

PROPERTIES OF GENERALIZED SHARP HÖLDER'S INEQUALITIES

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Abstract. Hölder's inequality and its various refinements are playing very important in mathematical analysis. In this paper, we give some new properties of generalized sharp Hölder's inequalities.

1. Introduction

The classical Hölder's inequality states that if $a_k \geq 0$, $b_k \geq 0$ ($k = 1, 2, \dots, n$), $p > 0$, $q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n b_k^q \right)^{\frac{1}{q}}. \quad (1)$$

The sign of inequality is reversed for $p < 0$ (for $p < 0$, we assume that $a_k, b_k > 0$).

As is well know, Hölder's inequality is one of the most important inequalities in mathematical analysis. Various generalizations, improvements, properties and applications of Hölder's inequality have been investigated by many authors. For example, Abramovich, Pečarić and Varošanc [1] gave an interesting refinement of Hölder's inequality. Matkowski [7] presented a new converse of Hölder's inequality. Tian [11] obtained some new improvements of Hölder's inequality by using a property of Hölder's inequality. For detailed expositions, the interested reader may consult [2], [6], [8] and [12] the references therein.

In 1981, Hu [4] established a new sharp version of Hölder's inequality as follows.

THEOREM A. Let $a_k \geq 0$, $b_k \geq 0$ ($k = 1, 2, \dots, n$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$), and let $p \geq q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n b_k^q \right)^{\frac{1}{q} - \frac{1}{p}} \left\{ \left[\left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p \right) \right]^2 - \left[\left(\sum_{k=1}^n b_k^q e_k \right) \left(\sum_{k=1}^n a_k^p \right) - \left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p e_k \right) \right]^2 \right\}^{\frac{1}{2p}}. \quad (2)$$

The integral form is as follows:

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THEOREM B. Let $f(x), g(x), e(x)$ be integrable functions defined on $[a, b]$ and $f(x) \geq 0, g(x) \geq 0, 1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, and let $p \geq q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} & \int_a^b f(x)g(x)dx \\ & \leq \left(\int_a^b g^q(x)dx \right)^{\frac{1}{q}-\frac{1}{p}} \left[\left(\int_a^b g^q(x)dx \int_a^b f^p(x)dx \right)^2 \right. \\ & \quad \left. - \left(\int_a^b g^q(x)e(x)dx \int_a^b f^p(x)dx - \int_a^b g^q(x)dx \int_a^b f^p(x)e(x)dx \right)^2 \right]^{\frac{1}{2p}}. \end{aligned} \tag{3}$$

In 2011, Tian [10] presented the reversed versions of inequalities (2) and (3) as follows.

THEOREM C. Let $a_k \geq 0, b_k > 0 (k = 1, 2, \dots, n), 1 - e_i + e_j \geq 0 (i, j = 1, 2, \dots, n)$, and let $p > 0, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} \sum_{k=1}^n a_k b_k & \geq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}-\frac{1}{q}} \left\{ \left[\left(\sum_{k=1}^n a_k^p \right) \left(\sum_{k=1}^n b_k^q \right) \right]^2 \right. \\ & \quad \left. - \left[\left(\sum_{k=1}^n a_k^p e_k \right) \left(\sum_{k=1}^n b_k^q \right) - \left(\sum_{k=1}^n a_k^p \right) \left(\sum_{k=1}^n b_k^q e_k \right) \right]^2 \right\}^{\frac{1}{2q}}. \end{aligned} \tag{4}$$

THEOREM D. Let $f(x), g(x), e(x)$ be integrable functions defined on $[a, b]$ and $f(x) \geq 0, g(x) > 0, 1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, and let $p > 0, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} & \int_a^b f(x)g(x)dx \\ & \geq \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int_a^b f^p(x)dx \int_a^b g^q(x)dx \right)^2 \right. \\ & \quad \left. - \left(\int_a^b f^p(x)e(x)dx \int_a^b g^q(x)dx - \int_a^b f^p(x)dx \int_a^b g^q(x)e(x)dx \right)^2 \right]^{\frac{1}{2q}}. \end{aligned} \tag{5}$$

In 2007, Wu [13] established the following generalizations of (2) and (3).

THEOREM E. Let $a_k \geq 0, b_k > 0 (k = 1, 2, \dots, n), 1 - e_i + e_j \geq 0 (i, j = 1, 2, \dots, n)$, and let $p \geq q > 0, \rho = \min\{\frac{1}{p} + \frac{1}{q}, 1\}$. Then

$$\begin{aligned} \sum_{k=1}^n a_k b_k & \leq n^{1-\rho} \left(\sum_{k=1}^n b_k^q \right)^{\frac{1}{q}-\frac{1}{p}} \left\{ \left[\left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p \right) \right]^2 \right. \\ & \quad \left. - \left[\left(\sum_{k=1}^n b_k^q e_k \right) \left(\sum_{k=1}^n a_k^p \right) - \left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p e_k \right) \right]^2 \right\}^{\frac{1}{2p}}. \end{aligned} \tag{6}$$

The integral form is as follows:

THEOREM F. Let $f(x), g(x), e(x)$ be integrable functions defined on $[a, b]$ and $f(x) \geq 0, g(x) > 0, 1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, and let $p \geq q > 0, \frac{1}{p} + \frac{1}{q} \leq 1$. Then

$$\begin{aligned} & \int_a^b f(x)g(x)dx \\ & \leq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b g^q(x)dx \right)^{\frac{1}{q}-\frac{1}{p}} \left[\left(\int_a^b g^q(x)dx \int_a^b f^p(x)dx \right)^2 \right. \\ & \quad \left. - \left(\int_a^b g^q(x)e(x)dx \int_a^b f^p(x)dx - \int_a^b g^q(x)dx \int_a^b f^p(x)e(x)dx \right)^2 \right]^{\frac{1}{2p}}. \end{aligned} \tag{7}$$

Later, Tian and Hu [9] presented the following reversed versions of inequalities (6) and (7).

