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WEIGHTED INTEGRAL INEQUALITY FOR THE SECOND DERIVATIVE OF 4–CONVEX FUNCTIONS

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Abstract. In this paper, we establish an energy estimate for the second derivative of 4-convex functions. Such kinds of estimates for the first derivative of 2-convex (convex) functions were obtained by Hussain, Pečarić and Shashivili [2].

1. Indroduction

Let f be a real valued function defined on a closed and bounded interval $[a,b]$, and let $x_0 < x_1 < \ldots < x_n$ be points in $[a,b]$. A nth order divided difference of f at these points is defined recursively as

$$[x_i]f = f(x_i) \quad (0 \leqslant i \leqslant n)$$

and

$$[x_0,x_1,\ldots,x_n]f = \frac{[x_1,x_2,\ldots,x_n]f - [x_0,x_1,\ldots,x_{n-1}]f}{x_n - x_0}.$$

The function f is called n-convex in $[a,b]$ if we have

$$[x_0,x_1,\ldots,x_n]f \geqslant 0$$

for all $a \leqslant x_0 < x_1 < \ldots < x_n \leqslant b$. We say f is n-concave if $-f$ is n-convex. We recall some facts from the theory of n-convex functions (for the proofs see, for example, [6, 5]). 1-convex function is simply non-decreasing and a 2-convex function is convex in the usual sense. If $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geqslant 0$. For a n-convex function, $f^{(k)}$ exists and is $(n-k)$-convex for $2 \leqslant k \leqslant n$.

The weighted integral inequalities for convex functions have applications in financial mathematics (see [3]). Therefore, it will be of interest to study these inequalities for functions having higher order convexity. In this paper, we derive a weighted integral inequality for the difference of two arbitrary 4-convex functions, not necessarily smooth. Similar weighted inequalities for convex functions have been obtained in [4, 2].

The paper is organized as follows. In the next section, we derive the inequality in a simpler case when functions are smooth, and then making use of approximation of general 4-functions by smooth 4-convex functions, we prove the inequality for general 4-convex functions in the second section.

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2. The case of smooth 4-convex functions

THEOREM 2.1. *Let f, $F \in C^4[a,b]$ be such that the following condition*

$$\left.\begin{aligned} |f''(x)| \leqslant F''(x) \qquad \forall\, x \in (a,b) \\ |f^{(iv)}(x)| \leqslant F^{(iv)}(x) \qquad \forall\, x \in (a,b) \end{aligned}\right\} \tag{2.1}$$

is fulfilled. Let h be a non-negative 2-concave function in $C^4[a,b]$ satisfying

$$h(x) = h'(x) = h''(x) = h'''(x) = 0, \quad x \in \{a,b\}. \tag{2.2}$$

Then the following energy estimate is valid:

$$\int_a^b |f''(x)|^2 h(x)dx \leqslant \int_a^b \left(\frac{(f(x))^2}{2} - \|f\|_{L^\infty} F(x)\right) h^{(iv)}(x)dx. \tag{2.3}$$

Proof. Put

$$I = \int_a^b |f''(x)|^2 h(x)dx.$$

Then write

$$I = \int_a^b f''(x)(f''(x)h(x))dx,$$

and integrate by parts to get

$$I = -\int_a^b f'(x)f'''(x)h(x)dx - \int_a^b f'(x)f''(x)h'(x)dx.$$

Note here that we have made use of the condition (2.2) on the weight function h. Again using integration by parts formula on first integral of right hand side, we obtain that

$$\begin{aligned} I = &\int_a^b f(x)f^{(iv)}(x)h(x)dx + \int_a^b f(x)f'''(x)h'(x)dx \\ &- \int_a^b f'(x)f''(x)h'(x)dx, \end{aligned}$$

next using integration by parts formula on the middle integral yields

$$\begin{aligned} I &= \int_a^b f(x)f^{(iv)}(x)h(x)dx - 2\int_a^b f'(x)f''(x)h'(x)dx \\ &\quad - \int_a^b f(x)f''(x)h''(x)dx \\ &= \int_a^b f(x)f^{(iv)}(x)h(x)dx - \int_a^b [(f'(x))^2]'h'(x)dx \\ &\quad - \int_a^b f(x)f''(x)h''(x)dx, \end{aligned}$$

once again using integration by parts formula on the middle integral, we achieve

$$\begin{aligned} I &= \int_a^b f(x)f^{(iv)}(x)h(x)dx + \int_a^b (f'(x))^2 h''(x)dx \\ &\quad - \int_a^b f(x)f''(x)h''(x)dx \\ &= \int_a^b f(x)f^{(iv)}(x)h(x)dx + \int_a^b f'(x)\left(f'(x)h''(x)\right)dx \\ &\quad - \int_a^b f(x)f''(x)h''(x)dx, \end{aligned}$$

at this point applying the integration by parts formula on the middle integral will lead us to

$$\begin{aligned} I &= \int_a^b f(x)f^{(iv)}(x)h(x)dx - 2\int_a^b f(x)f''(x)h''(x)dx \\ &\quad - \frac{1}{2}\int_a^b [f^2(x)]'h'''(x)dx, \end{aligned}$$

