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ENDPOINT ESTIMATES FOR COMMUTATORS OF SUBLINEAR OPERATORS IN THE MORREY–TYPE SPACES

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Abstract. Let $[b,\mathscr{T}_\alpha]$ $(0\leqslant\alpha<n)$ be the commutators generated by $BMO(\mathbb{R}^n)$ functions and a class of sublinear operators satisfying certain size conditions. The aim of this paper is to study the endpoint estimates of these commutators on the weighted Morrey spaces and the generalized Morrey spaces, under the assumptions that $[b,\mathscr{T}_\alpha]$ $(0\leqslant\alpha<n)$ satisfy (weighted or unweighted) endpoint inequalities on $\mathbb{R}^n$ or on bounded domains. Furthermore, as applications of our main results, we will obtain, in the endpoint case, the boundedness properties of many important operators in classical harmonic analysis on the weighted Morrey and the generalized Morrey spaces.

1. Introduction and main results

Suppose that $\mathscr{T}$ represents a linear or a sublinear operator, which satisfies that for any $f\in L^1(\mathbb{R}^n)$ with compact support and $x\notin \operatorname{supp} f$,

$$\big|\mathscr{T}f(x)\big|\leqslant c_0\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}dy, \tag{1.1}$$

where c_0 is a universal constant independent of f and $x\in\mathbb{R}^n$. The condition (1.1) was first introduced by Soria and Weiss in [28]. It can be proved that (1.1) is satisfied by many integral operators in Harmonic Analysis, such as the Hardy–Littlewood maximal operator, Calderón–Zygmund singular integral operators, Carleson's maximal operator, Ricci–Stein's oscillatory singular integrals and Bochner–Riesz means at the critical index and so on.

Similarly, for given $0<\alpha<n$, we assume that $\mathscr{T}_\alpha$ represents a linear or a sublinear operator with order α, which satisfies that for any $f\in L^1(\mathbb{R}^n)$ with compact support and $x\notin \operatorname{supp} f$,

$$\big|\mathscr{T}_\alpha f(x)\big|\leqslant c_1\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy, \tag{1.2}$$

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where c_1 is also a universal constant independent of f and $x \in \mathbb{R}^n$. It can be easily checked that (1.2) is satisfied by some important operators such as the fractional maximal operator, Riesz potential operators and fractional oscillatory singular integrals and so on.

Let b be a locally integrable function on $\mathbb{R}^n$, suppose that the commutator operator $[b,\mathscr{T}]$ stands for a linear or a sublinear operator, which satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$\big|[b,\mathscr{T}](f)(x)\big| \leqslant c_2 \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|\cdot|f(y)|}{|x-y|^n}\,dy, \tag{1.3}$$

where c_2 is an absolute constant independent of f and $x \in \mathbb{R}^n$. Similarly, for given $0<\alpha<n$, we assume that the commutator operator $[b,\mathscr{T}_\alpha]$ stands for a linear or a sublinear operator, which satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$\big|[b,\mathscr{T}_\alpha](f)(x)\big| \leqslant c_3 \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|\cdot|f(y)|}{|x-y|^{n-\alpha}}\,dy, \tag{1.4}$$

where c_3 is also an absolute constant independent of f and $x \in \mathbb{R}^n$.

The classical Morrey spaces $\mathscr{L}^{p,\lambda}$ were originally introduced by Morrey in [20] to study the local behavior of solutions to second order elliptic partial differential equations. Since then, these spaces played an important role in studying the regularity of solutions to partial differential equations. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on these spaces, we refer the reader to [1, 2, 23]. In [19], Mizuhara introduced the generalized Morrey space $L^{p,\Theta}$ which was later extended and studied by many authors (see [9, 10, 11, 18, 21]). In [14], Komori and Shirai defined the weighted Morrey space $L^{p,\kappa}(w)$ which may be viewed as an natural generalization of weighted Lebesgue space, and then discussed the boundedness of several classical operators in Harmonic Analysis on $L^{p,\kappa}(w)$.

In [18, 27], the authors investigated the boundedness of sublinear operators $\mathscr{T}_\alpha$ $(0 \leqslant \alpha < n)$ and their commutators with *BMO* functions on weighted Morrey spaces and generalized Morrey spaces. Motivated by [18, 27], in this paper, we will study the endpoint estimates of these commutators generated by $BMO(\mathbb{R}^n)$ functions and sublinear operators defined above in the weighted Morrey spaces $L^{1,\kappa}(w)$ for $0<\kappa<1$, and in the generalized Morrey spaces $L^{1,\Theta}$, where Θ is a growth function on $(0,+\infty)$ satisfying the doubling condition. In order to simplify the notation, for any given $\sigma>0$, we set

$$\Phi\left(\frac{|f(x)|}{\sigma}\right) = \frac{|f(x)|}{\sigma}\cdot\left(1+\log^+\frac{|f(x)|}{\sigma}\right)$$

when $\Phi(t)=t\cdot(1+\log^+ t)$ and $\log^+ t=\max\{\log t,0\}$. The main results of this paper can be stated as follows.

