Journal of
Mathematical
Inequalities
Volume 11, Number 2 (2017), 527–542
doi:10.7153/jmi-11-43

NEW INEQUALITIES FOR THE VOLUME OF THE UNIT BALL IN $\mathbb{R}^n$

TAO BAN AND CHAO-PING CHEN

(*Communicated by G. Nemes*)

Abstract. Many interesting monotonicity properties and inequalities for the volume of the unit ball in $\mathbb{R}^n$ have been established. The main object of this paper is to establish new inequalities for the volume of the unit ball in $\mathbb{R}^n$.

1. Introduction

In the recent past, several researchers established interesting monotonicity properties and inequalities of the volume of the unit ball in $\mathbb{R}^n$,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \qquad n \in \mathbb{N} := \{1,2,3,\ldots\},$$

where Γ denotes the gamma function.

Böhm and Hertel [7, p. 264] pointed out that the sequence $\left(\Omega_n\right)_{n\geqslant 1}$ is not monotonic for $n \geqslant 1$. Indeed, we have

$$\Omega_n < \Omega_{n+1} \quad \text{if} \quad 1 \leqslant n \leqslant 4 \quad \text{and} \quad \Omega_n > \Omega_{n+1} \quad \text{if} \quad n \geqslant 5.$$

Anderson et al. [5] showed that $\left(\Omega_n^{1/n}\right)_{n\geqslant 1}$ is monotonically decreasing to zero. Anderson and Qiu [6] proved that the sequence $\left(\Omega_n^{1/(n\ln n)}\right)_{n\geqslant 2}$ decreases to $e^{-1/2}$. Guo and Qi [12] proved that the sequence $\left(\Omega_n^{1/(n\ln n)}\right)_{n\geqslant 2}$ is logarithmically convex. Klain and Rota [13] proved that the sequence $\left(n\Omega_n/\Omega_{n-1}\right)_{n\geqslant 1}$ is increasing.

Many interesting inequalities for the volume of the unit ball in $\mathbb{R}^n$ have been established [3, 4, 8, 11, 16, 17, 18, 19]. For example, Alzer [3] proved that for all integers $n \geqslant 1$,

$$a_1\Omega_{n+1}^{n/(n+1)} \leqslant \Omega_n < b_1\Omega_{n+1}^{n/(n+1)}, \tag{1}$$

$$\sqrt{\frac{n+a_2}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} \leqslant \sqrt{\frac{n+b_2}{2\pi}} \tag{2}$$

Mathematics subject classification (2010): 33B15, 26D07.
Keywords and phrases: Volume of the unit n-dimensional ball, gamma function, inequality.

and

$$\left(1+\frac{1}{n}\right)^{a_3} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{b_3} \tag{3}$$

with the best possible constants

$$a_1 = \frac{2}{\sqrt{\pi}} = 1.1283\ldots, \quad b_1 = \sqrt{e} = 1.6487\ldots,$$
$$a_2 = \frac{1}{2}, \quad b_2 = \frac{\pi}{2} - 1 = 0.5707\ldots$$

and

$$a_3 = 2 - \frac{\ln\pi}{\ln 2} = 0.3485\ldots, \qquad b_3 = \frac{1}{2}.$$

Double inequality (2) was rediscovered by Qiu and Vuorinen [19]. Double inequality (2) refines a result due to Borgwardt [8, p. 253] who proved (2) with $a_2 = 0$ and $b_2 = 1$. Merkle [16] improved the left-hand side of (3) and obtained the following result:

$$\left(1+\frac{1}{n+1}\right)^{1/2} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}, \qquad n \geqslant 1. \tag{4}$$

Recently, Mortici [17] improved the inequalities (1) to (3) and obtained the following results:

$$\frac{\frac{64\cdot 720^{11/12}\cdot 2^{1/22}}{10395\cdot\pi^{5/11}}}{\sqrt[2n]{2\pi}} \leqslant \frac{\Omega_n}{\Omega_{n+1}^{n/(n+1)}} < \frac{\sqrt{e}}{\sqrt[2n]{2\pi}}, \qquad n \geqslant 4, \tag{5}$$

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n+\frac{1}{2}}{2\pi} + \frac{1}{16\pi n}}, \qquad n \geqslant 1 \tag{6}$$

and

$$\left(1+\frac{1}{n}\right)^{\frac{1}{2}-\frac{1}{4n}} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{\frac{1}{2}}, \qquad n \geqslant 4. \tag{7}$$

Very recently, Chen and Lin [11] presented sharp inequalities for the volume of the unit ball in $\mathbb{R}^n$. More precisely, the authors proved that, for all integers $n \geqslant 1$,

$$\left(1+\frac{1}{n+1}\right)^{\alpha} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leqslant \left(1+\frac{1}{n+1}\right)^{\beta}, \tag{8}$$

$$\left(1+\frac{1}{2n}-\frac{3}{8n^2}\right)^{\lambda} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leqslant \left(1+\frac{1}{2n}-\frac{3}{8n^2}\right)^{\mu}, \tag{9}$$

$$\sqrt{\frac{2\pi}{n+a}} \leqslant \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} < \sqrt{\frac{2\pi}{n+b}} \tag{10}$$

and

$$\frac{1}{\sqrt{\pi(n+\theta)}}\left(\frac{2\pi e}{n}\right)^{n/2} \leqslant \Omega_n < \frac{1}{\sqrt{\pi(n+\vartheta)}}\left(\frac{2\pi e}{n}\right)^{n/2} \tag{11}$$

with best possible constants

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2\ln 2 - \ln \pi}{\ln 3 - \ln 2} = 0.5957713\ldots,$$

$$\lambda = 1, \quad \mu = \frac{2\ln 2 - \ln \pi}{2\ln 3 - 3\ln 2} = 2.0509275\ldots,$$

$$a = \frac{\pi(1+\pi)^2}{2} - 1 = 25.94353\ldots, \quad b = \frac{1}{2} + 4\pi = 13.06637\ldots$$

and

$$\theta = \frac{e}{2} - 1 = 0.3591409\ldots, \quad \vartheta = \frac{1}{3}.$$

The main object of this paper is to establish new inequalities for the volume of the unit ball in $\mathbb{R}^n$.