THEOREM G. Let $a_k \geq 0, b_k > 0 (k = 1, 2, \dots, n), 1 - e_i + e_j \geq 0 (i, j = 1, 2, \dots, n)$, and let $p > 0, q < 0, \vartheta = \max\{\frac{1}{p} + \frac{1}{q}, 1\}$. Then

$$\begin{aligned} \sum_{k=1}^n a_k b_k & \geq n^{1-\vartheta} \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}-\frac{1}{q}} \left\{ \left[\left(\sum_{k=1}^n a_k^p \right) \left(\sum_{k=1}^n b_k^q \right) \right]^2 \right. \\ & \quad \left. - \left[\left(\sum_{k=1}^n a_k^p e_k \right) \left(\sum_{k=1}^n b_k^q \right) - \left(\sum_{k=1}^n a_k^p \right) \left(\sum_{k=1}^n b_k^q e_k \right) \right]^2 \right\}^{\frac{1}{2q}}. \end{aligned} \tag{8}$$

THEOREM H. Let $f(x), g(x), e(x)$ be integrable functions defined on $[a, b]$ and $f(x) \geq 0, g(x) > 0, 1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, and let $q < 0, \frac{1}{p} + \frac{1}{q} \geq 1$. Then

$$\begin{aligned} & \int_a^b f(x)g(x)dx \\ & \geq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int_a^b f^p(x)dx \int_a^b g^q(x)dx \right)^2 \right. \\ & \quad \left. - \left(\int_a^b f^p(x)e(x)dx \int_a^b g^q(x)dx - \int_a^b f^p(x)dx \int_a^b g^q(x)e(x)dx \right)^2 \right]^{\frac{1}{2q}}. \end{aligned} \tag{9}$$

In this paper, the above inequalities (6), (7), (8) and (9) are called as generalized sharp Hölder's inequalities.

In 1994, Hu [5] gave the following properties of inequalities (2) and (3)

THEOREM I. Let $a_k \geq 0, b_k \geq 0$ ($k = 1, 2, \dots$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots$), let $p \geq q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and let

$$\begin{aligned}
 F_s(n) &= \left(\sum_{k=1}^n b_k^q \right)^{\frac{2s}{q} - \frac{2s}{p}} \left\{ \left[\left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p \right) \right]^2 \right. \\
 &\quad - \left. \left[\left(\sum_{k=1}^n b_k^q e_k \right) \left(\sum_{k=1}^n a_k^p \right) - \left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p e_k \right) \right]^2 \right\}^{\frac{s}{p}} \\
 &\quad - \left(\sum_{k=1}^n a_k b_k \right)^{2s}, \quad s = 1, 2, \dots
 \end{aligned} \tag{10}$$

Then

$$F_s(n) \leq F_s(n + 1). \tag{11}$$

THEOREM J. Let $f(x), g(x), e(x)$ be integrable functions defined on $[a, b]$ and $f(x) \geq 0, g(x) \geq 0, 1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, let $p \geq q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and let

$$\begin{aligned}
 G_s(t) &= \left(\int_a^t g^q(x) dx \right)^{\frac{2s}{q} - \frac{2s}{p}} \left[\left(\int_a^t g^q(x) dx \int_a^t f^p(x) dx \right)^2 \right. \\
 &\quad - \left. \left(\int_a^t g^q(x) e(x) dx \int_a^t f^p(x) dx - \int_a^t g^q(x) dx \int_a^t f^p(x) e(x) dx \right)^2 \right]^{\frac{s}{p}} \\
 &\quad - \left(\int_a^t f(x) g(x) dx \right)^{2s}, \quad s = 1, 2, \dots
 \end{aligned} \tag{12}$$

Then we have

$$G_s(t_1) \leq G_s(t_2), \quad a \leq t_1 \leq t_2 \leq b. \tag{13}$$

In 2013, Tian [11] gave the similar properties of inequalities (4) and (5).

THEOREM K. Let $a_k \geq 0, b_k > 0$ ($k = 1, 2, \dots$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots$), let $p > 0, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and let

$$\begin{aligned}
 F(n) &= \left(\sum_{k=1}^n a_k^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_{k=1}^n a_k^p \right) \left(\sum_{k=1}^n b_k^q \right) \right]^2 \right. \\
 &\quad - \left. \left[\left(\sum_{k=1}^n a_k^p e_k \right) \left(\sum_{k=1}^n b_k^q \right) - \left(\sum_{k=1}^n a_k^p \right) \left(\sum_{k=1}^n b_k^q e_k \right) \right]^2 \right\}^{\frac{1}{q}} \\
 &\quad - \left(\sum_{k=1}^n a_k b_k \right)^2.
 \end{aligned} \tag{14}$$

Then

$$F(n) \geq F(n + 1). \tag{15}$$

THEOREM L. *Let $f(x)$, $g(x)$, $e(x)$ be integrable functions defined on $[a, b]$ and $f(x) \geq 0$, $g(x) > 0$, $1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, let $p > 0$, $q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and let*

$$\begin{aligned}
 G(t) = & \left(\int_a^t f^p(x) dx \right)^{\frac{2}{p} - \frac{2}{q}} \left[\left(\int_a^t f^p(x) dx \int_a^t g^q(x) dx \right)^2 \right. \\
 & - \left. \left(\int_a^t f^p(x) e(x) dx \int_a^t g^q(x) dx - \int_a^t f^p(x) dx \int_a^t g^q(x) e(x) dx \right)^2 \right]^{\frac{1}{q}} \\
 & - \left(\int_a^t f(x) g(x) dx \right)^2.
 \end{aligned} \tag{16}$$

Then we have

$$G(t_1) \geq G(t_2), \quad a \leq t_1 \leq t_2 \leq b. \tag{17}$$

Stimulated by the works of Hu [5] and Tian [11], in this paper, some similar properties of generalized sharp Hölder's inequalities (6), (7), (8) and (9) are given.

2. Main results

We begin this section with a lemma, which will be used in the sequel.

