{L^q(w)} := \sum_{a,b} \sum_{k=1}^{\infty} \Theta_{a,b}^k. \tag{4.1}$$

For each  $F \in \mathcal{F}$ , write

$$\Psi_F := \sum_{\substack{Q \in \mathcal{S}_{a,b}^k \\ \pi_{\mathcal{F}}(Q)=F}} |Q|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q, \quad \Psi_{F,j} := \Psi_F \mathbf{1}_{\{|\Psi_F| \simeq j 2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F\}}.$$

Making use of Hölder inequality, we may obtain that

$$\begin{aligned} \Theta_{a,b}^k &\leq \left\| \left( \sum_{j=1}^{\infty} j^{-\frac{2}{q'}} j^{\frac{2}{q}} \sum_{F \in \mathcal{F}} \Psi_{F,j} \right)^q \right\|_{L^1(w)}^{\frac{1}{q}} \\ &\leq \left( \sum_{j=1}^{\infty} j^{-\frac{2}{q'} q'} \right)^{\frac{1}{q}} \left\| \sum_{j=1}^{\infty} j^{\frac{2q}{q'}} \sum_{F \in \mathcal{F}} \Psi_{F,j}^q \right\|_{L^1(w)}^{\frac{1}{q}} \\ &\lesssim \left( \sum_{j=1}^{\infty} j^{2(q-1)} \sum_{F \in \mathcal{F}} \int_{Q_0} \Psi_{F,j}(x)^q w \, dx \right)^{\frac{1}{q}}. \end{aligned} \tag{4.2}$$

Therefore, we are in a position to consider the contribution of the integral in the above inequality. Before doing that, we first claim that the following estimate is true:

$$w(\{x; \Psi_F(x) > \lambda 2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F\}) \lesssim 2^{-\lambda} w(F). \tag{4.3}$$

By (4.3) and noticing the fact that the set  $\{x; \Psi_F(x) > \lambda 2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F\}$  coincides with  $F$  if  $0 < \lambda < j/2$  and is empty if  $\lambda > j$ , it is easy to get that

$$\int_0^{\infty} q \lambda^{q-1} w(\{x; \Psi_F(x) > \lambda 2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F\}) d\lambda \lesssim j^q 2^{-\frac{j}{2}} w(F).$$

Hence, it now follows that

$$\int_{Q_0} \Psi_{F,j}(x)^q w \, dx \lesssim 2^{-kq} |F|^{\frac{\alpha q}{n}} \langle \sigma_1 \rangle_F^q \langle \sigma_2 \rangle_F^q (j^q 2^{-\frac{j}{2}} w(F)). \tag{4.4}$$

Collecting the inequalities (4.2) and (4.4), we have

$$\begin{aligned} \left( \sum_{k=1}^{\infty} \Theta_{a,b}^k \right)^q &\lesssim \left( \sum_{k=1}^{\infty} 2^{-k} \right)^q \sum_{j=1}^{\infty} j^{3q-2} 2^{-\frac{j}{2}} \sum_{F \in \mathcal{F}} |F|^{\frac{\alpha q}{n}} \langle \sigma_1 \rangle_F^q \langle \sigma_2 \rangle_F^q w(F) \\ &\lesssim \frac{2^a}{2^b \varepsilon_1(2^b)} \sum_{F \in \mathcal{F}} \left( |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F \right)^{\frac{q}{p}} |F|. \end{aligned}$$