2. Lemmas and preliminaries

Euler's gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t, \qquad x > 0$$

is one of the most important functions in mathematical analysis and its applications in various diverse areas. The logarithmic derivative of the gamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$ is known as the psi (or digamma) function. The derivatives of the psi function $\psi(x)$:

$$\psi^{(n)}(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\{\psi(x)\}, \qquad n \in \mathbb{N}$$

are called the polygamma functions.

The following asymptotic formulas are well known [1, p. 257] that, as $x \to \infty$,

$$\ln\Gamma(x) \sim \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \cdots. \tag{12}$$

The following asymptotic expansion can be found (see [15, p. 33]):

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim x^{t-s} \sum_{n=0}^{\infty} (-1)^n \frac{(s-t)_n}{n!} B_n^{(t-s+1)}(t) \frac{1}{x^n}, \qquad x \to \infty, \tag{13}$$

where the symbol $B_n^{(a)}(t)$ stands for the generalized Bernoulli polynomials, defined by the following generating function:

$$\frac{x^a e^{tx}}{(e^x - 1)^a} = \sum_{n=0}^{\infty} B_n^{(a)}(t) \frac{x^n}{n!}, \tag{14}$$

and $(t)_n$ is Pochhammer's symbol defined as

$$(t)_n = t(t+1)\cdots(t+n-1) = \frac{\Gamma(t+n)}{\Gamma(t)}, \qquad (t)_0 = 1.$$

Formula (13) yields, as $x \to \infty$,

$$\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 \sim x + \frac{1}{4} + \frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3} + \frac{23}{8192x^4} + \frac{53}{65536x^5} - \cdots \tag{15}$$

and

$$\left[\frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)}\right]^2 \sim x + \frac{3}{4} + \frac{1}{32x} - \frac{3}{128x^2} + \frac{27}{2048x^3} - \frac{27}{8192x^4} - \frac{171}{65536x^5} + \cdots. \tag{16}$$

The following lemmas are required in our present investigation.

LEMMA 2.1. ([9, Corollary 1]) *Let $m,n \in \mathbb{N}$. Then for $x > 0$,*

$$\begin{aligned}
&\sum_{j=1}^{2m} \left(1 - \frac{1}{2^{2j}}\right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}} \\
&\quad < (-1)^n \left(\psi^{(n-1)}(x+1) - \psi^{(n-1)}\left(x+\frac{1}{2}\right)\right) + \frac{(n-1)!}{2x^n} \\
&\quad < \sum_{j=1}^{2m-1} \left(1 - \frac{1}{2^{2j}}\right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}},
\end{aligned} \tag{17}$$

where B_n are the Bernoulli numbers.

It follows from (17) that, for $x > 0$,

$$\begin{aligned}
\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} &< \psi(x+1) - \psi\left(x+\frac{1}{2}\right) \\
&< \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8}.
\end{aligned} \tag{18}$$

It was proved in [11] that for $x > 0$,

$$\frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} < \psi\left(x+\frac{3}{2}\right) - \psi(x+1) < \frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} + \frac{1}{16x^5} \tag{19}$$

and

$$\psi\left(x+\frac{3}{2}\right) - \psi(x+1) > \frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} + \frac{1}{16x^5} - \frac{3}{128x^6}. \tag{20}$$

LEMMA 2.2. *For $x \geqslant 2$,*

$$x + \frac{1}{4} + \frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3} < \left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2. \tag{21}$$

Proof. Consider the function $P(x)$ defined for $x>0$ by

$$P(x)=2\ln\Gamma(x+1)-2\ln\Gamma\left(x+\frac{1}{2}\right)-\ln\left(x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}-\frac{5}{2048x^3}\right).$$

We conclude from the asymptotic formula (15) that

$$\lim_{x\to\infty}P(x)=0.$$

Differentiating and applying the second inequality in (18), we find that for $x\geqslant 2$,

$$\begin{aligned}P'(x)&=2\left(\psi(x+1)-\psi\left(x+\frac{1}{2}\right)\right)-\frac{2048x^4-64x^2+32x+15)}{x(2048x^4+512x^3+64x^2-16x-5)}\\&<2\left(\frac{1}{2x}-\frac{1}{8x^2}+\frac{1}{64x^4}-\frac{1}{128x^6}+\frac{17}{2048x^8}\right)\\&\quad-\frac{2048x^4-64x^2+32x+15)}{x(2048x^4+512x^3+64x^2-16x-5)}\\&=-\Big(1548853+5160656(x-2)+6899824(x-2)^2+4780544(x-2)^3\\&\quad+1819808(x-2)^4+361984(x-2)^5+29440(x-2)^6\Big)\\&\qquad\Big/\Big(1024x^8(2048x^4+512x^3+64x^2-16x-5)\Big)\\&<0.\end{aligned}$$

Hence, we have for $x\geqslant 2$,

$$P(x)>\lim_{x\to\infty}P(x)=0.$$

By rearranging the terms in the last expression, inequality (21) follows. □

LEMMA 2.3. *For $x\geqslant 3$,*

$$x+\frac{3}{4}+\frac{1}{32x}-\frac{3}{128x^2}<\left[\frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)}\right]^2<x+\frac{3}{4}+\frac{1}{32x}. \tag{22}$$