LEMMA 2.1. [3] (Power means inequality) *If $x_i \geq 0$, $\lambda_i > 0$, $i = 1, 2, \dots, n$, $0 < p \leq 1$, then*

$$\sum_{i=1}^n \lambda_i x_i^p \leq \left(\sum_{i=1}^n \lambda_i \right)^{1-p} \left(\sum_{i=1}^n \lambda_i x_i \right)^p. \tag{18}$$

The sign of inequality is reversed for $p \geq 1$ or $p < 0$.

Next, we give some new properties of generalized sharp Hölder's inequalities (6) and (7).

THEOREM 2.2. *Let $a_k \geq 0$, $b_k > 0$ ($k = 1, 2, \dots$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots$), let $p \geq q > 0$, $\rho = \min\{\frac{1}{p} + \frac{1}{q}, 1\}$, and let*

$$\begin{aligned}
 F_s(n) = & n^{2s(1-\rho)} \left(\sum_{k=1}^n b_k^q \right)^{\frac{2s}{q} - \frac{2s}{p}} \left\{ \left[\left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p \right) \right]^2 \right. \\
 & - \left. \left[\left(\sum_{k=1}^n b_k^q e_k \right) \left(\sum_{k=1}^n a_k^p \right) - \left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p e_k \right) \right]^2 \right\}^{\frac{s}{p}} \\
 & - \left(\sum_{k=1}^n a_k b_k \right)^{2s}, \quad s = 1, 2, \dots.
 \end{aligned} \tag{19}$$

Then

$$F_s(n) \leq F_s(n + 1). \tag{20}$$

The integral form is as follows:

THEOREM 2.3. *Let $f(x), g(x), e(x)$ be integrable functions defined on $[a, b]$ and $f(x) \geq 0, g(x) \geq 0, 1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, let $p \geq q > 0, \frac{1}{p} + \frac{1}{q} \leq 1$, and let*

$$\begin{aligned}
 G_s(t) = & (t-a)^{2s-\frac{2s}{p}-\frac{2s}{q}} \left(\int_a^t g^q(x) dx \right)^{\frac{2s}{q}-\frac{2s}{p}} \left[\left(\int_a^t g^q(x) dx \int_a^t f^p(x) dx \right)^2 \right. \\
 & \left. - \left(\int_a^t g^q(x) e(x) dx \int_a^t f^p(x) dx - \int_a^t g^q(x) dx \int_a^t f^p(x) e(x) dx \right)^2 \right]^{\frac{s}{p}} \\
 & - \left(\int_a^t f(x) g(x) dx \right)^{2s}, \quad s = 1, 2, \dots.
 \end{aligned} \tag{21}$$

Then we have

$$G_s(t_1) \leq G_s(t_2), \quad a \leq t_1 \leq t_2 \leq b. \tag{22}$$

Proof of Theorem 2.2. We first consider the case $s = 1$. Denote

$$\begin{aligned}
 x_n = & n^{2(1-\rho)} \left(\sum_{k=1}^n b_k^q \right)^{\frac{2}{q}-\frac{2}{p}} \left\{ \left[\left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p \right) \right]^2 \right. \\
 & \left. - \left[\left(\sum_{k=1}^n b_k^q e_k \right) \left(\sum_{k=1}^n a_k^p \right) - \left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p e_k \right) \right]^2 \right\}^{\frac{1}{p}}
 \end{aligned}$$

We first consider the case $s = 1$. Let $\frac{1}{p} + \frac{1}{q} = t$ ($0 < t \leq 1$), which implies $\frac{1}{pt} + \frac{1}{qt} = 1$. By Theorem E since $\rho = t$ we have

$$\begin{aligned}
 y_n = & \left(\sum_{k=1}^n a_k b_k \right)^2 \leq n^{2(1-t)} \left(\sum_{k=1}^n b_k^q \right)^{\frac{2}{q}-\frac{2}{p}} \left\{ \left[\left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p \right) \right]^2 \right. \\
 & \left. - \left[\left(\sum_{k=1}^n b_k^q e_k \right) \left(\sum_{k=1}^n a_k^p \right) - \left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p e_k \right) \right]^2 \right\}^{\frac{1}{p}} = x_n,
 \end{aligned} \tag{23}$$

that is $y_n \leq x_n$. A simple calculation gives

$$\begin{aligned}
 & \sum_{k=1}^n a_k b_k \sum_{r=1}^n a_r b_r (1 - e_k + e_r) \\
 = & \sum_{k=1}^n \sum_{r=1}^n a_k b_k a_r b_r - \sum_{k=1}^n \sum_{r=1}^n a_r b_r a_k b_k e_k + \sum_{k=1}^n \sum_{r=1}^n a_k b_k a_r b_r e_r \\
 = & \left(\sum_{k=1}^n a_k b_k \right)^2,
 \end{aligned} \tag{24}$$

and so

$$\begin{aligned}
 x_n &= n^{2(1-t)} \left[\sum_{k=1}^n b_k^q \sum_{r=1}^n b_r^q (1 - e_k + e_r) \right]^{\frac{1}{q} - \frac{1}{p}} \\
 &\quad \times \left[\sum_{k=1}^n a_k^p \sum_{r=1}^n b_r^q - \sum_{k=1}^n a_k^p e_k \sum_{r=1}^n b_r^q + \sum_{k=1}^n a_k^p \sum_{r=1}^n b_r^q e_r \right]^{\frac{1}{p}} \\
 &\quad \times \left[\sum_{k=1}^n b_k^q \sum_{r=1}^n a_r^p - \sum_{k=1}^n b_k^q e_k \sum_{r=1}^n a_r^p + \sum_{k=1}^n b_k^q \sum_{r=1}^n a_r^p e_r \right]^{\frac{1}{p}} \\
 &= n^{2(1-t)} \left[\sum_{k=1}^n b_k^q \sum_{r=1}^n b_r^q (1 - e_k + e_r) \right]^{\frac{1}{q} - \frac{1}{p}} \\
 &\quad \times \left[\sum_{k=1}^n \sum_{r=1}^n a_k^p b_r^q (1 - e_k + e_r) \right]^{\frac{1}{p}} \left[\sum_{k=1}^n \sum_{r=1}^n b_k^q a_r^p (1 - e_k + e_r) \right]^{\frac{1}{p}}. \tag{25}
 \end{aligned}$$