Let  $\mathcal{F}^*$  be the maximal elements of  $\mathcal{F}$ , then we obtain

$$\begin{aligned} \left( \sum_{k=1}^{\infty} \Theta_{a,b}^k \right)^q &\lesssim \frac{2^a}{2^b \varepsilon_1(2^b)} \sum_{F^* \in \mathcal{F}^*} \sum_{F^* \supset F \in \mathcal{F}} \int_{E(F)} \mathcal{M}_{\alpha}(\mathbf{1}_{F^*} \sigma_1, \mathbf{1}_{F^*} \sigma_2)(x)^{\frac{q}{p}} dx \\ &\leq \frac{2^a}{2^b \varepsilon_1(2^b)} \sum_{F^* \in \mathcal{F}^*} \int_{F^*} \mathcal{M}_{\alpha}(\mathbf{1}_{F^*} \sigma_1, \mathbf{1}_{F^*} \sigma_2)(x)^{\frac{q}{p}} dx \\ &\leq \frac{2^a}{\varepsilon_1(2^b)} \sum_{F^* \in \mathcal{F}^*} v_{\bar{\sigma}}(F^*)^{\frac{q}{p}} \leq \frac{2^a}{\varepsilon_1(2^b)} \left( \sum_{F^* \in \mathcal{F}^*} v_{\bar{\sigma}}(F^*) \right)^{\frac{q}{p}} \\ &\leq \frac{2^a}{\varepsilon_1(2^b)} v_{\bar{\sigma}}(R)^{\frac{q}{p}} \leq \frac{2^a}{\varepsilon_1(2^b)} [\sigma_1(R)^{\frac{1}{p_1}} \sigma_2(R)^{\frac{1}{p_2}}]^q. \end{aligned}$$

Consequently, the equation (4.1) gives that

$$\begin{aligned} \mathcal{A}(R) &\lesssim \sum_{a,b} \frac{2^{a/q}}{\varepsilon_1(2^b)^{1/q}} \sigma_1(R)^{\frac{1}{p_1}} \sigma_2(R)^{\frac{1}{p_2}} \\ &\lesssim [[\bar{\sigma}]]_{(1,2,3),\varepsilon_1}^{1/q} \int_1^{\infty} \frac{dt}{t \varepsilon_1(t)^{1/q}} \sigma_1(R)^{\frac{1}{p_1}} \sigma_2(R)^{\frac{1}{p_2}}. \end{aligned}$$

This shows that

$$\mathfrak{T}_{\mathcal{A},(1,2,3)} \lesssim [[\bar{\sigma}]]_{(1,2,3),\varepsilon_1}^{1/q}.$$

We are left to prove the claim (4.3). If  $w$  is the Lebesgue measure, the inequality is obvious. For any  $Q \in \mathcal{S}_{a,b}^k$  satisfying  $\pi_{\mathcal{F}}(Q) = F$ , it holds that

$$2^a \simeq \mathcal{B}(Q) \simeq (2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F)^{\frac{q}{p'}} \langle w \rangle_Q 2^b \varepsilon_1(2^b).$$

Let  $\mathcal{S}_{a,b}^{k,*}$  be the maximal cubes in  $\mathcal{S}_{a,b}^k$  and

$$\Lambda_F := \frac{2^a}{2^b \varepsilon_1(2^b)} (2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F)^{-\frac{q}{p'}}.$$

Note that the set  $\{x; \Psi_F(x) > \lambda 2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F\}$  is the union of maximal cubes  $P \in \mathcal{S}_{a,b}^k$  with  $\pi_{\mathcal{F}}(P) = F$  and  $\inf_{x \in P} \Psi_F(x) > \lambda 2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F$ . Then, it yields

that

$$\begin{aligned}
 & w(\{x; \Psi_F(x) > \lambda 2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F\}) \\
 & \simeq \Lambda_F |\{x; \Psi_F(x) > \lambda 2^{-k} |F|^{\frac{\alpha}{n}} \langle \sigma_1 \rangle_F \langle \sigma_2 \rangle_F\}| \\
 & \lesssim \Lambda_F 2^{-\lambda} \sum_{Q^* \in \mathcal{S}_{a,b}^{k,*}} |Q^*| \lesssim 2^{-\lambda} \sum_{Q^* \in \mathcal{S}_{a,b}^{k,*}} w(Q^*) \\
 & \lesssim 2^{-\lambda} w(F). \quad \square
 \end{aligned}$$

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(Received March 28, 2016)

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