Proof. In [11, Lemma 2.2], it was proved that the first inequality in (22) is valid for all $x\geqslant 2$. We now prove the second inequality in (22) for $x\geqslant 3$. Consider the function $V(x)$ defined for $x>0$ by

$$V(x)=2\ln\Gamma\left(x+\frac{3}{2}\right)-2\ln\Gamma(x+1)-\ln\left(x+\frac{3}{4}+\frac{1}{32x}\right).$$

We conclude from the asymptotic formula (16) that

$$\lim_{x\to\infty}V(x)=0.$$

Differentiating and applying the first inequality in (19), we find that for $x \geqslant 3$,

$$\begin{aligned} V'(x) &= 2\left(\psi\left(x+\frac{3}{2}\right)-\psi(x+1)\right)-\frac{(32x^2-1)}{x(32x^2+24x+1)} \\ &> 2\left(\frac{1}{2x}-\frac{3}{8x^2}+\frac{1}{4x^3}-\frac{9}{64x^4}\right)-\frac{(32x^2-1)}{x(32x^2+24x+1)} \\ &= \frac{39+232(x-3)+72(x-3)^2}{32x^4(32x^2+24x+1)} > 0. \end{aligned}$$

Hence, we have for $x \geqslant 3$,

$$V(x) < \lim_{x\to\infty} V(x) = 0.$$

By rearranging the terms in the last expression, the second inequality in (22) follows. □

From (19), we obtain that for $x \geqslant 1$,

$$\begin{aligned} &(2x+1)\left(-\psi\left(x+\frac{3}{2}\right)+\psi(x+1)\right)+1 \\ &\quad > (2x+1)\left(-\frac{1}{2x}+\frac{3}{8x^2}-\frac{1}{4x^3}+\frac{9}{64x^4}-\frac{1}{16x^5}\right)+1 \\ &\quad = \frac{1}{4x}-\frac{1}{8x^2}+\frac{1}{32x^3}+\frac{1}{64x^4}-\frac{1}{16x^5} \\ &\quad = \frac{7+45(x-1)+74(x-1)^2+56(x-1)^3+16(x-1)^4}{64x^4} > 0 \end{aligned}$$

and

$$\begin{aligned} (2x+1)\left(-\psi\left(x+\frac{3}{2}\right)+\psi(x+1)\right)+1 &< (2x+1)\left(-\frac{1}{2x}+\frac{3}{8x^2}-\frac{1}{4x^3}+\frac{9}{64x^4}\right)+1 \\ &= \frac{1}{4x}-\frac{1}{8x^2}+\frac{1}{32x^3}+\frac{9}{64x^4}, \end{aligned}$$

we then obtain

$$0 < (2x+1)\left(-\psi\left(x+\frac{3}{2}\right)+\psi(x+1)\right)+1 < \frac{1}{4x}-\frac{1}{8x^2}+\frac{1}{32x^3}+\frac{9}{64x^4}, \qquad x \geqslant 1. \tag{23}$$

The proof of Theorem 3.2 makes use of the inequalities (22) and (23).

3. Main results

In view of the second inequality in (6), it is natural to ask: what is the smallest number ϑ_1 and what is the largest number ϑ_2 such that the inequality

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi}+\frac{1}{16\pi(n+\vartheta_1)}} \leqslant \frac{\Omega_{n-1}}{\Omega_n} \leqslant \sqrt{\frac{n+\frac{1}{2}}{2\pi}+\frac{1}{16\pi(n+\vartheta_2)}}$$

holds for all integers $n \geqslant 1$? Theorem 3.1 answers this question.

THEOREM 3.1. *For $n \geqslant 1$,*

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi}+\frac{1}{16\pi(n+\vartheta_1)}} \leqslant \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n+\frac{1}{2}}{2\pi}+\frac{1}{16\pi(n+\vartheta_2)}} \tag{24}$$

with best possible constants

$$\vartheta_1 = \frac{13-4\pi}{4\pi-12} = 0.7656283\ldots \quad \text{and} \quad \vartheta_2 = \frac{1}{2}.$$

Proof. Inequality (24) can be written as

$$\vartheta_1 \geqslant \frac{1}{16\pi\left(\frac{\Omega_{n-1}}{\Omega_n}\right)^2 - 8n - 4} - n > \vartheta_2.$$

We now show that the sequence

$$y_n = \frac{1}{16\pi\left(\frac{\Omega_{n-1}}{\Omega_n}\right)^2 - 8n - 4} - n = \frac{1}{16\left(\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\frac{1}{2})}\right)^2 - 8n - 4} - n$$

is strictly decreasing for $n \geqslant 1$. To this end, we consider the function $g(x)$ defined by

$$g(x) = \frac{1}{16\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right)^2 - 16x - 4} - 2x.$$

Elementary calculations show that

$$-\frac{1}{2}\left(\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right)^2 - x - \frac{1}{4}\right)^2 g'(x)$$
$$= \left(\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right)^2 - x - \frac{1}{4}\right)^2 + \frac{1}{16}\left(\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right)\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right)^2 - \frac{1}{32}.$$

By using inequalities (21) and the first inequality in (18), we obtain that for $x \geqslant 4$,

$$-\frac{1}{2}\left(\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right)^2 - x - \frac{1}{4}\right)^2 g'(x)$$
$$> \left(\frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3}\right)^2$$
$$+ \frac{1}{16}\left(\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6}\right)\left(x + \frac{1}{4} + \frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3}\right) - \frac{1}{32}$$
$$= \frac{125413+281712(x-4)+201058(x-4)^2+64761(x-4)^3+9840(x-4)^4+576(x-4)^5}{4194304x^9}$$
$$> 0.$$

Hence, the function $g(x)$ is strictly decreasing on $[4,\infty)$ and the sequence (x_n) is strictly decreasing for $n \geqslant 8$.