Additionally, performing some simple computations, we have

$$\begin{aligned}
 y_{n+1} - y_n + x_n &= a_{n+1} b_{n+1} \sum_{k=1}^{n+1} a_k b_k (1 - e_{n+1} + e_k) \\
 &\quad + \sum_{r=1}^n a_r b_r a_{n+1} b_{n+1} (1 - e_r + e_{n+1}) + x_n. \tag{26}
 \end{aligned}$$

By Hölder's inequality we have

$$\begin{aligned}
 &a_{n+1} b_{n+1} \sum_{k=1}^{n+1} a_k b_k (1 - e_{n+1} + e_k) + \sum_{r=1}^n a_r b_r a_{n+1} b_{n+1} (1 - e_r + e_{n+1}) \\
 &\leq a_{n+1} b_{n+1} \left[\sum_{k=1}^{n+1} a_k^{pt} (1 - e_{n+1} + e_k) \right]^{\frac{1}{pt}} \left[\sum_{k=1}^{n+1} b_k^{qt} (1 - e_{n+1} + e_k) \right]^{\frac{1}{qt}} \\
 &\quad + \sum_{r=1}^n a_r b_r \left[a_{n+1}^{pt} (1 - e_r + e_{n+1}) \right]^{\frac{1}{pt}} \left[b_{n+1}^{qt} (1 - e_r + e_{n+1}) \right]^{\frac{1}{qt}}, \tag{27}
 \end{aligned}$$

and so

$$\begin{aligned}
 &y_{n+1} - y_n + x_n \\
 &\leq a_{n+1} b_{n+1} \left[\sum_{k=1}^{n+1} a_k^{pt} (1 - e_{n+1} + e_k) \right]^{\frac{1}{pt}} \left[\sum_{k=1}^{n+1} b_k^{qt} (1 - e_{n+1} + e_k) \right]^{\frac{1}{qt}} \\
 &\quad + \sum_{r=1}^n a_r b_r \left[a_{n+1}^{pt} (1 - e_r + e_{n+1}) \right]^{\frac{1}{pt}} \left[b_{n+1}^{qt} (1 - e_r + e_{n+1}) \right]^{\frac{1}{qt}} + x_n \\
 &= b_{n+1}^{1 - \frac{q}{p}} \left[\sum_{k=1}^{n+1} b_k^{qt} (1 - e_{n+1} + e_k) \right]^{\frac{1}{qt} - \frac{1}{pt}} \times a_{n+1} \left[\sum_{k=1}^{n+1} b_k^{qt} (1 - e_{n+1} + e_k) \right]^{\frac{1}{pt}} \\
 &\quad \times b_{n+1}^{\frac{q}{p}} \left[\sum_{k=1}^{n+1} a_k^{pt} (1 - e_{n+1} + e_k) \right]^{\frac{1}{pt}}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{r=1}^n \left\{ b_r^{1-\frac{q}{p}} \left[b_{n+1}^{qt} (1-e_r+e_{n+1}) \right]^{\frac{1}{qt}-\frac{1}{pt}} \times a_r \left[b_{n+1}^{qt} (1-e_r+e_{n+1}) \right]^{\frac{1}{pt}} \right. \\
& \quad \left. \times b_r^{\frac{q}{p}} \left[a_{n+1}^{pt} (1-e_r+e_{n+1}) \right]^{\frac{1}{pt}} \right\} + x_n \\
& = b_{n+1}^{1-\frac{q}{p}} \left[\sum_{k=1}^{n+1} b_k^{qt} (1-e_{n+1}+e_k) \right]^{\frac{1}{qt}-\frac{1}{pt}} \times a_{n+1} \left[\sum_{k=1}^{n+1} b_k^{qt} (1-e_{n+1}+e_k) \right]^{\frac{1}{pt}} \\
& \quad \times b_{n+1}^{\frac{q}{p}} \left[\sum_{k=1}^{n+1} a_k^{pt} (1-e_{n+1}+e_k) \right]^{\frac{1}{pt}} \\
& + \sum_{r=1}^n \left\{ \left[b_r^{qt} b_{n+1}^{qt} (1-e_r+e_{n+1}) \right]^{\frac{1}{qt}-\frac{1}{pt}} \times \left[a_r^{pt} b_{n+1}^{qt} (1-e_r+e_{n+1}) \right]^{\frac{1}{pt}} \right. \\
& \quad \left. \times \left[b_r^{qt} a_{n+1}^{pt} (1-e_r+e_{n+1}) \right]^{\frac{1}{pt}} \right\} + x_n. \tag{28}
\end{aligned}$$

