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A BERNSTEIN TYPE INEQUALITY FOR NOD RANDOM VARIABLES AND APPLICATIONS

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Abstract. The Bernstein inequality is an exponential probability inequality for a sequence of bounded independent random variables. In this paper, we prove a Bernstein type inequality for unbounded negatively orthant dependent (NOD) random variables. As some applications, we obtain the convergence rates of the law of the iterated logarithm and law of the single logarithm for identically distributed NOD random variables. We also obtain a strong law for weighted sums of NOD random variables.

1. Introduction

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ and set $S_n = X_1 + \cdots + X_n, n \geqslant 1$. The exponential inequality for the partial sums $\sum_{i=1}^n (X_i - EX_i)$ is very useful in many probabilistic derivations. In particular, it provides a measure of convergence rate for the strong law of large numbers. There exist several versions available in the literature for independent random variables with assumptions of uniform boundedness or some, quite relaxed, control on their moments.

The following exponential inequality is well known as the Bernstein inequality. Its proof can be found in the literature, see for example Proposition 2.1 in Roussas [15].

THEOREM 1.1. *Let $X_1, \cdots, X_n$ be independent random variables such that $P(|X_i| \leqslant M) = 1$ for $1 \leqslant i \leqslant n$, where M is a positive constant. Set $S_n = \sum_{i=1}^n X_i$ and $v^2 = \sum_{i=1}^n EX_i^2$. Then, for any $\varepsilon > 0$,*

$$P(S_n - ES_n > \varepsilon) \leqslant \exp\left\{-\frac{\varepsilon^2}{2\left(v^2 + \frac{M\varepsilon}{3}\right)}\right\}, \tag{1.1}$$

$$P(|S_n - ES_n| > \varepsilon) \leqslant 2\exp\left\{-\frac{\varepsilon^2}{2\left(v^2 + \frac{M\varepsilon}{3}\right)}\right\}. \tag{1.2}$$

The inequality (1.1) of Theorem 1.1 follows from the following three inequalities:

$$P(S_n - ES_n > \varepsilon) \leqslant \exp(-t\varepsilon)E\exp(t(S_n - ES_n)) \text{ for any } t > 0, \tag{1.3}$$

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$$E\exp(t(S_n - ES_n)) \leqslant \prod_{i=1}^{n} E\exp(t(X_i - EX_i)) \text{ for any } t > 0, \tag{1.4}$$

$$\inf_{t>0} \exp(-t\varepsilon) \prod_{i=1}^{n} E\exp(t(X_i - EX_i)) < \exp\left\{ -\frac{\varepsilon^2}{2\left(v^2 + \frac{M\varepsilon}{3}\right)} \right\}. \tag{1.5}$$

The inequality (1.3) holds by the Markov inequality, (1.4) holds by independence condition, and (1.5) can be proved by expanding $\exp(tX_i)$ according to Taylor's formula. The independence condition is not used in (1.3) and (1.5). Since the inequality (1.1) also holds for $\{-X_i, 1 \leqslant i \leqslant n\}$, (1.2) holds.

In recent years, exponential inequalities, including the Bernstein inequality, for dependent random variables were obtained by many authors.

The concept of negatively associated random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [9]. A finite family of random variables $\{X_i, 1 \leqslant i \leqslant n\}$ is said to be negatively associated if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leqslant 0$$

whenever f_1 and f_2 are coordinatewise increasing (or coordinatewise decreasing) and the covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

Yang [25] obtained a Bernstein type inequality for bounded negatively associated random variables. Christofides and Hadjikyriakou [3], Jabbari et al. [7], and Roussas [15] obtained exponential inequalities for bounded negatively associated random variables. Kim and Kim [10], Nooghabi and Azarnoosh [14], Shao [16], Sung [18], Xing [22], Xing and Yang [23], and Xing et al. [24] obtained exponential inequalities for unbounded negatively associated random variables.

The concept of negatively orthant dependent (NOD) random variables was introduced by Lehmann [12] as follows. A finite family of random variables $\{X_1, \ldots, X_n\}$ is said to be NOD if the following two inequalities hold:

$$P(X_1 \leqslant x_1, \cdots, X_n \leqslant x_n) \leqslant \prod_{i=1}^{n} P(X_i \leqslant x_i)$$

and

$$P(X_1 > x_1, \cdots, X_n > x_n) \leqslant \prod_{i=1}^{n} P(X_i > x_i)$$

for all real numbers $x_1, \cdots, x_n$. An infinite family of random variables is NOD if every finite subfamily is NOD.

Note that negative association implies NOD, but the converse does not hold. The exponential inequalities for NOD are less studied than those for negative association. Sung [19] and Wang et al. [21] proved exponential inequalities for identically distributed NOD random variables with the finite Laplace transforms.

In this paper, we prove a Bernstein type inequality for unbounded NOD random variables. As some applications, we obtain the convergence rates of the law of the iterated logarithm and law of the single logarithm for identically distributed NOD random variables. We also obtain a strong law for weighted sums of NOD random variables.

Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance and $I(A)$ denotes the indicator function of the event A. It proves convenient to define $\log x = \max\{1, \ln x\}$, where $\ln x$ denotes the natural logarithm.

2. A Bernstein type inequality

We will prove a Bernstein type inequality for unbounded NOD random variables. To do this, the following lemmas are needed. The following lemma shows that Theorem 1.1 also holds for NOD random variables.

LEMMA 2.1. *Let $X_1, \cdots, X_n$ be a sequence of NOD random variables such that $P(|X_i| \leqslant M) = 1$ for $1 \leqslant i \leqslant n$, where M is a positive constant. Set $S_n = \sum_{i=1}^n X_i$ and $v^2 = \sum_{i=1}^n EX_i^2$. Then, for any $\varepsilon > 0$, (1.1) and (1.2) hold.*

Proof. Since (1.4) is well known (see, for example, Taylor et al. [20]), (1.1) also holds for NOD random variables. If $\{X_i, 1 \leqslant i \leqslant n\}$ is a sequence of NOD random variables, then $\{-X_i, 1 \leqslant i \leqslant n\}$ is still a sequence of NOD and so (1.1) also holds for $\{-X_i, 1 \leqslant i \leqslant n\}$. Thus, (1.2) also holds for NOD. □

LEMMA 2.2. *Let $X_1, \cdots, X_n$ be a sequence of NOD random variables such that $P(|X_i| \leqslant M) = 1$ for $1 \leqslant i \leqslant n$, where M is a positive constant. Set $S_n = \sum_{i=1}^n X_i$ and $v^2 = \sum_{i=1}^n EX_i^2$. Then, for any $\varepsilon > 0$ and $\gamma > 0$,*

$$P(S_n - ES_n > \varepsilon) \leqslant \exp\left\{-\frac{\varepsilon^2}{(2+\gamma)v^2}\right\} + \exp\left\{-\frac{\varepsilon}{D_\gamma M}\right\},$$

$$P(|S_n - ES_n| > \varepsilon) \leqslant 2\exp\left\{-\frac{\varepsilon^2}{(2+\gamma)v^2}\right\} + 2\exp\left\{-\frac{\varepsilon}{D_\gamma M}\right\},$$

where $D_\gamma = \frac{2}{3}(1 + 2/\gamma)$.

Proof. Observe that for any $\varepsilon > 0$ and $\gamma > 0$,

$$2\left(v^2 + \frac{M\varepsilon}{3}\right) \leqslant \begin{cases} (2+\gamma)v^2, & \text{if } 3\gamma v^2 \geqslant 2M\varepsilon, \\ \left(1 + \frac{2}{\gamma}\right)\frac{2M\varepsilon}{3}, & \text{if } 3\gamma v^2 < 2M\varepsilon. \end{cases}$$

Hence the result follows from Lemma 2.1. □

The following theorem is a Bernstein type inequality for unbounded NOD random variables.

THEOREM 2.1. *Let $X_1,\cdots,X_n$ be NOD random variables such that for some $s \geqslant 2$, $E|X_i|^s < \infty, 1 \leqslant i \leqslant n$. Set $S_n = \sum_{i=1}^n X_i$ and $v^2 = \sum_{i=1}^n EX_i^2$. Then, for any $\varepsilon > 0$ and $\delta > 0$,*

$$P(|S_n - ES_n| > \varepsilon) \leqslant 2\exp\left\{-\frac{\varepsilon^2}{(2+\delta)v^2}\right\} + C\sum_{i=1}^n E|X_i|^s/\varepsilon^s,$$

where C is a positive constant depending only on δ and s.

Proof. The proof is similar to that of Theorem 3.1 in Einmahl and Li [5]. For any $y > 0$, set

$$\beta(y) = \sum_{i=1}^n E|X_i|^s/y^s.$$

Assume first that $\beta(y) < 1$. For fixed $\gamma > 0$ and all $1 \leqslant i \leqslant n$, we let

$$\begin{aligned}
&Y_i = X_i I(|X_i| \leqslant \rho\gamma y) + \rho\gamma y I(X_i > \rho\gamma y) - \rho\gamma y I(X_i < -\rho\gamma y),\\
&Z_i(1) = (X_i - \rho\gamma y)I(\rho\gamma y < X_i \leqslant \gamma y), \quad Z_i(2) = (X_i + \rho\gamma y)I(-\gamma y \leqslant X_i < -\rho\gamma y),\\
&U_i = (X_i - \rho\gamma y)I(X_i > \gamma y) + (X_i + \rho\gamma y)I(X_i < -\gamma y),
\end{aligned}$$

where $\rho = \min\{1, 1/(\gamma D_\gamma \log(1/\beta(y)))\}$ and $D_\gamma = \frac{2}{3}(1+2/\gamma)$. Then $\{Y_i, 1 \leqslant i \leqslant n\}$ is still a sequence of NOD random variables, $\max_{1\leqslant i\leqslant n}|Y_i| \leqslant \rho\gamma y$, and $\sum_{i=1}^n EY_i^2 \leqslant v^2$. By Lemma 2.2, noting $\rho \leqslant 1/(\gamma D_\gamma \log(1/\beta(y)))$, we have

$$\begin{aligned}
P\left(\left|\sum_{i=1}^n (Y_i - EY_i)\right| > y\right) &\leqslant 2\exp\left(-\frac{y^2}{(2+\gamma)v^2}\right) + 2\exp\left(-\frac{y}{D_\gamma \rho\gamma y}\right)\\
&\leqslant 2\exp\left(-\frac{y^2}{(2+\gamma)v^2}\right) + 2\beta(y). \qquad (2.1)
\end{aligned}$$