Direct computation would yield

$$\begin{aligned} y_1 &= g\left(\tfrac{1}{2}\right) = 0.76562832\ldots, & y_2 &= g(1) = 0.68938142\ldots, \\ y_3 &= g\left(\tfrac{3}{2}\right) = 0.64519319\ldots, & y_4 &= g(2) = 0.61698646\ldots, \\ y_5 &= g\left(\tfrac{5}{2}\right) = 0.59764076\ldots, & y_6 &= g(3) = 0.58363594\ldots, \\ y_7 &= g\left(\tfrac{7}{2}\right) = 0.57306769\ldots, & y_8 &= g(4) = 0.56482818\ldots. \end{aligned}$$

Consequently, the sequence $(y_n)_{n\geqslant 1}$ is strictly decreasing. This leads to

$$\lim_{n\to\infty} y_n < y_n \leqslant y_1 = \frac{13-4\pi}{4\pi-12} \quad \text{for all} \quad n \geqslant 1.$$

It remains to prove that

$$\lim_{n\to\infty} y_n = \frac{1}{2}. \tag{25}$$

We conclude from the asymptotic formula (15) that

$$g(x) = \frac{1}{2} + O\left(\frac{1}{x}\right) \quad \text{as} \quad x \to \infty,$$

which implies (25). This completes the proof of Theorem 3.1. □

As numerical computations confirm, the ratio $\Omega_n^2/(\Omega_{n-1}\Omega_{n+1})$ in (8) becomes closer to $\left(1+1/(n+1)\right)^{1/2}$, as n approaches infinity. This fact motivated us to pose the following question: what is the smallest number θ_1 and what is the largest number θ_2 such that the inequality

$$\left(1+\frac{1}{n+\theta_1}\right)^{1/2} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leqslant \left(1+\frac{1}{n+\theta_2}\right)^{1/2}$$

holds for all integers $n \geqslant 1$? Theorem 3.2 answers this question.

THEOREM 3.2. *For* $n \geqslant 1$,

$$\left(1+\frac{1}{n+\theta_1}\right)^{1/2} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n+\theta_2}\right)^{1/2} \tag{26}$$

with best possible constants

$$\theta_1 = \frac{2\pi^2-16}{16-\pi^2} = 0.60994576\ldots \quad \textit{and} \quad \theta_2 = \frac{1}{2}.$$

Proof. Inequality (26) can be written as

$$\theta_1 \geqslant \frac{1}{\left(\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right)^2 - 1} - n > \theta_2.$$

We now show that the sequence

$$x_n = \frac{1}{\left(\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right)^2 - 1} - n = \frac{1}{\frac{4}{(n+1)^2}\left(\frac{\Gamma(\frac{n}{2}+\frac{3}{2})}{\Gamma(\frac{n}{2}+1)}\right)^4 - 1} - n$$

is strictly decreasing for $n \geqslant 1$. To this end, we consider the function $f(x)$ defined by

$$f(x) = \frac{1}{\frac{4}{(2x+1)^2}\left(\frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)}\right)^4 - 1} - 2x.$$

Elementary calculations show that

$$\begin{aligned}
&-\frac{1}{2}\left(\frac{4}{(2x+1)^2}\left(\frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)}\right)^4 - 1\right)^2 f'(x)\\
&\quad= \left(\frac{4}{(2x+1)^2}\left(\frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)}\right)^4 - 1\right)^2\\
&\qquad - \frac{8\Big((2x+1)\big(-\psi(x+\frac{3}{2})+\psi(x+\frac{1}{2})\big)+1\Big)}{(2x+1)^3}\left(\frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)}\right)^4.
\end{aligned}$$

By using inequalities (22) and (23), we obtain that for $x \geqslant 12$,

$$\begin{aligned}
&-\frac{1}{2}\left(\frac{4}{(2x+1)^2}\left(\frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)}\right)^4 - 1\right)^2 f'(x)\\
&\quad> \left(\frac{4}{(2x+1)^2}\left(x+\frac{3}{4}+\frac{1}{32x}-\frac{3}{128x^2}\right)^2 - 1\right)^2\\
&\qquad - \frac{8\left(\frac{1}{4x}-\frac{1}{8x^2}+\frac{1}{32x^3}+\frac{9}{64x^4}\right)}{(2x+1)^3}\left(x+\frac{3}{4}+\frac{1}{32x}\right)^2\\
&\quad= \frac{\lambda(x-12)}{16777216x^8(2x+1)^4},
\end{aligned}$$

with

$$\begin{aligned}
\lambda(x) = {}& 45704971569681 + 200145001247568x + 105831697102560x^2\\
&+ 25217680734464x^3 + 3382791477504x^4 + 273737302016x^5\\
&+ 13326352384x^6 + 360972288x^7 + 4194304x^8.
\end{aligned}$$

Hence, $f(x)$ is strictly decreasing on $[12,\infty)$ and the sequence (x_n) is strictly decreasing for $n \geqslant 24$.