THEOREM 1.1. *Let* $b \in BMO(\mathbb{R}^n)$ *and* $[b,\mathscr{T}]$ *satisfy the condition* (1.3). *Suppose that* $0<\kappa<1$, $w \in A_1$, *and for any given* $\sigma>0$,

$$w\big(\{x\in\mathbb{R}^n : \big|[b,\mathscr{T}](f)(x)\big|>\sigma\}\big) \leqslant C_0 \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx, \tag{1.5}$$

where $\Phi(t)=t\cdot(1+\log^+ t)$ *and* C_0 *depends only on* n,w *and* $\|b\|_*$, *but not on* f *and* σ. *Then for the above given* $\sigma>0$ *and any ball* $B\subset\mathbb{R}^n$, *there exists a constant* $C>0$ *independent of* f, B *and* σ *such that*

$$\frac{1}{w(B)^{\kappa}}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{T}](f)(x)\big|>\sigma\big\}\big)$$
$$\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\big(w(B)^{1-\kappa}\big)}{w(B)^{\kappa}}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\right\}.$$

THEOREM 1.2. *Let* $b\in BMO(\mathbb{R}^n)$ *and* $[b,\mathscr{T}]$ *satisfy the condition* (1.3). *Suppose that* Θ *satisfies* (2.3) *with* $1\leqslant D(\Theta)<2^n$, *and for any given* $\sigma>0$,

$$\big|\big\{x\in\mathbb{R}^n:\big|[b,\mathscr{T}](f)(x)\big|>\sigma\big\}\big|\leqslant C_0\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx, \tag{1.6}$$

where $\Phi(t)=t\cdot(1+\log^+ t)$ *and* C_0 *depends only on* $n,D(\Theta)$ *and* $\|b\|_*$, *but not on* f *and* σ. *Then for the above given* $\sigma>0$ *and any ball* $B(x_0,r)\subset\mathbb{R}^n$, *there exists a constant* $C>0$ *independent of* f, $B(x_0,r)$ *and* σ *such that*

$$\frac{1}{\Theta(r)}\cdot\big|\big\{x\in B(x_0,r):\big|[b,\mathscr{T}](f)(x)\big|>\sigma\big\}\big|$$
$$\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\big(\frac{|B(x_0,r)|}{\Theta(r)}\big)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\right\}.$$

THEOREM 1.3. *Let* $b\in BMO(\mathbb{R}^n)$ *and* $[b,\mathscr{T}_\alpha]$ *satisfy the condition* (1.4). *Suppose that* $0<\alpha<n$, $q=n/(n-\alpha)$, $0<\kappa<1/q$, $w^q\in A_1$, *and for any given* $\sigma>0$ *and any bounded domain* $\Omega\subset\mathbb{R}^n$,

$$\Big[w^q\big(\big\{x\in\Omega:\big|[b,\mathscr{T}_\alpha](f)(x)\big|>\sigma\big\}\big)\Big]^{1/q}\leqslant C_0\int_\Omega\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx, \tag{1.7}$$

where $\Phi(t)=t\cdot(1+\log^+ t)$ *and* C_0 *depends only on* n,α,w *and* $\|b\|_*$, *but not on* f, Ω *and* σ. *Then for the above given* $\sigma>0$ *and any ball* $B\subset\mathbb{R}^n$, *there exists a constant* $C>0$ *independent of* f, B *and* σ *such that*

$$\left(\frac{1}{w^q(B)^{\kappa q}}\cdot w^q\big(\big\{x\in B:\big|[b,\mathscr{T}_\alpha](f)(x)\big|>\sigma\big\}\big)\right)^{1/q}$$
$$\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\big(\frac{w(B)}{w^q(B)^{\kappa}}\big)}{w^q(B)^{\kappa}}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\right\}.$$

THEOREM 1.4. *Let* $b\in BMO(\mathbb{R}^n)$ *and* $[b,\mathscr{T}_\alpha]$ *satisfy the condition* (1.4). *Suppose that* $0<\alpha<n$, $q=n/(n-\alpha)$, Θ *satisfies* (2.3) *with* $1\leqslant D(\Theta)<2^{n/q}$, *and for any given* $\sigma>0$ *and any bounded domain* $\Omega\subset\mathbb{R}^n$,

$$\big|\big\{x\in\Omega:\big|[b,\mathscr{T}_\alpha](f)(x)\big|>\sigma\big\}\big|^{1/q}\leqslant C_0\int_\Omega\Phi\left(\frac{|f(x)|}{\sigma}\right)dx, \tag{1.8}$$

where $\Phi(t) = t\cdot(1+\log^+ t)$ *and* C_0 *depends only on* $n,\alpha,D(\Theta)$ *and* $\|b\|_*$*, but not on* f*,* Ω *and* σ*. Then for the above given* $\sigma > 0$ *and any ball* $B(x_0,r)\subset\mathbb{R}^n$*, there exists a constant* $C>0$ *independent of* f*,* $B(x_0,r)$ *and* σ *such that*

$$\left(\frac{1}{\Theta^q(r)}\cdot\big|\big\{x\in B(x_0,r):\big|[b,\mathscr{T}_\alpha](f)(x)\big|>\sigma\big\}\big|\right)^{1/q} \leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\big(\frac{|B(x_0,r)|}{\Theta(r)}\big)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\right\}.$$

REMARK 1.5. It should be pointed out that the conclusions of our main theorems are natural generalizations of the corresponding endpoint estimates on the weighted or unweighted Lebesgue spaces. The operators satisfying the assumptions of the above theorems include θ-type Calderón–Zygmund operators, Marcinkiewicz integral operators, Littlewood–Paley operators, Bochner–Riesz means, fractional maximal functions and fractional integrals, which will be discussed in the last section.