By using Hölder's inequality and Power means inequality (18), we obtain

$$\begin{aligned}
& b_{n+1}^{1-\frac{q}{p}} \left[\sum_{k=1}^{n+1} b_k^{qt} (1-e_{n+1}+e_k) \right]^{\frac{1}{qt}-\frac{1}{pt}} \times a_{n+1} \left[\sum_{k=1}^{n+1} b_k^{qt} (1-e_{n+1}+e_k) \right]^{\frac{1}{pt}} \\
& \quad \times b_{n+1}^{\frac{q}{p}} \left[\sum_{k=1}^{n+1} a_k^{pt} (1-e_{n+1}+e_k) \right]^{\frac{1}{pt}} \\
& + \sum_{r=1}^n \left\{ \left[b_r^{qt} b_{n+1}^{qt} (1-e_r+e_{n+1}) \right]^{\frac{1}{qt}-\frac{1}{pt}} \times \left[a_r^{pt} b_{n+1}^{qt} (1-e_r+e_{n+1}) \right]^{\frac{1}{pt}} \right. \\
& \quad \left. \times \left[b_r^{qt} a_{n+1}^{pt} (1-e_r+e_{n+1}) \right]^{\frac{1}{pt}} \right\} \\
& \leq b_{n+1}^{1-\frac{q}{p}} \left[\sum_{k=1}^{n+1} b_k^{qt} (1-e_{n+1}+e_k) \right]^{\frac{1}{qt}-\frac{1}{pt}} \times a_{n+1} \left[\sum_{k=1}^{n+1} b_k^{qt} (1-e_{n+1}+e_k) \right]^{\frac{1}{pt}} \\
& \quad \times b_{n+1}^{\frac{q}{p}} \left[\sum_{k=1}^{n+1} a_k^{pt} (1-e_{n+1}+e_k) \right]^{\frac{1}{pt}} \\
& + \left[\sum_{r=1}^n b_r^{qt} b_{n+1}^{qt} (1-e_r+e_{n+1}) \right]^{\frac{1}{qt}-\frac{1}{pt}} \times \left[\sum_{r=1}^n a_r^{pt} b_{n+1}^{qt} (1-e_r+e_{n+1}) \right]^{\frac{1}{pt}} \\
& \quad \times \left[\sum_{r=1}^n b_r^{qt} a_{n+1}^{pt} (1-e_r+e_{n+1}) \right]^{\frac{1}{pt}} \\
& \leq b_{n+1}^{1-\frac{q}{p}} \left[\sum_{k=1}^{n+1} (1-e_{n+1}+e_k) \right]^{(1-t)(\frac{1}{qt}-\frac{1}{pt})} \times \left[\sum_{k=1}^{n+1} b_k^q (1-e_{n+1}+e_k) \right]^{\frac{1}{q}-\frac{1}{p}} \\
& \quad \times a_{n+1} \left[\sum_{k=1}^{n+1} (1-e_{n+1}+e_k) \right]^{(1-t)\frac{1}{pt}} \times \left[\sum_{k=1}^{n+1} b_k^q (1-e_{n+1}+e_k) \right]^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
 & \times b_{n+1}^{\frac{q}{p}} \left[\sum_{k=1}^{n+1} (1 - e_{n+1} + e_k) \right]^{(1-t)\frac{1}{pt}} \times \left[\sum_{k=1}^{n+1} a_k^p (1 - e_{n+1} + e_k) \right]^{\frac{1}{p}} \\
 & + \left[\sum_{r=1}^n (1 - e_r + e_{n+1}) \right]^{(1-t)(\frac{1}{q} - \frac{1}{p})} \left[\sum_{r=1}^n b_r^q b_{n+1}^q (1 - e_r + e_{n+1}) \right]^{\frac{1}{q} - \frac{1}{p}} \\
 & \times \left[\sum_{r=1}^n (1 - e_r + e_{n+1}) \right]^{(1-t)\frac{1}{pt}} \left[\sum_{r=1}^n a_r^p b_{n+1}^q (1 - e_r + e_{n+1}) \right]^{\frac{1}{p}} \\
 & \times \left[\sum_{r=1}^n (1 - e_r + e_{n+1}) \right]^{(1-t)\frac{1}{pt}} \left[\sum_{r=1}^n b_r^q a_{n+1}^p (1 - e_r + e_{n+1}) \right]^{\frac{1}{p}} \\
 = & \left[\sum_{k=1}^{n+1} (1 - e_{n+1} + e_k) \right]^{(1-t)} \times b_{n+1}^{1-\frac{q}{p}} \left[\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right]^{\frac{1}{q} - \frac{1}{p}} \\
 & \times a_{n+1} \left[\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right]^{\frac{1}{p}} \times b_{n+1}^{\frac{q}{p}} \left[\sum_{k=1}^{n+1} a_k^p (1 - e_{n+1} + e_k) \right]^{\frac{1}{p}} \\
 & + \left[\sum_{r=1}^n (1 - e_r + e_{n+1}) \right]^{(1-t)} \left[\sum_{r=1}^n b_r^q b_{n+1}^q (1 - e_r + e_{n+1}) \right]^{\frac{1}{q} - \frac{1}{p}} \\
 & \times \left[\sum_{r=1}^n a_r^p b_{n+1}^q (1 - e_r + e_{n+1}) \right]^{\frac{1}{p}} \left[\sum_{r=1}^n b_r^q a_{n+1}^p (1 - e_r + e_{n+1}) \right]^{\frac{1}{p}} \\
 \leq & \left[\sum_{k=1}^{n+1} \sum_{r=1}^{n+1} (1 - e_r + e_k) \right]^{(1-t)} \times \left\{ b_{n+1}^{1-\frac{q}{p}} \left[\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right]^{\frac{1}{q} - \frac{1}{p}} \right. \\
 & \times a_{n+1} \left[\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right]^{\frac{1}{p}} \times b_{n+1}^{\frac{q}{p}} \left[\sum_{k=1}^{n+1} a_k^p (1 - e_{n+1} + e_k) \right]^{\frac{1}{p}} \\
 & + \left[\sum_{r=1}^n b_r^q b_{n+1}^q (1 - e_r + e_{n+1}) \right]^{\frac{1}{q} - \frac{1}{p}} \left[\sum_{r=1}^n a_r^p b_{n+1}^q (1 - e_r + e_{n+1}) \right]^{\frac{1}{p}} \\
 & \left. \times \left[\sum_{r=1}^n b_r^q a_{n+1}^p (1 - e_r + e_{n+1}) \right]^{\frac{1}{p}} \right\} \\
 = & (n+1)^{2(1-t)} \left\{ b_{n+1}^{1-\frac{q}{p}} \left[\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right]^{\frac{1}{q} - \frac{1}{p}} \right. \\
 & \times a_{n+1} \left[\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right]^{\frac{1}{p}} \times b_{n+1}^{\frac{q}{p}} \left[\sum_{k=1}^{n+1} a_k^p (1 - e_{n+1} + e_k) \right]^{\frac{1}{p}} \\
 & + \left[\sum_{r=1}^n b_r^q b_{n+1}^q (1 - e_r + e_{n+1}) \right]^{\frac{1}{q} - \frac{1}{p}} \left[\sum_{r=1}^n a_r^p b_{n+1}^q (1 - e_r + e_{n+1}) \right]^{\frac{1}{p}} \\
 & \left. \times \left[\sum_{r=1}^n b_r^q a_{n+1}^p (1 - e_r + e_{n+1}) \right]^{\frac{1}{p}} \right\}.
 \end{aligned}$$