Direct computation would yield

$$
\begin{aligned}
x_1 &= f\left(\tfrac{1}{2}\right) = 0.609945759\ldots, & x_2 &= f(1) = 0.577896831\ldots,\\
x_3 &= f\left(\tfrac{3}{2}\right) = 0.559987003\ldots, & x_4 &= f(2) = 0.548648634\ldots,\\
x_5 &= f\left(\tfrac{5}{2}\right) = 0.540861512\ldots, & x_6 &= f(3) = 0.535197735\ldots,\\
x_7 &= f\left(\tfrac{7}{2}\right) = 0.530899584\ldots, & x_8 &= f(4) = 0.527529486\ldots,\\
x_9 &= f\left(\tfrac{9}{2}\right) = 0.524817824\ldots, & x_{10} &= f(5) = 0.522589737\ldots,\\
x_{11} &= f\left(\tfrac{11}{2}\right) = 0.520727001\ldots, & x_{12} &= f(6) = 0.519146891\ldots,\\
x_{13} &= f\left(\tfrac{13}{2}\right) = 0.517789828\ldots, & x_{14} &= f(7) = 0.516611833\ldots,\\
x_{15} &= f\left(\tfrac{15}{2}\right) = 0.515579748\ldots, & x_{16} &= f(8) = 0.514668106\ldots,\\
x_{17} &= f\left(\tfrac{17}{2}\right) = 0.513857031\ldots, & x_{18} &= f(9) = 0.513130781\ldots,\\
x_{19} &= f\left(\tfrac{19}{2}\right) = 0.512476735\ldots, & x_{20} &= f(10) = 0.511884651\ldots,\\
x_{21} &= f\left(\tfrac{21}{2}\right) = 0.511346135\ldots, & x_{22} &= f(11) = 0.510854241\ldots,\\
x_{23} &= f\left(\tfrac{23}{2}\right) = 0.510403173\ldots, & x_{24} &= f(12) = 0.509988057\ldots.
\end{aligned}
$$

Consequently, the sequence $\left(x_n\right)_{n\geqslant 1}$ is strictly decreasing. This leads to

$$\lim_{n\to\infty} x_n < x_n \leqslant x_1 = \frac{2\pi^2-16}{16-\pi^2} \quad \text{for all} \quad n \geqslant 1.$$

It remains to prove that

$$\lim_{n\to\infty} x_n = \frac{1}{2}. \tag{27}$$

We conclude from the asymptotic formula (16) that

$$f(x) = \frac{1}{2} + O\left(\frac{1}{x}\right) \quad \text{as} \quad x \to \infty,$$

which implies (27). This completes the proof of Theorem 3.2. □

REMARK 3.1. The lower bound in (26) is sharper than one in (8) for $n \geqslant 1$; the upper bound in (26) is sharper than one in (8) for $n \geqslant 2$.

Theorem 3.3 improves the inequality (7).

THEOREM 3.3. *For* $n \geqslant 1$,

$$\left(1+\frac{1}{n+1}\right)^{\frac{1}{2}+\frac{1}{4n}-\frac{3}{8n^2}} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n+1}\right)^{\frac{1}{2}+\frac{1}{4n}}. \tag{28}$$

Proof. In order to prove the first inequality in (28), it suffices to show that the sequence

$$y_n = \ln 2 + 2\ln\Gamma\left(\frac{n+3}{2}\right) - 2\ln\Gamma\left(\frac{n+2}{2}\right) - \ln(n+1) - \left(\frac{1}{2}+\frac{1}{4n}-\frac{3}{8n^2}\right)\ln\left(\frac{n+2}{n+1}\right)$$

is positive for $n \geqslant 1$.

We consider the function $F(x)$ defined by

$$F(x) = \ln 2 + 2\ln\Gamma\left(x+\tfrac{3}{2}\right) - 2\ln\Gamma(x+1) - \ln(2x+1) - \left(\frac{1}{2} + \frac{1}{8x} - \frac{3}{32x^2}\right)\ln\left(\frac{2x+2}{2x+1}\right).$$

We conclude from the asymptotic formula (16) that

$$\lim_{x\to\infty} F(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} y_n = 0.$$

It is not difficult to show that

$$\frac{1}{2x} - \frac{3}{8x^2} < \ln\left(\frac{2x+2}{2x+1}\right) < \frac{1}{2x} - \frac{3}{8x^2} + \frac{7}{24x^3}, \qquad x > 0. \tag{29}$$

Differentiating $F(x)$ and applying inequalities (19) and (29), we obtain that for $x \geqslant 3/2$,

$$\begin{aligned} F'(x) &= 2\left(\psi\left(x+\frac{3}{2}\right) - \psi(x+1)\right) + \left(\frac{2x-3}{16x^3}\right)\ln\left(\frac{2x+2}{2x+1}\right) - \frac{64x^3+48x^2-4x+3}{32x^2(2x+1)(x+1)} \\ &< 2\left(\frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} + \frac{1}{16x^5}\right) + \left(\frac{2x-3}{16x^3}\right)\left(\frac{1}{2x} - \frac{3}{8x^2} + \frac{7}{24x^3}\right) \\ &\quad - \frac{64x^3+48x^2-4x+3}{32x^2(2x+1)(x+1)} \\ &= -\frac{24+124(x-1)+213(x-1)^2+92(x-1)^3}{384x^6(2x+1)(x+1)} < 0. \end{aligned}$$

Hence, $F(x)$ is strictly decreasing on $[3/2,\infty)$, and therefore, the sequence (y_n) is strictly decreasing for $n \geqslant 3$.

Direct computation would yield

$$\begin{aligned} y_1 &= F\left(\frac{1}{2}\right) = -\frac{3}{8}\ln 3 + \frac{19}{8}\ln 2 - \ln\pi = 0.089515\ldots, \\ y_2 &= F(1) = -\frac{65}{16}\ln 2 + \frac{49}{32}\ln 3 + \ln\pi = 0.0110695\ldots, \\ y_3 &= F\left(\frac{3}{2}\right) = \frac{73}{12}\ln 2 - \frac{13}{24}\ln 5 - 2\ln 3 - \ln\pi = 0.002912\ldots. \end{aligned}$$

Consequently, the sequence $(y_n)_{n\geqslant 1}$ is strictly decreasing. This leads to

$$y_n > \lim_{n\to\infty} y_n = 0, \quad n \geqslant 1.$$

In order to prove the second inequality in (28), it suffices to show that the sequence

$$z_n = \ln 2 + 2\ln\Gamma\left(\frac{n+3}{2}\right) - 2\ln\Gamma\left(\frac{n+2}{2}\right) - \ln(n+1) - \left(\frac{1}{2} + \frac{1}{4n}\right)\ln\left(\frac{n+2}{n+1}\right)$$

is negative for $n \geqslant 1$.