2. Notation and preliminaries

A weight w will always mean a non-negative, locally integrable function on $\mathbb{R}^n$ which is positive on a set of positive measure, $B=B(x_0,r_B)=\{x\in\mathbb{R}^n:|x-x_0|<r_B\}$ denotes the open ball centered at x_0 and with radius $r_B>0$. Given a ball B and $\lambda>0$, λB denotes the ball with the same center as B whose radius is λ times that of B. Given a Lebesgue measurable set E and a weight function w, $|E|$ will denote the Lebesgue measure of E and $w(E)=\int_E w(x)\,dx$. For $1<p<\infty$, a weight function w is said to belong to the Muckenhoupt's class A_p, if there is a constant $C>0$ such that for every ball $B\subseteq\mathbb{R}^n$ (see [8, 22]),

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}\,dx\right)^{p-1}\leqslant C.$$

For the case $p=1$, $w\in A_1$, if there is a constant $C>0$ such that for every ball $B\subseteq\mathbb{R}^n$,

$$\frac{1}{|B|}\int_B w(x)\,dx\leqslant C\cdot\operatorname*{ess\,inf}_{x\in B} w(x).$$

We also define $A_\infty=\cup_{1\leqslant p<\infty}A_p$. It is well known that if $w\in A_p$ with $1\leqslant p<\infty$, then for any ball B, there exists an absolute constant $C>0$ such that

$$w(2B)\leqslant C\,w(B). \tag{2.1}$$

In general, for $w\in A_1$ and any $\lambda>1$, there exists an absolute constant $C>0$ such that (see [8])

$$w\big(\lambda B\big)\leqslant C\cdot\lambda^n w(B).$$

Moreover, if w is in A_∞, then for all balls B and all measurable subsets E of B, there exists a number $\delta > 0$ independent of E and B such that (see [8])

$$\frac{w(E)}{w(B)} \leqslant C\left(\frac{|E|}{|B|}\right)^{\delta}. \tag{2.2}$$

We say that a weight w is in the reverse Hölder class RH_s, if there exist two constants $s > 1$ and $C > 0$ such that the following reverse Hölder inequality with exponent $s > 1$ holds for every ball $B \subseteq \mathbb{R}^n$.

$$\left(\frac{1}{|B|}\int_B w(x)^s\,dx\right)^{1/s} \leqslant C\left(\frac{1}{|B|}\int_B w(x)\,dx\right).$$

Given a weight function w on $\mathbb{R}^n$, for $1 \leqslant p < \infty$, the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ is defined as the set of all functions f such that

$$\|f\|_{L^p_w} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty.$$

In particular, when $w \equiv 1$, we will denote $L^p_w(\mathbb{R}^n)$ simply by $L^p(\mathbb{R}^n)$.

Let $0 < \kappa < 1$ and u, v be two weight functions on $\mathbb{R}^n$. Then the weighted Morrey space $L^{1,\kappa}(u,v)$ is defined by (see [14])

$$L^{1,\kappa}(u,v) := \left\{ f \in L^1_{loc}(u) : \|f\|_{L^{1,\kappa}(u,v)} = \sup_B \frac{1}{v(B)^\kappa}\int_B |f(x)|u(x)\,dx < \infty \right\},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. If $u = v = w$, then we set $L^{1,\kappa}(w,w) = L^{1,\kappa}(w)$. Define

$$\mathscr{M}^{1,\kappa}_{L\log L}(u,v) := \left\{ f \in L^1_{loc}(u) : \|f\|_{\mathscr{M}^{1,\kappa}_{L\log L}(u,v)} < \infty \right\},$$

where $\Phi(t) = t \cdot (1 + \log^+ t)$ and

$$\|f\|_{\mathscr{M}^{1,\kappa}_{L\log L}(u,v)} := \sup_B \frac{\Phi\left(\frac{u(B)}{v(B)^\kappa}\right)}{u(B)}\int_B |f(x)|u(x)\,dx < \infty.$$

Obviously, for $0 < \kappa < 1$, $L^{1,\kappa}(u,v) \supseteq \mathscr{M}^{1,\kappa}_{L\log L}(u,v)$. When $u = v = w$, then we set $\mathscr{M}^{1,\kappa}_{L\log L}(w,w) = \mathscr{M}^{1,\kappa}_{L\log L}(w)$.

Let $\Theta = \Theta(r)$, $r > 0$, be a growth function, that is, a positive increasing function in $(0,+\infty)$ and satisfy the following doubling condition:

$$\Theta(2r) \leqslant D \cdot \Theta(r), \quad \text{for all } r > 0, \tag{2.3}$$

where $D = D(\Theta) \geqslant 1$ is a doubling constant independent of r. The generalized Morrey space $L^{1,\Theta}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f for which (see [19])

$$\sup_{r>0;B(x_0,r)} \frac{1}{\Theta(r)}\int_{B(x_0,r)} |f(x)|\,dx < \infty,$$

where the supremum is taken over all balls $B(x_0,r)$ in $\mathbb{R}^n$. We also define

$$\mathscr{M}^{1,\Theta}_{L\log L}(\mathbb{R}^n) := \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{\mathscr{M}^{1,\Theta}_{L\log L}} < \infty \right\},$$

where $\Phi(t) = t \cdot (1+\log^+ t)$ and

$$\|f\|_{\mathscr{M}^{1,\Theta}_{L\log L}} := \sup_{r>0;B(x_0,r)} \frac{\Phi\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{|B(x_0,r)|} \int_{B(x_0,r)} |f(x)|\,dx < \infty.$$