Since $0 \leqslant Z_i(1) \leqslant \gamma y$, $|\sum_{i=1}^n Z_i(1)| = \sum_{i=1}^n Z_i(1) > 2\gamma y$ implies that there exist at least two indices i such that $Z_i(1) \neq 0$. In this case, $\rho < 1$ (otherwise, $Z_i(1) = 0$ for $1 \leqslant i \leqslant n$), namely, $\rho = 1/(\gamma D_\gamma \log(1/\beta(y)))$ and $\gamma D_\gamma \log(1/\beta(y)) > 1$. It follows by the definition of NOD that

$$\begin{aligned}
P\left(\left|\sum_{i=1}^n Z_i(1)\right| > 2\gamma y\right) &\leqslant \sum_{1\leqslant i_1 < i_2 \leqslant n} P(Z_{i_1}(1) \neq 0, Z_{i_2}(1) \neq 0)\\
&\leqslant \sum_{1\leqslant i_1 < i_2 \leqslant n} P(X_{i_1} > \rho\gamma y, X_{i_2} > \rho\gamma y)\\
&\leqslant \sum_{1\leqslant i_1 < i_2 \leqslant n} P(X_{i_1} > \rho\gamma y)P(X_{i_2} > \rho\gamma y)\\
&\leqslant \left(\sum_{i=1}^n P(|X_i| > \rho\gamma y)\right)^2 \leqslant \left(\frac{\sum_{i=1}^n E|X_i|^s}{(\rho\gamma y)^s}\right)^2\\
&= D_\gamma^{2s}\beta^2(y)(\log(1/\beta(y)))^{2s} \leqslant K_s D_\gamma^{2s}\beta(y), \qquad (2.2)
\end{aligned}$$

where K_s is a positive constant such that $(\log x)^{2s} \leqslant K_s x$ for all $x \geqslant 1$.

Similarly,

$$P\left(\left|\sum_{i=1}^{n} Z_i(2)\right| > 2\gamma y\right) \leqslant K_s D_\gamma^{2s} \beta(y). \tag{2.3}$$

By the Markov inequality,

$$\begin{aligned}
P\left(\left|\sum_{i=1}^{n} U_i\right| > 0\right) &\leqslant \sum_{i=1}^{n} P(|X_i| > \gamma y) \\
&\leqslant (\gamma y)^{-s} \sum_{i=1}^{n} E|X_i|^s = \gamma^{-s}\beta(y).
\end{aligned} \tag{2.4}$$

Combining (2.1)–(2.4), we see that if $\beta(y) < 1$, then

$$\begin{aligned}
&P\left(\left|\sum_{i=1}^{n} (X_i - EY_i)\right| > y + 4\gamma y\right) \\
&\leqslant P\left(\left|\sum_{i=1}^{n} (Y_i - EY_i)\right| > y\right) + P\left(\left|\sum_{i=1}^{n} Z_i(1)\right| > 2\gamma y\right) \\
&\quad + P\left(\left|\sum_{i=1}^{n} Z_i(2)\right| > 2\gamma y\right) + P\left(\left|\sum_{i=1}^{n} U_i\right| > 0\right) \\
&\leqslant 2\exp\left(-\frac{y^2}{(2+\gamma)v^2}\right) + C_{\gamma,s}\beta(y),
\end{aligned} \tag{2.5}$$

where $C_{\gamma,s} = 2 + 2K_s D_\gamma^{2s} + \gamma^{-s}$.

If $\beta(y)/(\gamma^s \rho^{s-1}) \leqslant 1$, then

$$\begin{aligned}
\left|\sum_{i=1}^{n} (X_i - EX_i)\right| &\leqslant \left|\sum_{i=1}^{n} (X_i - EY_i)\right| + \left|\sum_{i=1}^{n} (EZ_i(1) + EZ_i(2) + EU_i)\right| \\
&\leqslant \left|\sum_{i=1}^{n} (X_i - EY_i)\right| + \sum_{i=1}^{n} E|X_i| I(|X_i| > \rho\gamma y) \\
&\leqslant \left|\sum_{i=1}^{n} (X_i - EY_i)\right| + \frac{1}{(\rho\gamma y)^{s-1}} \sum_{i=1}^{n} E|X_i|^s \\
&\leqslant \left|\sum_{i=1}^{n} (X_i - EY_i)\right| + \gamma y.
\end{aligned}$$