(29)

Hence, from (29), (28) and (25), we have

$$\begin{aligned}
& y_{n+1} - y_n + x_n \\
& \leq (n+1)^{2(1-t)} \left\{ \left[b_{n+1}^q \left(\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right) \right]^{\frac{1}{q} - \frac{1}{p}} \right. \\
& \quad \times \left[a_{n+1}^p \left(\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right) \right]^{\frac{1}{p}} \left[b_{n+1}^q \left(\sum_{k=1}^{n+1} a_k^p (1 - e_{n+1} + e_k) \right) \right]^{\frac{1}{p}} \\
& \quad + \left[\sum_{k=1}^n b_k^q b_{n+1}^q (1 - e_k + e_{n+1}) \right]^{\frac{1}{q} - \frac{1}{p}} \left[\sum_{k=1}^n a_k^p b_{n+1}^q (1 - e_k + e_{n+1}) \right]^{\frac{1}{p}} \\
& \quad \times \left[\sum_{k=1}^n b_k^q a_{n+1}^p (1 - e_k + e_{n+1}) \right]^{\frac{1}{p}} + \left[\sum_{k=1}^n b_k^q \sum_{r=1}^n b_r^q (1 - e_k + e_r) \right]^{\frac{1}{q} - \frac{1}{p}} \\
& \quad \times \left[\sum_{k=1}^n \sum_{r=1}^n a_k^p b_r^q (1 - e_k + e_r) \right]^{\frac{1}{p}} \left[\sum_{k=1}^n \sum_{r=1}^n b_k^q a_r^p (1 - e_k + e_r) \right]^{\frac{1}{p}} \left. \right\} \\
& \leq (n+1)^{2(1-t)} \left\{ \left[b_{n+1}^q \left(\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right) \right] \right. \\
& \quad + \sum_{k=1}^n \left[b_k^q b_{n+1}^q (1 - e_k + e_{n+1}) + b_k^q \sum_{r=1}^n b_r^q (1 - e_k + e_r) \right] \left. \right\}^{\frac{1}{q} - \frac{1}{p}} \\
& \quad \times \left\{ a_{n+1}^p \left(\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right) \right. \\
& \quad + \sum_{k=1}^n \left[a_k^p b_{n+1}^q (1 - e_k + e_{n+1}) + a_k^p \sum_{r=1}^n b_r^q (1 - e_k + e_r) \right] \left. \right\}^{\frac{1}{p}} \\
& \quad \times \left\{ b_{n+1}^q \left(\sum_{k=1}^{n+1} a_k^p (1 - e_{n+1} + e_k) \right) \right. \\
& \quad + \sum_{k=1}^n \left[b_k^q a_{n+1}^p (1 - e_k + e_{n+1}) + b_k^q \sum_{r=1}^n a_r^p (1 - e_k + e_r) \right] \left. \right\}^{\frac{1}{p}} \\
& = (n+1)^{2(1-t)} \left\{ b_{n+1}^q \left[\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right] \right. \\
& \quad + \sum_{k=1}^n b_k^q (1 - e_k + e_{n+1}) \left. \right] + \left(\sum_{k=1}^n b_k^q \right)^2 \left. \right\}^{\frac{1}{q} - \frac{1}{p}} \\
& \quad \times \left\{ a_{n+1}^p \left(\sum_{k=1}^{n+1} b_k^q (1 - e_{n+1} + e_k) \right) + \sum_{k=1}^n a_k^p \left[\sum_{r=1}^{n+1} b_r^q (1 - e_k + e_r) \right] \right\}^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
 & \times \left\{ b_{n+1}^q \left(\sum_{k=1}^{n+1} a_k^p (1 - e_{n+1} + e_k) \right) + \sum_{k=1}^n b_k^q \left[\sum_{r=1}^{n+1} a_r^p (1 - e_k + e_r) \right] \right\}^{\frac{1}{p}} \\
 &= (n+1)^{2(1-t)} \left[\left(b_{n+1}^q \right)^2 + 2b_{n+1}^q \sum_{k=1}^n b_k^q + \left(\sum_{k=1}^n b_k^q \right)^2 \right]^{\frac{1}{q} - \frac{1}{p}} \\
 & \times \left[\sum_{k=1}^{n+1} \sum_{r=1}^{n+1} a_k^p b_r^q (1 - e_k + e_r) \right]^{\frac{1}{p}} \left[\sum_{k=1}^{n+1} \sum_{r=1}^{n+1} b_k^q a_r^p (1 - e_k + e_r) \right]^{\frac{1}{p}} \\
 &= (n+1)^{2(1-t)} \left(\sum_{k=1}^{n+1} b_k^q \right)^{\frac{2}{q} - \frac{2}{p}} \left\{ \left[\left(\sum_{k=1}^{n+1} b_k^q \right) \left(\sum_{k=1}^{n+1} a_k^p \right) \right]^2 \right. \\
 & \quad \left. - \left[\left(\sum_{k=1}^{n+1} b_k^q e_k \right) \left(\sum_{k=1}^{n+1} a_k^p \right) - \left(\sum_{k=1}^{n+1} b_k^q \right) \left(\sum_{k=1}^{n+1} a_k^p e_k \right) \right]^2 \right\}^{\frac{1}{p}} \\
 &= x_{n+1}.
 \end{aligned} \tag{30}$$