Obviously, we have $L^{1,\Theta}(\mathbb{R}^n) \supseteq \mathscr{M}^{1,\Theta}_{L\log L}(\mathbb{R}^n)$. From the above two definitions, for given $\sigma > 0$, we may rewrite the right-hand side of the inequalities in Theorems 1.1–1.4 as $\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\kappa}_{L\log L}(w)}$, $\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\Theta}_{L\log L}}$, $\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\kappa}_{L\log L}(w,w^q)}$ and $\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\Theta}_{L\log L}}$, respectively.

We next recall some basic definitions and facts about Orlicz spaces needed for the proof of the main results. For more information on the subject, one can see [26]. A function Φ is called a Young function if it is continuous, nonnegative, convex and strictly increasing on $[0,+\infty)$ with $\Phi(0) = 0$ and $\Phi(t) \to +\infty$ as $t \to +\infty$. We define the Φ-average of a function f over a ball B by means of the following Luxemburg norm:

$$\|f\|_{\Phi,B} = \inf\left\{ \sigma > 0 : \frac{1}{|B|}\int_B \Phi\left(\frac{|f(x)|}{\sigma}\right) dx \leqslant 1 \right\}.$$

An equivalent norm that is often useful in calculations is as follows (see [26, 24]):

$$\|f\|_{\Phi,B} \leqslant \inf_{\eta>0}\left\{ \eta + \frac{\eta}{|B|}\int_B \Phi\left(\frac{|f(x)|}{\eta}\right) dx \right\} \leqslant 2\|f\|_{\Phi,B}. \tag{2.4}$$

Given a Young function Φ, we use $\bar{\Phi}$ to denote the complementary Young function associated to Φ. Then the following generalized Hölder's inequality holds for any given ball B (see [24, 25]).

$$\frac{1}{|B|}\int_B |f(x)\cdot g(x)|dx \leqslant 2\|f\|_{\Phi,B}\|g\|_{\bar{\Phi},B}.$$

In order to deal with the weighted case, for $w \in A_\infty$, we also need to define the weighted Φ-average of a function f over a ball B by means of the weighted Luxemburg norm:

$$\|f\|_{\Phi(w),B} = \inf\left\{ \sigma > 0 : \frac{1}{w(B)}\int_B \Phi\left(\frac{|f(x)|}{\sigma}\right) w(x)\,dx \leqslant 1 \right\}.$$

It can be shown that for $w \in A_\infty$ (see [26, 35]),

$$\|f\|_{\Phi(w),B} \approx \inf_{\eta>0}\left\{ \eta + \frac{\eta}{w(B)}\int_B \Phi\left(\frac{|f(x)|}{\eta}\right) w(x)\,dx \right\}, \tag{2.5}$$

and

$$\frac{1}{w(B)}\int_B |f(x)\cdot g(x)|w(x)\,dx \leqslant C\|f\|_{\Phi(w),B}\|g\|_{\bar{\Phi}(w),B}.$$

Here, and in what follows, $A \approx B$ means that there exist two positive constants C_1 and C_2 such that $C_1 \leqslant \frac{A}{B} \leqslant C_2$. The young function that we are going to use is $\Phi(t) = t\cdot(1+\log^+ t)$ with its complementary Young function $\bar{\Phi}(t) \approx e^t - 1$. In the present situation, we denote

$$\big\|f\big\|_{L\log L,B} = \big\|f\big\|_{\Phi,B}, \qquad \big\|g\big\|_{\exp L,B} = \big\|g\big\|_{\bar{\Phi},B};$$

and

$$\big\|f\big\|_{L\log L(w),B} = \big\|f\big\|_{\Phi(w),B}, \qquad \big\|g\big\|_{\exp L(w),B} = \big\|g\big\|_{\bar{\Phi}(w),B}.$$

By the (unweighted or weighted) generalized Hölder's inequality, we have (see [24, 35])

$$\frac{1}{|B|}\int_B |f(x)\cdot g(x)|\,dx \leqslant 2\big\|f\big\|_{L\log L,B}\big\|g\big\|_{\exp L,B}, \tag{2.6}$$

and

$$\frac{1}{w(B)}\int_B |f(x)\cdot g(x)|w(x)\,dx \leqslant C\big\|f\big\|_{L\log L(w),B}\big\|g\big\|_{\exp L(w),B}. \tag{2.7}$$

Let us now recall the definition of the space of $BMO(\mathbb{R}^n)$ (Bounded Mean Oscillation) (see [7, 12]). A locally integrable function b is said to be in $BMO(\mathbb{R}^n)$, if

$$\|b\|_* = \sup_B \frac{1}{|B|}\int_B |b(x)-b_B|\,dx < \infty,$$

where b_B stands for the average of b on B, i.e., $b_B = \frac{1}{|B|}\int_B b(y)\,dy$ and the supremum is taken over all balls B in $\mathbb{R}^n$. Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$. By the John–Nirenberg's inequality, it is not difficult to see that for any given ball B (see [24, 25])

$$\big\|b-b_B\big\|_{\exp L,B} \leqslant C\|b\|_*. \tag{2.8}$$

Furthermore, we can also prove that for any $w \in A_\infty$ and any given ball B (see [35]),

$$\big\|b-b_B\big\|_{\exp L(w),B} \leqslant C\|b\|_*. \tag{2.9}$$

In the sequel, the letter C always denotes a positive constant which is independent of the main parameters involved, but whose value may be different from line to line. We also use C_0, c_2, c_3 appearing in the first section of this paper to denote certain constants. For convenience, we write $p' = p/(p-1)$ for given $1 < p < \infty$.