If $\rho < 1$, then

$$\frac{\beta(y)}{\gamma^s \rho^{s-1}} \leqslant \frac{\beta(y)}{\gamma^s \rho^s} = D_\gamma^s \beta(y) \left(\log(1/\beta(y))\right)^s \leqslant D_\gamma^s \beta(y) \left(K_s/\beta(y)\right)^{1/2} \leqslant \left(C_{\gamma,s}\beta(y)\right)^{1/2}.$$

If $\rho = 1$, then

$$\frac{\beta(y)}{\gamma^s \rho^{s-1}} = \frac{\beta(y)}{\gamma^s} \leqslant C_{\gamma,s}\beta(y).$$

Thus we have that if $C_{\gamma,s}\beta(y) \leqslant 1$, then $\beta(y)/(\gamma^s\rho^{s-1}) \leqslant 1$ and hence

$$\left|\sum_{i=1}^{n}(X_i - EX_i)\right| \leqslant \left|\sum_{i=1}^{n}(X_i - EY_i)\right| + \gamma y,$$

which implies by (2.5), noting $\beta(y) < 1$ (since $C_{\gamma,s} > 2$), that

$$P\left(\left|\sum_{i=1}^{n}(X_i - EX_i)\right| > y + 5\gamma y\right) \leqslant 2\exp\left(-\frac{y^2}{(2+\gamma)v^2}\right) + C_{\gamma,s}\beta(y). \tag{2.6}$$

The above inequality is trivial if $C_{\gamma,s}\beta(y) > 1$ and therefore (2.6) holds for all $y > 0$ and $\gamma > 0$.

Finally, for any $\varepsilon > 0$ and $\delta > 0$, we can take $y > 0$ and $\gamma > 0$ such that $y(1+5\gamma) = \varepsilon$ and $(2+\gamma)(1+5\gamma)^2 = 2+\delta$. Note that γ depends only on δ. Then

$$P\left(\left|\sum_{i=1}^{n}(X_i - EX_i)\right| > \varepsilon\right) \leqslant 2\exp\left(-\frac{\varepsilon^2}{(2+\delta)v^2}\right) + C_{\gamma,s}(1+5\gamma)^s \sum_{i=1}^{n} E|X_i|^s/\varepsilon^s.$$

Setting $C = C_{\gamma,s}(1+5\gamma)^s$, we obtain the result. □

3. Convergence rates of LIL and LSL

As some applications of Theorem 2.1, we can obtain the convergence rates of the law of the iterated logarithm (LIL) and law of the single logarithm (LSL) for identically distributed NOD random variables.

THEOREM 3.1. *Let $\{X, X_n, n \geqslant 1\}$ be a sequence of identically distributed NOD random variables. Set $S_n = \sum_{i=1}^n X_i$ for $n \geqslant 1$. If $EX = 0$ and $EX^2 < \infty$, then for any $\varepsilon > \sqrt{2EX^2}$,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(|S_n| > \varepsilon\sqrt{n\log\log n}\right) < \infty. \tag{3.1}$$

Proof. For $1 \leqslant i \leqslant n$, $n \geqslant 1$, let

$$X_{ni} = -\sqrt{n}I(X_i < -\sqrt{n}) + X_i I(|X_i| \leqslant \sqrt{n}) + \sqrt{n}I(X_i > \sqrt{n}).$$

Then $\{X_{ni}, 1 \leqslant i \leqslant n\}$ is a sequence of NOD random variables. Set $S_{nk} = \sum_{i=1}^k X_{ni}$ for $1 \leqslant k \leqslant n$ and $n \geqslant 1$. For $\varepsilon > \sqrt{2EX^2}$, let $\varepsilon = \varepsilon_1 + \varepsilon_2$, where $\varepsilon_1 > \sqrt{2EX^2}$ and $\varepsilon_2 > 0$.

In view of $EX = 0$ and $EX^2 < \infty$, we obtain

$$\frac{|ES_{nn}|}{\sqrt{n\log\log n}} = \frac{|\sum_{i=1}^n E(X_i - X_{ni})|}{\sqrt{n\log\log n}} \leqslant \frac{\sum_{i=1}^n E|X_i|I(|X_i| > \sqrt{n})}{\sqrt{n\log\log n}} \leqslant \frac{EX^2 I(|X| > \sqrt{n})}{\sqrt{\log\log n}} \to 0.$$

It follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\left(|S_n| > \varepsilon\sqrt{n\log\log n}\right) \\
&\leqslant \sum_{n=1}^{\infty} n^{-1} P\left(|S_{nn}| > \varepsilon\sqrt{n\log\log n}\right) + \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant i\leqslant n} |X_i| > \sqrt{n}\right) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} P\left(|S_{nn} - ES_{nn}| > \varepsilon_1\sqrt{n\log\log n}\right) + \sum_{n=1}^{\infty} P\left(|X| > \sqrt{n}\right) \\
&:= CI_1 + I_2.
\end{aligned}$$