So

$$F_1(n) \leq F_1(n+1).$$

From $y_{n+1} - y_n \leq x_{n+1} - x_n$, it is easy to see that the equality

$$y_{n+1}^s - y_n^s \leq x_{n+1}^s - x_n^s$$

holds with $s = 2, 3, \dots$. That is, $F_s(n) \leq F_s(n+1)$, $s = 2, 3, \dots$. The proof of Theorem 2.2 is completed. \square

Proof of Theorem 2.3. Since $a < t_1 < t_2 < b$, there exists $\beta \in (0, 1)$ such that

$$\frac{t_1 - a}{t_2 - a} = \beta \text{ or } t_1 = (1 - \beta)a + \beta t_2.$$

Case 1. Let β is a rational number. Then $\beta = \frac{m_1}{m_2}$, where $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$. Let $n_1 = m_1 n$ and $n_2 = m_2 n$. Then we have an equidistant partition of $[a, t_1]$:

$$a = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n_1-1} < x_{n_1} = t_1,$$

where

$$x_i = a + \frac{t_1 - a}{n_1} i \text{ and } \Delta x_i = \frac{t_1 - a}{n_1}, \quad i = 1, 2, \dots, n_1.$$

Therefore we have

$$\begin{aligned}
 & \left(\frac{t_1 - a}{n_1} \right)^{2s} F_s(n_1) \\
 & \triangleq \left(\frac{t_1 - a}{n_1} \right)^{2s} n_1^{2s - \frac{2s}{p} - \frac{2s}{q}} \left(\sum_{i=1}^{n_1} g^q(x_i) \right)^{\frac{2s}{q} - \frac{2s}{p}} \left[\left(\sum_{i=1}^{n_1} g^q(x_i) \sum_{i=1}^{n_1} f^p(x_i) \right)^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 & - \left(\sum_{i=1}^{n_1} g^q(x_i) e(x_i) \sum_{i=1}^{n_1} f^p(x_i) - \sum_{i=1}^{n_1} g^q(x_i) \sum_{i=1}^{n_1} f^p(x_i) e(x_i) \right)^2 \Bigg]^{\frac{s}{p}} \\
 & - \left(\frac{t_1 - a}{n_1} \right)^{2s} \left(\sum_{i=1}^{n_1} f(x_i) g(x_i) \right)^{2s} \\
 = & (t_1 - a)^{2s - \frac{2s}{p} - \frac{2s}{q}} \left(\sum_{i=1}^{n_1} g^q(x_i) \frac{t_1 - a}{n_1} \right)^{\frac{2s}{q} - \frac{2s}{p}} \\
 & \times \left[\left(\sum_{i=1}^{n_1} g^q(x_i) \frac{t_1 - a}{n_1} \sum_{i=1}^{n_1} f^p(x_i) \frac{t_1 - a}{n_1} \right)^2 \right. \\
 & - \left(\sum_{i=1}^{n_1} g^q(x_i) e(x_i) \frac{t_1 - a}{n_1} \sum_{i=1}^{n_1} f^p(x_i) \frac{t_1 - a}{n_1} \right. \\
 & \left. \left. - \sum_{i=1}^{n_1} g^q(x_i) \frac{t_1 - a}{n_1} \sum_{i=1}^{n_1} f^p(x_i) e(x_i) \frac{t_1 - a}{n_1} \right)^2 \right]^{\frac{s}{p}} \\
 & - \left(\sum_{i=1}^{n_1} f(x_i) g(x_i) \frac{t_1 - a}{n_1} \right)^{2s}.
 \end{aligned}$$

Similarly, for the equidistant partition of $[a, t_2]$:

$$a = x'_0 < x'_1 < x'_2 < \dots < x'_i < \dots < x'_{n_2-1} < x'_{n_2} = t_2,$$

where

$$x'_i = a + \frac{t_2 - a}{n_2} i \quad \text{and} \quad \Delta x'_i = \frac{t_2 - a}{n_2}, \quad i = 1, 2, \dots, n_2,$$

we have

$$\begin{aligned}
 & \left(\frac{t_2 - a}{n_2} \right)^{2s} F_s(n_2)' \\
 \triangleq & \left(\frac{t_2 - a}{n_2} \right)^{2s} \left\{ n_2^{2s - \frac{2s}{p} - \frac{2s}{q}} \left(\sum_{i=1}^{n_2} g^q(x'_i) \right)^{\frac{2s}{q} - \frac{2s}{p}} \left[\left(\sum_{i=1}^{n_2} g^q(x'_i) \sum_{i=1}^{n_2} f^p(x'_i) \right)^2 \right. \right. \\
 & - \left(\sum_{i=1}^{n_2} g^q(x'_i) e(x'_i) \sum_{i=1}^{n_2} f^p(x'_i) - \sum_{i=1}^{n_2} g^q(x'_i) \sum_{i=1}^{n_2} f^p(x'_i) e(x'_i) \right)^2 \Bigg]^{\frac{s}{p}} \\
 & \left. - \left(\sum_{i=1}^{n_2} f(x'_i) g(x'_i) \right)^{2s} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= (t_2 - a)^{2s - \frac{2s}{p} - \frac{2s}{q}} \left(\sum_{i=1}^{n_2} g^q(x'_i) \frac{t_2 - a}{n_2} \right)^{\frac{2s}{q} - \frac{2s}{p}} \\
 &\quad \times \left[\left(\sum_{i=1}^{n_2} g^q(x'_i) \frac{t_2 - a}{n_2} \sum_{i=1}^{n_2} f^p(x'_i) \frac{t_2 - a}{n_2} \right)^2 \right. \\
 &\quad - \left(\sum_{i=1}^{n_2} g^q(x'_i) e(x'_i) \frac{t_2 - a}{n_2} \sum_{i=1}^{n_2} f^p(x'_i) \frac{t_2 - a}{n_2} \right. \\
 &\quad \left. \left. - \sum_{i=1}^{n_2} g^q(x'_i) \frac{t_2 - a}{n_2} \sum_{i=1}^{n_2} f^p(x'_i) e(x'_i) \frac{t_2 - a}{n_2} \right)^2 \right]^{\frac{s}{p}} \\
 &\quad - \left(\sum_{i=1}^{n_2} f(x'_i) g(x'_i) \frac{t_2 - a}{n_2} \right)^{2s}.
 \end{aligned}$$