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ and decompose $f = f_1 + f_2$, where $f_1 = f\cdot\chi_{2B}$, χ_{2B} denotes the characteristic function of $2B = B(x_0, 2r_B)$. For any

$0<\kappa<1$, $w\in A_1$ and any given $\sigma>0$, one writes

$$\begin{aligned}
&\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{T}](f)(x)\big|>\sigma\big\}\big)\\
\leqslant&\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{T}](f_1)(x)\big|>\sigma/2\big\}\big)\\
&+\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{T}](f_2)(x)\big|>\sigma/2\big\}\big)\\
:=&I_1+I_2.
\end{aligned}$$

Using the inequalities (1.5) and (2.1), we get

$$\begin{aligned}
I_1&\leqslant C_0\cdot\frac{1}{w(B)^\kappa}\int_{\mathbb{R}^n}\Phi\left(\frac{|f_1(x)|}{\sigma}\right)\cdot w(x)\,dx\\
&=C_0\cdot\frac{1}{w(B)^\kappa}\int_{2B}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\\
&=C_0\cdot\frac{w(2B)^\kappa}{w(B)^\kappa}\cdot\frac{1}{w(2B)^\kappa}\int_{2B}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\\
&\leqslant C\cdot\sup_B\left\{\frac{1}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\right\}\\
&\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\big(w(B)^{1-\kappa}\big)}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\right\}.
\end{aligned}$$

For any $x\in B$, from the definition of (1.3), it follows that

$$\begin{aligned}
\big|[b,\mathscr{T}](f_2)(x)\big|&\leqslant c_2\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|\cdot|f_2(y)|}{|x-y|^n}dy\\
&\leqslant c_2\big|b(x)-b_B\big|\cdot\int_{\mathbb{R}^n}\frac{|f_2(y)|}{|x-y|^n}dy+c_2\int_{\mathbb{R}^n}\frac{|b(y)-b_B|\cdot|f_2(y)|}{|x-y|^n}dy\\
&:=\mu(x)+\nu(x).
\end{aligned}$$

So we have

$$\begin{aligned}
I_2\leqslant&\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x\in B:\mu(x)>\sigma/4\big\}\big)+\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x\in B:\nu(x)>\sigma/4\big\}\big)\\
:=&I_3+I_4.
\end{aligned}$$

For the term I_3, we can easily see that for every $x\in B$,

$$\int_{\mathbb{R}^n}\frac{|f_2(y)|}{|x-y|^n}dy=\int_{(2B)^c}\frac{|f(y)|}{|x-y|^n}dy\leqslant C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|\,dy. \tag{3.1}$$

Since $w\in A_1$, then there exists a number $s>1$ such that $w\in RH_s$. Hence, by using the above pointwise estimate (3.1), Chebyshev's inequality together with Hölder's inequality and John–Nirenberg's inequality (see [12]), we conclude that

$$\begin{aligned} I_3 &\leqslant \frac{1}{w(B)^\kappa}\cdot\frac{4}{\sigma}\int_B \mu(x)\cdot w(x)\,dx \\ &\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy \\ &\quad\times\frac{1}{w(B)^\kappa}\cdot\left(\int_B\left|b(x)-b_B\right|^{s'}dx\right)^{1/s'}\left(\int_B w(x)^s dx\right)^{1/s} \\ &\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\times w(B)^{1-\kappa}. \end{aligned}$$

Furthermore, it follows directly from the A_1 condition and the fact $t\leqslant\Phi(t)=t\cdot(1+\log^+ t)$ that

$$\begin{aligned} I_3 &= C\sum_{j=1}^{\infty}\frac{1}{w(2^{j+1}B)}\cdot\frac{w(2^{j+1}B)}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\times w(B)^{1-\kappa} \\ &\leqslant C\sum_{j=1}^{\infty}\frac{1}{w(2^{j+1}B)}\cdot\operatorname*{ess\,inf}_{y\in 2^{j+1}B} w(y)\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\times w(B)^{1-\kappa} \\ &\leqslant C\sum_{j=1}^{\infty}\frac{1}{w(2^{j+1}B)}\cdot\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\cdot w(y)\,dy\times w(B)^{1-\kappa} \\ &\leqslant C\cdot\sup_B\left\{\frac{1}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy\right\}\times\sum_{j=1}^{\infty}\frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}} \\ &\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\left(w(B)^{1-\kappa}\right)}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy\right\}\times\sum_{j=1}^{\infty}\frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}}. \end{aligned}$$