For I_1, we will use Theorem 2.1. Since $\varepsilon_1 > \sqrt{2EX^2}$, we can take $\delta > 0$ sufficiently small such that $t = \varepsilon_1^2/((2+\delta)EX^2) > 1$. By Theorem 2.1,

$$\begin{aligned}
I_1 &\leqslant 2\sum_{n=1}^{\infty} n^{-1}\exp\left(-\frac{\varepsilon_1^2 n\log\log n}{(2+\delta)\sum_{i=1}^n EX_{ni}^2}\right) + C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n}\frac{E|X_{ni}|^6}{(n\log\log n)^3} \\
&\leqslant 2\sum_{n=1}^{\infty} n^{-1}\exp\left(-\frac{\varepsilon_1^2\log\log n}{(2+\delta)EX^2}\right) + C\sum_{n=1}^{\infty}\frac{E|X|^6 I(|X| \leqslant \sqrt{n})}{n^3} + CI_2 \\
&\leqslant 2\sum_{n=1}^{\infty}(n\log^t n)^{-1} + C\sum_{n=1}^{\infty}\frac{E|X|^6 I(|X| \leqslant \sqrt{n})}{n^3} + CI_2.
\end{aligned}$$

The last inequality holds by the fact that $\exp(-t\log\log n) \leqslant (\log n)^{-t}$, since $\log x = \max\{1, \ln x\}$. The convergence of the last series is equivalent to $EX^2 < \infty$. It is also true that $I_2 < \infty$ is equivalent to $EX^2 < \infty$. Hence the result is proved. □

REMARK 3.1. For a sequence of i.i.d. random variables $\{X, X_n, n \geqslant 1\}$ with $EX = 0$ and $EX^2 < \infty$, Davis [4] proved that (3.1) holds for all $\varepsilon > \sqrt{2EX^2}$, and

$$\sum_{n=1}^{\infty} n^{-1} P(|S_n| > \varepsilon\sqrt{n\log\log n}) = \infty$$

for all $0 < \varepsilon < \sqrt{2EX^2}$. Gut [6] proved the converse part, i.e., if (3.1) holds for some $\varepsilon > 0$, then $EX = 0$ and $EX^2 < \infty$.

THEOREM 3.2. *Let $\{X, X_n, n \geqslant 1\}$ be a sequence of identically distributed NOD random variables. Set $S_n = \sum_{i=1}^n X_i$ for $n \geqslant 1$. If $EX = 0$ and $E[|X|^2/(\log|X|)]^{r+1} < \infty$ for some $r > 0$, then for any $\varepsilon > \sqrt{2rEX^2}$,*

$$\sum_{n=1}^{\infty} n^{r-1} P\left(|S_n| > \varepsilon\sqrt{n\log n}\right) < \infty. \tag{3.2}$$

Proof. For $1 \leqslant i \leqslant n$, $n \geqslant 1$, let

$$X_{ni} = -\sqrt{n\log n} I(X_i < -\sqrt{n\log n}) + X_i I(|X_i| \leqslant \sqrt{n\log n}) + \sqrt{n\log n} I(X_i > \sqrt{n\log n}).$$

Then $\{X_{ni}, 1 \leqslant i \leqslant n\}$ is a sequence of NOD random variables. Set $S_{nk} = \sum_{i=1}^k X_{ni}$ for $1 \leqslant k \leqslant n$ and $n \geqslant 1$. For $\varepsilon > \sqrt{2rEX^2}$, let $\varepsilon = \varepsilon_1 + \varepsilon_2$, where $\varepsilon_1 > \sqrt{2rEX^2}$ and $\varepsilon_2 > 0$. In view of $EX = 0$ and $EX^2 < \infty$, we obtain

$$\begin{aligned} \frac{|ES_{nn}|}{\sqrt{n\log n}} &= \frac{|\sum_{i=1}^n E(X_i - X_{ni})|}{\sqrt{n\log n}} \leqslant \frac{\sum_{i=1}^n E|X_i| I(|X_i| > \sqrt{n\log n})}{\sqrt{n\log n}} \\ &\leqslant \frac{EX^2 I(|X| > \sqrt{n\log n})}{\log n} \to 0. \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-1} P\left(|S_n| > \varepsilon\sqrt{n\log n}\right) \\ &\leqslant \sum_{n=1}^{\infty} n^{r-1} P\left(|S_{nn}| > \varepsilon\sqrt{n\log n}\right) + \sum_{n=1}^{\infty} n^{r-1} \sum_{i=1}^{n} P\left(|X_i| > \sqrt{n\log n}\right) \\ &\leqslant C\sum_{n=1}^{\infty} n^{r-1} P\left(|S_{nn} - ES_{nn}| > \varepsilon_1\sqrt{n\log n}\right) + \sum_{n=1}^{\infty} n^{r} P\left(|X| > \sqrt{n\log n}\right) \\ &:= CJ_1 + J_2. \end{aligned}$$