next we use integration by parts formula on the last integral to obtain

$$\begin{aligned} I &= \int_a^b f(x)f^{(iv)}(x)h(x)dx - 2\int_a^b f(x)f''(x)h''(x)dx \\ &\quad + \frac{1}{2}\int_a^b f^2(x)h^{(iv)}(x)dx, \end{aligned}$$

whence we get

$$\begin{aligned} I \leqslant & \sup_{a\leqslant x\leqslant b}|f(x)|\int_a^b |f^{(iv)}(x)|h(x)dx \\ & + 2\sup_{a\leqslant x\leqslant b}|f(x)|\int_a^b |f''(x)||h''(x)|dx \\ & + \frac{1}{2}\int_a^b f^2(x)h^{(iv)}(x)dx, \end{aligned}$$

which further turns into

$$\begin{aligned} I \leqslant & \sup_{a\leqslant x\leqslant b}|f(x)|\int_a^b F^{(iv)}(x)h(x)dx \\ & - 2\sup_{a\leqslant x\leqslant b}|f(x)|\int_a^b F''(x)h''(x)dx \\ & + \frac{1}{2}\int_a^b f^2(x)h^{(iv)}(x)dx, \end{aligned}$$

in view of (2.1) and the fact that h is 2-concave. Finally, using integration by parts formula four times on the first integral and twice on the second integral, we get

$$\begin{aligned} I \leqslant & \sup_{a\leqslant x\leqslant b} |f(x)| \int_a^b F(x)h^{(iv)}(x)dx \\ & -2 \sup_{a\leqslant x\leqslant b} |f(x)| \int_a^b F(x)h^{(iv)}(x)dx \\ & +\frac{1}{2}\int_a^b f^2(x)h^{(iv)}(x)dx, \end{aligned}$$

from which follows the desired inequality (2.3). □

The following weighted energy inequality for the smooth 4-convex functions simply follows from the previous theorem by taking $F = f$.

COROLLARY 2.2. *Let $f \in C^4[a,b]$ be 4-convex as well as 2-convex, and let h be the same as in the previous theorem. Then the following energy estimate is valid:*

$$\int_a^b |f''(x)|^2 h(x)dx \leqslant \int_a^b \left(\frac{(f(x))^2}{2} - \|f\|_{L^\infty} f(x) \right) h^{(iv)}(x)dx. \tag{2.4}$$

The next result describes the energy estimates for the difference of two 4-convex functions.

COROLLARY 2.3. *Let $f_0, f_1 \in C^4[a,b]$ be both 4-convex as well as 2-convex functions. Let the weight function h satisfies the conditions of Theorem 2.1. Then we have*

$$\int_a^b |f_2''(x)-f_1''(x)|^2 h(x)dx \leqslant \left(\frac{1}{2}\|f_2-f_1\|_{L^\infty}^2 + \|f_2-f_1\|_{L^\infty} (\|f_1\|_{L^\infty}+\|f_2\|_{L^\infty}) \right) \|h^{(iv)}\|_{L^1}.$$

Proof. Apply Theorem 2.1 to $f = f_2 - f_1$ and $F = f_1 + f_2$ to get

$$\begin{aligned} \int_a^b |f_2''(x) - f_1''(x)|^2 h(x)dx \leqslant \int_a^b \Bigg(& \frac{(f_2(x) - f_1(x))^2}{2} \\ & - \|f_2 - f_1\|_{L^\infty} (f_2(x) + f_1(x)) \Bigg) h^{(iv)}(x)dx, \end{aligned}$$

whence we get the desired inequality. □

We conclude this section with the following remark.

REMARK 2.4. Let f_0, f_1 and h be the same as in the previous corollary. Then using Hölder inequality, we obtain that

$$\int_a^b |f_2''(x) - f_1''(x)|^2 h(x)dx \leqslant \|\tilde{f}\|_{L^p} \|h^{(iv)}\|_{L^q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\tilde{f}(x) = \frac{(f_2(x) - f_1(x))^2}{2} - \|f_2 - f_1\|_{L^\infty} (f_1(x) + f_2(x)).$$

3. The case of arbitrary 4-convex functions

First we define the mollification of arbitrary 4-convex function in $[a,b]$. The mollification of arbitrary function is very well explained in the book by L. C. Evans [1]. Let f be an arbitrary 4-convex function and also 2-convex function. Then by the property of the differentiability of the 4-convex function, $f \in C^2[a,b]$. Let $\theta_\varepsilon \in C^\infty(\mathbb{R})$ have support on interval $I_\varepsilon = I(x_0, r_\varepsilon)$. The θ is called approximation identity or mollifier. Take

$$\theta_\varepsilon(x) = \begin{cases} C exp\frac{1}{x^2-1}, & x \leqslant 0, \\ 0, & x > 0, \end{cases}$$

where C is a constant such that

$$\int_{\mathbb{R}} \theta_\varepsilon(x)dx = 1.$$

Now using θ_ε as a kernal, we define the convolution of f as

$$f_\varepsilon(x) = \int_{\mathbb{R}} f(x-y)\theta_\varepsilon(y)dy = \int_{\mathbb{R}} f(y)\theta_\varepsilon(x-y)dy$$

Since $\theta\varepsilon \in C^\infty(\mathbb{R})$, so $f_\varepsilon \in C^\infty(\mathbb{R})$.