Note that $w\in A_1\subset A_\infty$, by the inequality (2.2), we get

$$\begin{aligned} \sum_{j=1}^{\infty}\frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}} &\leqslant C\sum_{j=1}^{\infty}\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta(1-\kappa)} \\ &\leqslant C\sum_{j=1}^{\infty}\left(\frac{1}{2^{(j+1)n}}\right)^{\delta(1-\kappa)}\leqslant C, \end{aligned}\tag{3.2}$$

which in turn implies that

$$I_3\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\left(w(B)^{1-\kappa}\right)}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy\right\}.$$

Similar to the proof of (3.1), for all $x\in B$, we can show the following pointwise estimate as well.

$$\nu(x)\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\left|b(y)-b_B\right|\cdot\left|f(y)\right|dy. \tag{3.3}$$

Applying the above pointwise estimate (3.3) and Chebyshev's inequality, we have

$$\begin{aligned} I_4 &\leqslant \frac{1}{w(B)^\kappa}\cdot\frac{4}{\sigma}\int_B \nu(x)\cdot w(x)\,dx \\ &\leqslant \frac{w(B)}{w(B)^\kappa}\cdot\frac{C}{\sigma}\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b(y)-b_B\big|\cdot\big|f(y)\big|\,dy \\ &\leqslant \frac{w(B)}{w(B)^\kappa}\cdot\frac{C}{\sigma}\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|\cdot\big|f(y)\big|\,dy \\ &\quad + \frac{w(B)}{w(B)^\kappa}\cdot\frac{C}{\sigma}\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b_{2^{j+1}B}-b_B\big|\cdot\big|f(y)\big|\,dy \\ &:= I_5+I_6. \end{aligned}$$

To estimate the term I_5, we first use the generalized Hölder's inequality with weight (2.7), (2.9) and (2.5) together with the A_1 condition to obtain

$$\begin{aligned} I_5 &\leqslant \frac{C}{\sigma}\cdot w(B)^{1-\kappa}\sum_{j=1}^\infty\frac{1}{w(2^{j+1}B)}\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|\cdot\big|f(y)\big|w(y)\,dy \\ &\leqslant \frac{C}{\sigma}\cdot w(B)^{1-\kappa}\sum_{j=1}^\infty\big\|b-b_{2^{j+1}B}\big\|_{\exp L(w),2^{j+1}B}\big\|f\big\|_{L\log L(w),2^{j+1}B} \\ &\leqslant \frac{C\|b\|_*}{\sigma}\cdot w(B)^{1-\kappa}\sum_{j=1}^\infty\inf_{\eta>0}\left\{\eta+\frac{\eta}{w(2^{j+1}B)}\int_{2^{j+1}B}\Phi\left(\frac{|f(z)|}{\eta}\right)\cdot w(z)\,dz\right\}. \end{aligned}$$

Moreover, observe that for any $a,b>0$, $\Phi(a\cdot b)\leqslant\Phi(a)\cdot\Phi(b)$ when $\Phi(t)=t\cdot(1+\log^+ t)$. For $j=1,2,\dots$, we may take $\eta=\dfrac{\sigma}{w(2^{j+1}B)^{1-\kappa}}$ and then use the estimate (3.2) to obtain

$$\begin{aligned} I_5 &\leqslant \frac{C\|b\|_*}{\sigma}\cdot w(B)^{1-\kappa} \\ &\quad\times\sum_{j=1}^\infty\left\{\frac{\sigma}{w(2^{j+1}B)^{1-\kappa}}+\frac{\sigma}{w(2^{j+1}B)}\cdot\frac{\Phi\big(w(2^{j+1}B)^{1-\kappa}\big)}{w(2^{j+1}B)^{1-\kappa}}\int_{2^{j+1}B}\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\} \\ &\leqslant C\|b\|_*\cdot\left[1+\sup_B\left\{\frac{1+\log^+\big(w(B)^{1-\kappa}\big)}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}\right]\times\sum_{j=1}^\infty\frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}} \\ &\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\big(w(B)^{1-\kappa}\big)}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}. \end{aligned}$$

For the last term I_6 we proceed as follows. Since $b\in BMO(\mathbb{R}^n)$, then a simple calculation shows that

$$\big|b_{2^{j+1}B}-b_B\big|\leqslant C\cdot(j+1)\|b\|_*. \tag{3.4}$$

Applying the inequality (3.4) and the facts that $w \in A_1$ and $t \leqslant \Phi(t)$, we get

$$\begin{aligned}
I_6 &\leqslant C \cdot w(B)^{1-\kappa} \sum_{j=1}^{\infty} (j+1)\|b\|_* \cdot \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} \frac{|f(y)|}{\sigma}\, dy \\
&\leqslant C \cdot w(B)^{1-\kappa} \sum_{j=1}^{\infty} (j+1)\|b\|_* \cdot \frac{1}{w(2^{j+1}B)} \int_{2^{j+1}B} \frac{|f(y)|}{\sigma} \cdot w(y)\, dy \\
&\leqslant C \cdot \sup_B \left\{ \frac{1+\log^+ \left(w(B)^{1-\kappa}\right)}{w(B)^{\kappa}} \int_B \Phi\left(\frac{|f(y)|}{\sigma}\right) \cdot w(y)\, dy \right\} \times \sum_{j=1}^{\infty} (j+1) \cdot \frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}}.
\end{aligned}$$