For J_1, we will use Theorem 2.1. Since $\varepsilon_1 > \sqrt{2rEX^2}$, we can take $\delta > 0$ sufficiently small such that $t = \varepsilon_1^2/((2+\delta)EX^2) > r$. Taking $s > 2(r+1)$, we have by Theorem 2.1 that

$$\begin{aligned} J_1 &\leqslant 2\sum_{n=1}^{\infty} n^{r-1} \exp\left(-\frac{\varepsilon_1^2 n\log n}{(2+\delta)\sum_{i=1}^n EX_{ni}^2}\right) + C\sum_{n=1}^{\infty} n^{r-1} \sum_{i=1}^{n} \frac{E|X_{ni}|^s}{(n\log n)^{s/2}} \\ &\leqslant 2\sum_{n=1}^{\infty} n^{r-1} \exp\left(-\frac{\varepsilon_1^2 \log n}{(2+\delta)EX^2}\right) + C\sum_{n=1}^{\infty} \frac{E|X|^s I(|X| \leqslant \sqrt{n\log n})}{n^{s/2-r}(\log n)^{s/2}} + CJ_2 \\ &\leqslant 2\sum_{n=1}^{\infty} n^{r-t-1} + C\sum_{n=1}^{\infty} \frac{E|X|^s I(|X| \leqslant \sqrt{n\log n})}{n^{s/2-r}(\log n)^{s/2}} + CJ_2. \end{aligned}$$

The convergence of the last series is equivalent to $E[|X|^2/(\log|X|)]^{r+1} < \infty$. It is also true that $J_2 < \infty$ is equivalent to $E[|X|^2/(\log|X|)]^{r+1} < \infty$. Hence the result is proved. □

REMARK 3.2. When $r = 1$, Wang et al. [21] proved that (3.2) holds for $\varepsilon = 3\sqrt{2\alpha e EX^2}$ under the stronger moment condition $E\exp(\alpha|X|) < \infty$ for some $\alpha > 1$. Sung [19] improved the result of Wang et al. [21] by showing that $\alpha > 1$ can be weaken to $\alpha > 1/9$. Hence Theorem 3.2 improves the results of Wang et al. [21] and Sung [19].

REMARK 3.3. For a sequence of i.i.d. random variables $\{X, X_n, n \geqslant 1\}$ with $EX = 0$ and $E[|X|^2/(\log|X|)]^{r+1} < \infty$, Lai [11] proved that (3.2) holds for any $\varepsilon > \sqrt{2rEX^2}$, and Chen et al. [2] pointed out that

$$\sum_{n=1}^{\infty} n^{r-1} P\left(|S_n| > \varepsilon\sqrt{n\log n}\right) = \infty \quad \text{if } 0 < \varepsilon < \sqrt{2rEX^2}.$$

Lai [11] also proved that if (3.2) holds for some $\varepsilon > 0$, then $E[|X|^2/(\log|X|)]^{r+1} < \infty$ and $EX = 0$.

4. A strong law for weighted sums

Many useful linear statistics, including the least squares estimators, nonparametric regression function estimators and jackknife estimators, are of the form of the weighted sums. In this section, we establish a strong law for weighted sums of NOD random variables by using Theorem 2.1.

THEOREM 4.1. *Let $0 < \alpha < 1/2$, $g(x)$ be a positive increasing and regularly varying function at infinity with index α, i.e., for all $\lambda > 0$,*

$$\lim_{x\to\infty} \frac{g(\lambda x)}{g(x)} = \lambda^{\alpha},$$

and let $h(x)$ be the inverse function of $g(x)$. Let $\{X, X_n, n \geqslant 1\}$ be a sequence of identically distributed NOD random variables with $EX = 0$ and $Eh(|X|) < \infty$. Suppose that $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of non-negative constants such that

$$\max_{1\leqslant i\leqslant n} |a_{ni}| \leqslant K g^{-1}(n) \tag{4.1}$$

for some constant $K > 0$ and

$$\limsup_{n\to\infty} \log n \sum_{i=1}^{n} a_{ni}^2 = \rho \tag{4.2}$$

for some constant $0 \leqslant \rho < \infty$. Then

$$\limsup_{n\to\infty} |\sum_{i=1}^{n} a_{ni} X_i| \leqslant \sqrt{2\rho EX^2} \ \text{a.s.} \tag{4.3}$$