Due to

$$\frac{t_1 - a}{t_2 - a} = \frac{m_1 n}{m_2 n} = \frac{n_1}{n_2},$$

we see that $\Delta x_i = \Delta x'_i$ and $x_i = x'_i$ for $i = 1, 2, \dots, n_1$. By Theorem 2.2, we deduce that

$$\left(\frac{t_1 - a}{n_1} \right)^{2s} F_s(n_1) < \left(\frac{t_2 - a}{n_2} \right)^{2s} F_s(n_2)',$$

which, by letting $n \rightarrow \infty$, implies that $G_s(t_1) \leq G_s(t_2)$ for $a \leq t_1 \leq t_2 \leq b$.

Case 2. Let β is an irrational number. Then there exists an rational sequence $\{\beta_j\}$ such that $\lim_{j \rightarrow \infty} \beta_j = \beta$. From Case 1 we have

$$G_s((1 - \beta_j)a + \beta_j t_2) \leq G_s(t_2),$$

then letting $j \rightarrow \infty$ gives

$$G_s((1 - \beta)a + \beta t_2) = G_s(t_1) \leq G_s(t_2),$$

which completes the proof. \square

By the same way as in Theorem 2.2 and Theorem 2.3, we can obtain the following properties of (8) and (9).

THEOREM 2.4. Let $a_k \geq 0$, $b_k > 0$ ($k = 1, 2, \dots$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots$), let $p > 0$, $q < 0$, $\vartheta = \max\{\frac{1}{p} + \frac{1}{q}, 1\}$, and let

$$F_s(n) = n^{2s(1-\vartheta)} \left(\sum_{k=1}^n a_k^p \right)^{\frac{2s}{p} - \frac{2s}{q}} \left\{ \left[\left(\sum_{k=1}^n a_k^p \right) \left(\sum_{k=1}^n b_k^q \right) \right]^2 - \left[\left(\sum_{k=1}^n a_k^p e_k \right) \left(\sum_{k=1}^n b_k^q \right) - \left(\sum_{k=1}^n a_k^p \right) \left(\sum_{k=1}^n b_k^q e_k \right) \right]^2 \right\}^{\frac{s}{q}} - \left(\sum_{k=1}^n a_k b_k \right)^{2s}, \quad s = 1, 2, \dots \quad (31)$$

Then

$$F_s(n) \geq F_s(n+1). \quad (32)$$

The integral form is as follows:

THEOREM 2.5. Let $f(x)$, $g(x)$, $e(x)$ be integrable functions defined on $[a, b]$ and $f(x) \geq 0$, $g(x) > 0$, $1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, let $p > 0$, $q < 0$, $\frac{1}{p} + \frac{1}{q} \geq 1$, and let

$$G_s(t) = (t-a)^{2s - \frac{2s}{p} - \frac{2s}{q}} \left(\int_a^t f^p(x) dx \right)^{\frac{2s}{p} - \frac{2s}{q}} \left[\left(\int_a^t f^p(x) dx \int_a^t g^q(x) dx \right)^2 - \left(\int_a^t f^p(x) e(x) dx \int_a^t g^q(x) dx - \int_a^t f^p(x) dx \int_a^t g^q(x) e(x) dx \right)^2 \right]^{\frac{s}{q}} - \left(\int_a^t f(x) g(x) dx \right)^{2s}, \quad s = 1, 2, \dots \quad (33)$$

Then we have

$$G_s(t_1) \geq G_s(t_2), \quad a \leq t_1 \leq t_2 \leq b. \quad (34)$$

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REFERENCES

- [1] S. ABRAMOVICH, J. PEČARIĆ AND S. VAROŠANEC, *Continuous sharpening of Hölder's and Minkowski's inequalities*, Math. Inequal. Appl. **8**, 2 (2005), 179–190.
- [2] P. CHUNAEV, *Hölder and Minkowski type inequalities with alternating signs*, J. Math. Inequal. **9**, 1 (2015), 61–71.
- [3] G. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, second ed., Cambridge University Press, UK, 1952.
- [4] K. HU, *On an inequality and its applications*, Sci. Sinica **24**, (1981), 1047–1055.
- [5] K. HU, *On an inequality and its applications*, J. Jiangxi Norm. Univ. **18**, 4 (1994), 330–333.
- [6] E. G. KWON, J. E. BAE, *On a refined Hölder's inequality*, J. Math. Inequal. Preprint.

- [7] J. MATKOWSKI, *A converse of the Hölder inequality theorem*, Math. Inequal. Appl. **12**, 1 (2009), 21–32.
- [8] L. NIKOLOVA AND S. VAROŠANEC, *Refinements of Hölder's inequality derived from functions $\Psi_{p,q,\lambda}$ and $\Phi_{p,q,\lambda}$* , Ann. Funct. Anal. **2**, 1 (2011), 72–83.
- [9] J. TIAN, X.-M. HU, *A new reversed version of a generalized sharp Hölder's inequality and its Applications*, Abstr. Appl. Anal. **2013**, (2013), Article ID 901824, 9 pages.
- [10] J.-F. TIAN, *Extension of Hu Ke's inequality and its applications*, J. Inequal. Appl. **2011**, (2011), 77.
- [11] J.-F. TIAN, *Property of a Hölder-type inequality and its application*, Math. Inequal. Appl. **16**, 3 (2013), 831–841.
- [12] J.-F. TIAN, *Properties of generalized Hölder's inequalities*, J. Math. Inequal. **9**, 2 (2015), 473–480.
- [13] S. WU, *Generalization of a sharp Hölder's inequality and its application*, J. Math. Anal. Appl. **332**, (2007), 741–750.

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