Since $w \in A_1 \subset A_\infty$, by using the inequality (2.2) again, we have

$$\begin{aligned}
\sum_{j=1}^{\infty} (j+1) \cdot \frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}} &\leqslant C \sum_{j=1}^{\infty} (j+1) \cdot \left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta(1-\kappa)} \\
&\leqslant C \sum_{j=1}^{\infty} (j+1) \cdot \left(\frac{1}{2^{(j+1)n}}\right)^{\delta(1-\kappa)} \leqslant C.
\end{aligned}$$

Therefore

$$I_6 \leqslant C \cdot \sup_B \left\{ \frac{1+\log^+ \left(w(B)^{1-\kappa}\right)}{w(B)^{\kappa}} \int_B \Phi\left(\frac{|f(y)|}{\sigma}\right) \cdot w(y)\, dy \right\}.$$

Summarizing the above discussions, we obtain the conclusion of the theorem. $\square$

Proof of Theorem 1.2. For any ball $B = B(x_0, r) \subseteq \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r > 0$, we write f as $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{2B}$. Then for each fixed $\sigma > 0$, we have

$$\begin{aligned}
&\frac{1}{\Theta(r)} \cdot \left|\left\{x \in B : \left|[b,\mathscr{T}](f)(x)\right| > \sigma\right\}\right| \\
\leqslant & \frac{1}{\Theta(r)} \cdot \left|\left\{x \in B : \left|[b,\mathscr{T}](f_1)(x)\right| > \sigma/2\right\}\right| + \frac{1}{\Theta(r)} \cdot \left|\left\{x \in B : \left|[b,\mathscr{T}](f_2)(x)\right| > \sigma/2\right\}\right| \\
:= & J_1 + J_2.
\end{aligned}$$

We consider the term J_1 first. The condition (1.6) and the inequality (2.3) imply that

$$\begin{aligned}
J_1 &\leqslant C_0 \cdot \frac{1}{\Theta(r)} \int_{\mathbb{R}^n} \Phi\left(\frac{|f_1(x)|}{\sigma}\right) dx \\
&= C_0 \cdot \frac{1}{\Theta(r)} \int_{2B} \Phi\left(\frac{|f(x)|}{\sigma}\right) dx \\
&= C_0 \cdot \frac{\Theta(2r)}{\Theta(r)} \cdot \frac{1}{\Theta(2r)} \int_{B(x_0,2r)} \Phi\left(\frac{|f(x)|}{\sigma}\right) dx \\
&\leqslant C \cdot \sup_{r>0} \left\{ \frac{1}{\Theta(r)} \int_{B(x_0,r)} \Phi\left(\frac{|f(x)|}{\sigma}\right) dx \right\} \\
&\leqslant C \cdot \sup_{r>0} \left\{ \frac{1+\log^+ \left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)} \int_{B(x_0,r)} \Phi\left(\frac{|f(x)|}{\sigma}\right) dx \right\}.
\end{aligned}$$

We now turn our attention to the estimate of J_2. Recall that the following estimate holds for any $x\in B$,

$$\big|[b,\mathscr{T}](f_2)(x)\big| \leqslant \mu(x)+\nu(x),$$

where

$$\mu(x)=c_2\big|b(x)-b_B\big|\cdot\int_{\mathbb{R}^n}\frac{|f_2(y)|}{|x-y|^n}dy,$$

and

$$\nu(x)=c_2\int_{\mathbb{R}^n}\frac{|b(y)-b_B|\cdot|f_2(y)|}{|x-y|^n}dy.$$

Thus, we have

$$\begin{aligned}J_2 \leqslant &\frac{1}{\Theta(r)}\cdot\big|\big\{x\in B:\mu(x)>\sigma/4\big\}\big|+\frac{1}{\Theta(r)}\cdot\big|\big\{x\in B:\nu(x)>\sigma/4\big\}\big|\\ :=&J_3+J_4.\end{aligned}$$

By using the previous pointwise estimate (3.1), Chebyshev's inequality and the definition of BMO, we can deduce that

$$\begin{aligned}J_3 &\leqslant \frac{1}{\Theta(r)}\cdot\frac{4}{\sigma}\int_B\mu(x)\,dx\\ &\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\times\left\{\frac{|B|}{\Theta(r)}\cdot\frac{1}{|B|}\int_B\big|b(x)-b_B\big|\,dx\right\}\\ &\leqslant C\|b\|_*\sum_{j=1}^{\infty}\frac{|B|}{|2^{j+1}B|}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\cdot\frac{1}{\Theta(2^{j+1}r)}\int_{B(x_0,2^{j+1}r)}\frac{|f(y)|}{\sigma}\,dy\\ &\leqslant C\cdot\sup_{r>0}\left\{\frac{1}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\right\}\times\sum_{j=1}^{\infty}\frac{|B|}{|2^{j+1}B|}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\\ &\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\big(\frac{|B(x_0,r)|}{\Theta(r)}\big)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\right\}\times\sum_{j=1}^{\infty}\frac{|B|}{|2^{j+1}B|}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}.\end{aligned}$$

Note that $1\leqslant D(\Theta)<2^n$, then by using the doubling condition (2.3) of Θ, we can see that

$$\sum_{j=1}^{\infty}\frac{|B|}{|2^{j+1}B|}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\leqslant\sum_{j=1}^{\infty}\left(\frac{D(\Theta)}{2^n}\right)^{j+1}\leqslant C, \tag{3.5}$$

which in turn gives that

$$J_3\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\big(\frac{|B(x_0,r)|}{\Theta(r)}\big)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\right\}.$$