Proof. Let $\gamma > 0$ be given. For $1 \leqslant i \leqslant n$, $n \geqslant 1$, let

$$X_{ni}(1) = -n^{-\Delta} I(a_{ni}X_i < -n^{-\Delta}) + a_{ni}X_i I(|a_{ni}X_i| \leqslant n^{-\Delta}) + n^{-\Delta} I(a_{ni}X_i > n^{-\Delta}),$$

$$X_{ni}(2) = (a_{ni}X_i + n^{-\Delta})I(-\gamma/N \leqslant a_{ni}X_i < -n^{-\Delta}),$$
$$X_{ni}(3) = (a_{ni}X_i - n^{-\Delta})I(n^{-\Delta} < a_{ni}X_i \leqslant \gamma/N),$$
$$X_{ni}(4) = (a_{ni}X_i + n^{-\Delta})I(a_{ni}X_i < -\gamma/N) + (a_{ni}X_i - n^{-\Delta})I(a_{ni}X_i > \gamma/N),$$

where $\Delta > 0$ and integer N will be specified below. Then $\{X_{ni}(j), 1 \leqslant i \leqslant n\},\ j = 1,4,$ are all sequences of NOD random variables. Let

$$T_n(j) = \sum_{i=1}^{n} X_{ni}(j), \quad j = 1, \cdots, 4.$$

Since $\gamma > 0$ is arbitrary, to prove (4.3), it is enough to show that

$$\limsup_{n\to\infty} |T_n(1) - ET_n(1)| \leqslant \sqrt{2\rho EX^2} \text{ a.s.,} \tag{4.4}$$
$$ET_n(1) \to 0, \tag{4.5}$$
$$\limsup_{n\to\infty} |T_n(2)| \leqslant \gamma \text{ a.s.,} \tag{4.6}$$
$$\limsup_{n\to\infty} |T_n(3)| \leqslant \gamma \text{ a.s.,} \tag{4.7}$$
$$T_n(4) \to 0 \text{ a.s.} \tag{4.8}$$

We will use Theorem 2.1. For any $\varepsilon > \sqrt{2\rho EX^2}$, we can take $\delta > 0$ sufficiently small such that $\varepsilon^2/((2+\delta)EX^2) > \rho$. Let $\delta' > 1$ be sufficiently close to 1 such that $\varepsilon^2/(\delta'(2+\delta)EX^2) > \rho$. By (4.2), $\log n \sum_{i=1}^{n} a_{ni}^2 \leqslant \varepsilon^2/(\delta'(2+\delta)EX^2)$ for all n large enough. Then, we have by Theorem 2.1 that for any $s \geqslant 2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(|T_n(1) - ET_n(1)| > \varepsilon\right) \\
&\leqslant 2\sum_{n=1}^{\infty} \exp\left\{-\frac{\varepsilon^2}{(2+\delta)\sum_{i=1}^{n} E|X_{ni}(1)|^2}\right\} + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} E|X_{ni}(1)|^s \\
&\leqslant 2\sum_{n=1}^{\infty} \exp\left\{-\frac{\varepsilon^2}{(2+\delta)EX^2\sum_{i=1}^{n} a_{ni}^2}\right\} + C\sum_{n=1}^{\infty} n^{1-s\Delta} \\
&\leqslant C\sum_{n=1}^{\infty} \exp\{-\delta' \log n\} + C\sum_{n=1}^{\infty} n^{1-s\Delta} \leqslant C\sum_{n=1}^{\infty} n^{-\delta'} + C\sum_{n=1}^{\infty} n^{1-s\Delta}.
\end{aligned}$$

If we take $s > \max\{2, 2/\Delta\}$, then the last series converges. Hence, (4.4) holds by the Borel-Cantelli lemma.

Taking $r \in (2, 1/\alpha)$, we get by $EX = 0$, (4.1) and (4.2) that

$$\begin{aligned}
|ET_n(1)| &\leqslant \sum_{i=1}^{n} E|a_{ni}X_i| I(|a_{ni}X_i| > n^{-\Delta}) \leqslant n^{(r-1)\Delta} \sum_{i=1}^{n} E|a_{ni}X_i|^r I(|a_{ni}X_i| > n^{-\Delta}) \\
&\leqslant E|X|^r n^{(r-1)\Delta} \sum_{i=1}^{n} |a_{ni}|^r \leqslant K^{r-2} E|X|^r n^{(r-1)\Delta} (g(n))^{-(r-2)} \sum_{i=1}^{n} a_{ni}^2 \\
&\leqslant Cn^{(r-1)\Delta} (g(n))^{-(r-2)}.
\end{aligned}$$

We take $\Delta > 0$ such that $0 < \Delta < \alpha(r-2)/(r-1)$. Then the function $x^{(r-1)\Delta}(g(x))^{-(r-2)}$ corresponding to $n^{(r-1)\Delta}(g(n))^{-(r-2)}$ is regularly varying at infinity with index $(r-1)\Delta - \alpha(r-2) < 0$. Since the index is negative, we have that $n^{(r-1)\Delta}(g(n))^{-(r-2)} \to 0$ (see, for example, Lemma 2.3 in Zhou [26]). Hence (4.5) holds.