If f is continous, then f_ε converges uniformly to f in any compact subset $K \subseteq I$.

$$|f_\varepsilon - f| \underset{\varepsilon\to 0}{\longrightarrow} 0$$

if $\varepsilon = \frac{1}{m}$ then $|f_m - f| \underset{m\to\infty}{\longrightarrow} 0$.

Since

$$\begin{aligned} f_\varepsilon(\lambda x_1 + (1-\lambda)x_2) &= \int_{\mathbb{R}} f(\lambda x_1 + (1-\lambda)x_2 - y)\,\theta_\varepsilon(y)dy \\ &= \int_{\mathbb{R}} f(\lambda(x_1-y) + (1-\lambda)(x_2-y))\,\theta_\varepsilon y dy \\ &\leqslant \int_{\mathbb{R}} [\lambda f(x_1-y) + (1-\lambda)f(x_2-y)]\,\theta_\varepsilon y dy \\ &= \int_{\mathbb{R}} \lambda f(x_1-y)\theta\varepsilon y dy + \int_{\mathbb{R}} (1-\lambda)f(x_2-y)\theta\varepsilon y dy \\ &= \lambda f_\varepsilon(x_1) + (1-\lambda) f_\varepsilon(x_2), \end{aligned}$$

so f_ε is 2-convex. Similarly, the 2-convexity of $f^{(2)}$ yields the 2-convexity of $f_\varepsilon^{(2)}$. Therefore, f_ε is 4-convex.

THEOREM 3.1. *Let f_0, f_1 be both 4-convex as well as 2-convex functions. Let the weight function h satisfies the conditions of Theorem 2.1. Then we have*

$$\int_a^b |f_2''(x) - f_1''(x)|^2 h(x)dx \leqslant \left(\frac{1}{2}\|f_2 - f_1\|_{L^\infty}^2 + \|f_2 - f_1\|_{L^\infty}(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty})\right)\|h^{(iv)}\|_{L^1}.$$

Proof. Let $I = [a,b]$, and choose an increasing sequence (I_k) of subintervals of I such that $\cup I_k = I$. Consider the smooth approximations $f_{m,i}(x)$, $i = 1,2$. Then, for the interval I_{k+l} there exist an integer m_{k+l} such that $f_{m,i}(x)$ converges uniformly to $f_i(x)$, $i = 1,2$. and also $f_{m,i}(x)$ is smooth for $m \geqslant m_{k+l}$.

Now using the Corollary 2.3 for the functions $f_{m,1}$ and $f_{m,2}$ over interval I_{k+l}, we get

$$\int_{I_{k+l}} \left|f_{m,2}''(x) - f_{m,1}''(x)\right|^2 h_{k+l}(x)dx$$
$$\leqslant c_{k+l}\left[\frac{1}{2}\|f_{m,2} - f_{m,1}\|_{L^\infty}^2 + \|f_{m,2} - f_{m,1}\|_{L^\infty}\left(\|f_{m,1}\|_{L^\infty} + \|f_{m,2}\|_{L^\infty}\right)\right],$$

where $c_{k+l} = \int_{I_{k+l}} \left|h_{k+l}^{(iv)}(x)\right| dx$. Now taking limit $m \to \infty$, we obtain

$$\int_{I_{k+l}} \left|f_2''(x) - f_1''(x)\right|^2 h_{k+l}(x)dx$$
$$\leqslant c_{k+l}\left[\frac{1}{2}\|f_2 - f_1\|_{L^\infty(I_{k+l})}^2 + \|f_2 - f_1\|_{L^\infty(I_{k+l})} \times \left(\|f_1\|_{L^\infty(I_{k+l})} + \|f_2\|_{L^\infty(I_{k+l})}\right)\right].$$

Now writing left hand integral for the smaller interval $I_k \subset I_{k+l}$ and also taking limit $l \to \infty$ we obtain

$$\int_{I_k} \left|f_2''(x) - f_1''(x)\right|^2 h(x)dx$$
$$\leqslant c_\infty\left[\frac{1}{2}\|f_2 - f_1\|_{L^\infty(I)}^2 + \|f_2 - f_1\|_{L^\infty(I)}\left(\|f_1\|_{L^\infty(I)} + \|f_2\|_{L^\infty(I)}\right)\right].$$

Since we have

$$\int_I \left|f_i''(x)\right|^2 h(x)dx < \infty \;\; i = 1,2,$$

Taking limit as $k \to \infty$ we obtain the required result. □

REMARK 3.2. The previous theorem implies that if two 4-convex functions are closed in L^∞-norm then their second derivatives are also closed in weighted L^2-norm.

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