We consider the function $G(x)$ defined by

$$G(x)=\ln 2+2\ln\Gamma\left(x+\frac{3}{2}\right)-2\ln\Gamma(x+1)-\ln(2x+1)-\left(\frac{1}{2}+\frac{1}{8x}\right)\ln\left(\frac{2x+2}{2x+1}\right).$$

We conclude from the asymptotic formula (16) that

$$\lim_{x\to\infty}G(x)=0 \quad\text{and}\quad \lim_{n\to\infty}z_n=0.$$

Differentiating $G(x)$ and applying inequalities (19) and (29), we obtain that for $x\geqslant 3$,

$$\begin{aligned}G'(x)&=2\left(\psi\left(x+\frac{3}{2}\right)-\psi(x+1)\right)+\frac{1}{8x^2}\ln\left(\frac{2x+2}{2x+1}\right)-\frac{16x^2+12x-1}{8x(2x+1)(x+1)}\\&>2\left(\frac{1}{2x}-\frac{3}{8x^2}+\frac{1}{4x^3}-\frac{9}{64x^4}\right)+\frac{1}{8x^2}\left(\frac{1}{2x}-\frac{3}{8x^2}\right)-\frac{16x^2+12x-1}{8x(2x+1)(x+1)}\\&=\frac{3\left(20+27(x-3)+6(x-3)^2\right)}{64x^4(2x+1)(x+1)}>0.\end{aligned}$$

Hence, $G(x)$ is strictly increasing on $[3,\infty)$, and therefore, the sequence (z_n) is strictly increasing for $n\geqslant 6$.

Direct computation would yield

$$\begin{aligned}z_1&=G\left(\tfrac{1}{2}\right)=-0.0625343\ldots, & z_2&=G(1)=-0.0159006\ldots,\\z_3&=G\left(\tfrac{3}{2}\right)=-0.0063856\ldots, & z_4&=G(2)=-0.0031937\ldots,\\z_5&=G\left(\tfrac{5}{2}\right)=-0.0018234\ldots, & z_6&=G(3)=-0.0011382\ldots.\end{aligned}$$

Consequently, the sequence $(z_n)_{n\geqslant 1}$ is strictly increasing. This leads to

$$z_n<\lim_{n\to\infty}z_n=0,\quad n\geqslant 1.$$

The proof of Theorem 3.3 is thus completed. □

REMARK 3.2. The lower bound in (28) is sharper than one in (7) for $n\geqslant 2$, and the upper bound in (28) is better than one in (7) for all $n\geqslant 1$.

Theorem 3.4 provides new bounds for $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$.

THEOREM 3.4. *For $n\geqslant 1$,*

$$1+\frac{1}{2n}-\frac{3}{8n^2}<\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}<1+\frac{1}{2n}-\frac{3}{8n^2}+\frac{3}{16n^3}+\frac{3}{128n^4}. \tag{30}$$

Proof. Inequality (30) can be written as

$$1+\frac{1}{2n}-\frac{3}{8n^2}<\frac{2\left[\Gamma\left(\frac{n+3}{2}\right)\right]^2}{(n+1)\left[\Gamma\left(\frac{n}{2}+1\right)\right]^2}<1+\frac{1}{2n}-\frac{3}{8n^2}+\frac{3}{16n^3}+\frac{3}{128n^4}. \tag{31}$$

The lower bound in (31) is obtained by considering the function $J(x)$ defined for $x>0$ by

$$J(x)=\ln 2+2\ln\Gamma\left(x+\frac{3}{2}\right)-\ln(2x+1)-2\ln\Gamma(x+1)-\ln\left(1+\frac{1}{4x}-\frac{3}{8(2x)^2}\right).$$

We conclude from the asymptotic formula (16) that

$$\lim_{x\to\infty}J(x)=0.$$

Differentiating and using the second inequality in (19), we find that for $x\geqslant 1$,

$$\begin{aligned}J'(x)&=2\left(\psi\left(x+\frac{3}{2}\right)-\psi(x+1)\right)-\frac{2(32x^3-x+3)}{x(2x+1)(32x^2+8x-3)}\\&<2\left(\frac{1}{2x}-\frac{3}{8x^2}+\frac{1}{4x^3}-\frac{9}{64x^4}+\frac{1}{16x^5}\right)-\frac{2(32x^3-x+3)}{x(2x+1)(32x^2+8x-3)}\\&=-\frac{67+505(x-1)+954(x-1)^2+648(x-1)^3+144(x-1)^4}{32x^5(2x+1)(32x^2+8x-3)}<0.\end{aligned}$$

Hence, $J(x)$ is strictly decreasing on $[1,\infty)$, and therefore, the sequence $\left\{J\left(\frac{n}{2}\right)\right\}_{n=2}^{\infty}$ is strictly decreasing.

Direct computation would yield

$$\begin{aligned}J\left(\tfrac{1}{2}\right)&=5\ln 2-\ln\pi-2\ln 3=0.1237814\ldots,\\J(1)&=2\ln 2+\ln 3+\ln\pi-\ln 37=0.0187186\ldots.\end{aligned}$$

Consequently, the sequence $\left\{J\left(\frac{n}{2}\right)\right\}_{n=1}^{\infty}$ is strictly decreasing. This leads to

$$J\left(\tfrac{n}{2}\right)=\ln\left(\frac{2\left[\Gamma\left(\frac{n+3}{2}\right)\right]^2}{(n+1)\left[\Gamma\left(\frac{n}{2}+1\right)\right]^2}\right)-\ln\left(1+\frac{1}{2n}-\frac{3}{8n^2}\right)>\lim_{n\to\infty}J\left(\tfrac{n}{2}\right)=0$$

for $n\geqslant 1$. By rearranging the terms in the last expression, the first inequality in (31) follows.