The proofs of (4.6) and (4.7) are similar, and so we only prove (4.7). To prove (4.7), by the Borel-Cantelli lemma, it is enough to show that

$$\sum_{n=1}^{\infty} P(|T_n(3)| > \gamma) < \infty.$$

Since $X_{ni}(3) = (a_{ni}X_i - n^{-\Delta})I(n^{-\Delta} < a_{ni}X_i \leqslant \gamma/N)$, $|T_n(3)| = |\sum_{i=1}^{n} X_{ni}(3)| > \gamma$ implies that there must exist at least N indices i such that $X_{ni}(3) \neq 0$. Hence, we have by Bonferroni's inequality (see, for example, Lemma 4.1.2 in Stout [17]), the definition of NOD, and Markov's inequality that

$$\begin{aligned}
P(|T_n(3)| > \gamma) &\leqslant P(\text{there exist at least } N \text{ indices } i \text{ such that } X_{ni}(3) \neq 0) \\
&\leqslant \sum_{1\leqslant i_1<\cdots<i_N\leqslant n} P(X_{ni_1}(3) \neq 0, \cdots, X_{ni_N}(3) \neq 0) \\
&\leqslant \sum_{1\leqslant i_1<\cdots<i_N\leqslant n} P\left(a_{ni_1}X_{i_1} > n^{-\Delta}, \cdots, a_{ni_N}X_{i_N} > n^{-\Delta}\right) \\
&\leqslant \sum_{1\leqslant i_1<\cdots<i_N\leqslant n} \prod_{j=1}^{N} P\left(a_{ni_j}X_{i_j} > n^{-\Delta}\right) \leqslant \left(\sum_{i=1}^{n} P\left(|a_{ni}X_i| > n^{-\Delta}\right)\right)^N \\
&\leqslant \left(n^{r\Delta}\sum_{i=1}^{n} |a_{ni}|^r E|X|^r\right)^N \leqslant Cn^{r\Delta N}(g(n))^{-N(r-2)},
\end{aligned}$$

where $r \in (2, 1/\alpha)$. Then

$$\sum_{n=1}^{\infty} P(|T_n(3)| > \gamma) \leqslant C\sum_{n=1}^{\infty} n^{r\Delta N}[g(n)]^{-N(r-2)}.$$

We now choose Δ sufficiently small and N sufficiently large such that $N[\alpha(r-2) - r\Delta] > 1$. Then the function $x^{r\Delta N}[g(x)]^{-N(r-2)}$ corresponding to the summands of the last series is regularly varying at infinity with index $-N[\alpha(r-2) - r\Delta] < -1$. Since the index is less than -1, the last series converges (see, for example, Lemma 2.4 in Zhou [26]), and hence the first series also converges.

Finally, we show that (4.8) holds. By $Eh(|X|) < \infty$,

$$\sum_{k=1}^{\infty} P(|X| > (\gamma/NK)g(k)) < \infty,$$

which implies that the series $\sum_{i=1}^{\infty} |X_i|I(|X_i| > (\gamma/NK)g(i))$ converges a.s. by the Borel-Cantelli lemma. Note that, by (4.1),

$$|X_{ni}(4)| \leqslant |a_{ni}X_i|I(|a_{ni}X_i| > \gamma/N) \leqslant Kg^{-1}(n)|X_i|I(|X_i| > (\gamma/NK)g(n)).$$

Hence

$$\begin{aligned}|T_n(4)| &\leqslant Kg^{-1}(n)\sum_{i=1}^{n}|X_i|I(|X_i|>(\gamma/NK)g(n))\\ &\leqslant Kg^{-1}(n)\sum_{i=1}^{n}|X_i|I(|X_i|>(\gamma/NK)g(i))\\ &\leqslant Kg^{-1}(n)\sum_{i=1}^{\infty}|X_i|I(|X_i|>(\gamma/NK)g(i))\to 0 \text{ a.s.}\end{aligned}$$

as $n\to\infty$, i.e., (4.8) holds. □

REMARK 4.1. The non-negative condition on the weights $\{a_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ ensures that $\{a_{ni}X_i, 1\leqslant i\leqslant n\}$ is a sequence of NOD random variables. If the non-negative condition is deleted, we can apply Theorem 4.1 to the weights $\{a_{ni}^{+}, 1\leqslant i\leqslant n, n\geqslant 1\}$ and $\{a_{ni}^{-}, 1\leqslant i\leqslant n, n\geqslant 1\}$, respectively, and hence (4.3) can be replaced by

$$\limsup_{n\to\infty}|\sum_{i=1}^{n}a_{ni}X_i|\leqslant 2\sqrt{2\rho EX^2} \text{ a.s.}$$

Although the upper bound has increased by a factor of 2, it remains the same when $\rho=0$.

REMARK 4.2. When $g(x)=x^{\alpha}$ and $\rho=0$, Li et al. [13] proved Theorem 4.1 for i.i.d. random variables under the stronger condition $\sum_{i=1}^{n}a_{ni}^2=O(n^{-\delta})$, $\delta>0$, than (4.2). Jing and Liang [8] extended the result of Li et al. [13] to negatively associated random variables under condition (4.2) instead of $\sum_{i=1}^{n}a_{ni}^2=O(n^{-\delta})$. Recently, Chen et al. [2] proved Theorem 4.1 for i.i.d. random variables without the non-negative condition on the weights. Hence, Theorem 4.1 extends and improves the corresponding results of Chen et al. [2], Li et al. [13], and Jing and Liang [8].

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