The upper bound in (31) is obtained by considering the function $K(x)$ defined for $x>0$ by

$$\begin{aligned}K(x)=\ln 2+2\ln\Gamma\left(x+\frac{3}{2}\right)-\ln(2x+1)-2\ln\Gamma(x+1)\\-\ln\left(1+\frac{1}{4x}-\frac{3}{8(2x)^2}+\frac{3}{16(2x)^3}+\frac{3}{128(2x)^4}\right).\end{aligned}$$

We conclude from the asymptotic formula (16) that

$$\lim_{x\to\infty}K(x)=0.$$

Differentiating and using (20), we find that for $x \geqslant \frac{3}{2}$,

$$
\begin{aligned}
K'(x) &= 2\left(\psi\left(x+\frac{3}{2}\right) - \psi(x+1)\right) - \frac{2(2048x^5-64x^3+96x^2-81x-6)}{x(2x+1)(2048x^4+512x^3-192x^2+48x+3)} \\
&> 2\left(\frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} + \frac{1}{16x^5} - \frac{3}{128x^6}\right) \\
&\quad - \frac{2(2048x^5-64x^3+96x^2-81x-6)}{x(2x+1)(2048x^4+512x^3-192x^2+48x+3)} \\
&= \frac{9828+55941(x-\frac{3}{2})+103860(x-\frac{3}{2})^2+87492(x-\frac{3}{2})^3+34720(x-\frac{3}{2})^4+5280(x-\frac{3}{2})^5}{64x^6(2x+1)(2048x^4+512x^3-192x^2+48x+3)} \\
&> 0.
\end{aligned}
$$

Hence, $K(x)$ is strictly increasing on $[\frac{3}{2}, \infty)$, and therefore, the sequence $\left\{K\left(\frac{n}{2}\right)\right\}_{n=3}^{\infty}$ is strictly increasing.

Direct computation would yield

$$
\begin{aligned}
K\left(\tfrac{1}{2}\right) &= 9\ln 2 - \ln\pi - 2\ln 3 - \ln 19 = -0.0480688\ldots, \\
K(1) &= 8\ln 2 + \ln 3 + \ln\pi - \ln 41 - \ln 59 = -0.0025898\ldots, \\
K\left(\tfrac{3}{2}\right) &= 12\ln 2 + \ln 3 - \ln\pi - \ln 7 - \ln 13 - \ln 43 = -0.0004110\ldots.
\end{aligned}
$$

Consequently, the sequence $\left\{K\left(\frac{n}{2}\right)\right\}_{n=1}^{\infty}$ is strictly increasing. This leads to

$$
\begin{aligned}
K\left(\tfrac{n}{2}\right) &= \ln\left(\frac{2\left[\Gamma\left(\frac{n+3}{2}\right)\right]^2}{(n+1)\left[\Gamma\left(\frac{n}{2}+1\right)\right]^2}\right) - \ln\left(1 + \frac{1}{2n} - \frac{3}{8n^2} + \frac{3}{16n^3} + \frac{3}{128n^4}\right) \\
&< \lim_{n\to\infty} K\left(\tfrac{n}{2}\right) = 0, \qquad n \geqslant 1.
\end{aligned}
$$

By rearranging the terms in the last expression, the second inequality in (31) follows. This completes the proof of Theorem 3.4. □

Theorem 3.5 provides new bounds for Ω_n.

THEOREM 3.5. *For* $n \geqslant 1$,

$$
\sqrt{\frac{1}{\pi}\left(\frac{1}{n} - \frac{1}{3n^2}\right)}\left(\frac{2\pi e}{n}\right)^{n/2} < \Omega_n < \sqrt{\frac{1}{\pi}\left(\frac{1}{n} - \frac{1}{3n^2} + \frac{1}{18n^3} + \frac{31}{810n^4}\right)}\left(\frac{2\pi e}{n}\right)^{n/2}. \tag{32}
$$

Proof. The lower bound is obtained by considering the function $C(x)$ defined for $x > 0$ by

$$
C(x) = x\ln\pi - \ln\Gamma(x+1) - \left[\frac{1}{2}\ln\left(\frac{1}{\pi}\right) + \frac{1}{2}\ln\left(\frac{1}{2x} - \frac{1}{3(2x)^2}\right) + x(\ln\pi + 1 - \ln x)\right].
$$

We conclude from the asymptotic formula (12) that

$$
\lim_{x\to\infty} C(x) = 0.
$$

It follows from the known result (see [2, Theorem 8]) that, for $x>0$,

$$\ln x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^6}<\psi(x)<\ln x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}. \qquad (33)$$

Differentiating and applying the first inequality in (33), we find that for $x\geqslant\frac{1}{2}$,

$$\begin{aligned} C'(x)&=\ln x-\psi(x)-\frac{1}{x}+\frac{3x-1}{x(6x-1)}\\ &<-\frac{1}{2x}+\frac{1}{12x^2}-\frac{1}{120x^4}+\frac{1}{252x^6}+\frac{3x-1}{x(6x-1)}\\ &=-\frac{\frac{29}{8}+\frac{237}{2}(x-\frac{1}{2})+483(x-\frac{1}{2})^2+546(x-\frac{1}{2})^3+210(x-\frac{1}{2})^4}{2520x^6(6x-1)}<0. \end{aligned}$$

Hence, $C(x)$ is strictly decreasing for $x\geqslant\frac{1}{2}$, and we have for $n\geqslant 1$,

$$C\left(\tfrac{n}{2}\right)=\ln\left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\right)-\ln\left[\sqrt{\frac{1}{\pi}\left(\frac{1}{n}-\frac{1}{3n^2}\right)}\left(\frac{2\pi e}{n}\right)^{n/2}\right]>\lim_{n\to\infty}C\left(\tfrac{n}{2}\right)=0.$$

By rearranging the terms in the last expression, the first inequality in (32) follows.

The upper bound is obtained by considering the function $D(x)$ defined for $x>0$ by

$$\begin{aligned} D(x)&=x\ln\pi-\ln\Gamma(x+1)\\ &\quad-\left[\frac{1}{2}\ln\left(\frac{1}{\pi}\right)+\frac{1}{2}\ln\left(\frac{1}{2x}-\frac{1}{3(2x)^2}+\frac{1}{18(2x)^3}+\frac{31}{810(2x)^4}\right)+x(\ln\pi+1-\ln x)\right]. \end{aligned}$$

We conclude from the asymptotic formula (12) that

$$\lim_{x\to\infty}D(x)=0.$$

Differentiating and applying the second inequality in (33), we find that for $x\geqslant\frac{1}{2}$,

$$\begin{aligned} D'(x)&=\ln x-\psi(x)-\frac{1}{x}+\frac{3240x^3-1080x^2+135x+62}{x(6480x^3-1080x^2+90x+31)}\\ &>-\frac{1}{2x}+\frac{1}{12x^2}-\frac{1}{120x^4}+\frac{3240x^3-1080x^2+135x+62}{x(6480x^3-1080x^2+90x+31)}\\ &=\frac{\frac{543}{2}+1300(x-\frac{1}{2})+1390(x-\frac{1}{2})^2}{120x^4\left[616+3870(x-\frac{1}{2})+8640(x-\frac{1}{2})^2+6480(x-\frac{1}{2})^3\right]}>0. \end{aligned}$$

Hence, $D(x)$ is strictly increasing for $x\geqslant\frac{1}{2}$, and we have for $n\geqslant 1$,

$$\begin{aligned} D\left(\tfrac{n}{2}\right)&=\ln\left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\right)-\ln\left[\sqrt{\frac{1}{\pi}\left(\frac{1}{n}-\frac{1}{3n^2}+\frac{1}{18n^3}+\frac{31}{810n^4}\right)}\left(\frac{2\pi e}{n}\right)^{n/2}\right]\\ &<\lim_{n\to\infty}D\left(\tfrac{n}{2}\right)=0. \end{aligned}$$

By rearranging the terms in the last expression, the second inequality in (32) follows. □

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Dover, New York, 1972.

[2] H. ALZER, *On some inequalities for the gamma and psi functions*, Math. Comput. **66** (1997), 373–389.

[3] H. ALZER, *Inequalities for the volume of the unit ball in* $\mathbb{R}^n$, J. Math. Anal. Appl. **252** (2000), 353–363.

[4] H. ALZER, *Inequalities for the volume of the unit ball in* $\mathbb{R}^n$*, II*, Mediterr. J. Math. **5** (2008), 395–413.

[5] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Special functions of quasiconformal theory*, Exposition. Math. **7** (1989), 97–136.

[6] G. D. ANDERSON AND S.-L. QIU, *A monotoneity property of the gamma function*, Proc. Amer. Math. Soc. **125** (1997), 3355–3362.

[7] J. BÖHM AND E. HERTEL, *Polyedergeometrie in* n*-dimensionalen Räumen konstanter Krmmung*, Birkhäuser, Basel, 1981.

[8] K. H. BORGWARDT, *The Simplex Method*, Springer, Berlin, 1987.

[9] C.-P. CHEN AND R. B. PARIS, *Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function*, Appl. Math. Comput. **250** (2015), 514–529.

[10] C.-P. CHEN AND R. B. PARIS, *Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function*, Appl. Math. Comput. **250** (2015), 514–529.

[11] C.-P. CHEN AND L. LIN, *Inequalities for the volume of the unit ball in* $\mathbb{R}^n$, Mediterr. J. Math. **11** (2014), 299–314.

[12] B.-N. GUO AND F. QI, *Monotonicity and logarithmic convexity relating to the volume of the unit ball*, Optim. Lett. **7** (2013), 1139–1153.

[13] D. A. KLAIN AND G.-C. ROTA, *A continuous analogue of Sperner's theorem*, Comm. Pure Appl. Math. **50** (1997), 205–223.

[14] L. LIN, *Further refinements of Gurland's formula for* π, J. Inequal. Appl. **2013**, 2013: 48, doi:10.1186/1029-242X-2013-48.

[15] Y. L. LUKE, *The Special Functions and their Approximations*, vol. I, Academic Press, New York, 1969.

[16] M. MERKLE, *Gurland's ratio for the gamma function*, Comp. Math. Appl. **49** (2005), 389–406.

[17] C. MORTICI, *Monotonicity properties of the volume of the unit ball in* $\mathbb{R}^n$, Optim. Lett. **4** (2010), 457–464.

[18] C. MORTICI, *Series associated to some expressions involving the volume of the unit ball and applications*, Appl. Math. Comput. **294** (2017), 121–138.

[19] S.-L. QIU AND M. VUORINEN, *Some properties of the gamma and psi functions, with applications*, Math. Comput. **74** (2004), 723–742.

(Received February 26, 2016)

Tao Ban
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000, Henan Province, China
e-mail: bantao454000@sohu.com

Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454003, Henan Province, People's Republic of China
e-mail: chenchaoping@sohu.com