Applying the previous pointwise estimate (3.3) and Chebyshev's inequality, we have

$$
\begin{aligned}
J_4 &\leqslant \frac{1}{\Theta(r)}\cdot\frac{4}{\sigma}\int_B \nu(x)\,dx\\
&\leqslant \frac{|B|}{\Theta(r)}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b(y)-b_B\big|\cdot\big|f(y)\big|\,dy\\
&\leqslant \frac{|B|}{\Theta(r)}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|\cdot\big|f(y)\big|\,dy\\
&\quad+\frac{|B|}{\Theta(r)}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b_{2^{j+1}B}-b_B\big|\cdot\big|f(y)\big|\,dy\\
&:= J_5+J_6.
\end{aligned}
$$

For the term J_5, we first use the generalized Hölder's inequality (2.6), (2.8) and (2.4) to obtain

$$
\begin{aligned}
J_5 &\leqslant \frac{|B|}{\Theta(r)}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\big\|b-b_{2^{j+1}B}\big\|_{\exp L,2^{j+1}B}\big\|f\big\|_{L\log L,2^{j+1}B}\\
&\leqslant \frac{C\|b\|_*}{\sigma}\cdot\frac{|B|}{\Theta(r)}\sum_{j=1}^{\infty}\inf_{\eta>0}\left\{\eta+\frac{\eta}{|2^{j+1}B|}\int_{2^{j+1}B}\Phi\left(\frac{|f(z)|}{\eta}\right)dz\right\}.
\end{aligned}
$$

Moreover, notice that the inequality $\Phi(a\cdot b)\leqslant\Phi(a)\cdot\Phi(b)$ holds for any $a,b>0$, when $\Phi(t)=t\cdot(1+\log^+ t)$. For $j=1,2,\dots$, in this case, we may take $\eta=\dfrac{\sigma\cdot\Theta(2^{j+1}r)}{|2^{j+1}B|}$ and then use the estimate (3.5) to obtain

$$
\begin{aligned}
J_5 &\leqslant \frac{C\|b\|_*}{\sigma}\cdot\frac{|B|}{\Theta(r)}\\
&\quad\times\sum_{j=1}^{\infty}\left\{\frac{\sigma\cdot\Theta(2^{j+1}r)}{|2^{j+1}B|}+\frac{\sigma}{|2^{j+1}B|}\cdot\frac{\Theta(2^{j+1}r)}{|2^{j+1}B|}\cdot\Phi\left(\frac{|2^{j+1}B|}{\Theta(2^{j+1}r)}\right)\int_{B(x_0,2^{j+1}r)}\Phi\left(\frac{|f(z)|}{\sigma}\right)dz\right\}\\
&\leqslant C\|b\|_*\cdot\left[1+\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(z)|}{\sigma}\right)dz\right\}\right]\\
&\quad\times\sum_{j=1}^{\infty}\frac{|B|}{|2^{j+1}B|}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\\
&\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(z)|}{\sigma}\right)dz\right\}.
\end{aligned}
$$

For the last term J_6, an application of the inequality (3.4) leads to that

$$\begin{aligned}
J_6 &\leqslant C\cdot\frac{|B|}{\Theta(r)}\sum_{j=1}^{\infty}(j+1)\|b\|_*\cdot\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\\
&= C\cdot\frac{|B|}{\Theta(r)}\sum_{j=1}^{\infty}(j+1)\|b\|_*\cdot\frac{\Theta(2^{j+1}r)}{|2^{j+1}B|}\cdot\frac{1}{\Theta(2^{j+1}r)}\int_{B(x_0,2^{j+1}r)}\frac{|f(y)|}{\sigma}\,dy\\
&\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\right\}\\
&\quad\times\sum_{j=1}^{\infty}(j+1)\cdot\frac{|B|}{|2^{j+1}B|}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}.
\end{aligned}$$

Moreover, by using the doubling condition (2.3) of Θ again and the fact that $1\leqslant D(\Theta)<2^n$, we find that

$$\sum_{j=1}^{\infty}(j+1)\cdot\frac{|B|}{|2^{j+1}B|}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\leqslant C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{D(\Theta)}{2^n}\right)^{j+1}\leqslant C. \tag{3.6}$$

Substituting the above inequality (3.6) into the term J_6, we thus obtain

$$J_6\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\right\}.$$

Summing up all the above estimates, we therefore conclude the proof of the main theorem. □

4. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Fix a ball $B=B(x_0,r_B)\subseteq\mathbb{R}^n$ and $x\in B$, we split f as usual by $f=f\cdot\chi_{2B}+f\cdot\chi_{(2B)^c}:=f_1+f_2$. For any $0<\kappa<1/q$, $w^q\in A_1$ with $q=n/(n-\alpha)>1$ and any given $\sigma>0$, we then write

$$\begin{aligned}
&\left(\frac{1}{w^q(B)^{\kappa q}}\cdot w^q\big(\big\{x\in B:\big|[b,\mathscr{T}_\alpha](f)(x)\big|>\sigma\big\}\big)\right)^{1/q}\\
\leqslant&\left(\frac{1}{w^q(B)^{\kappa q}}\cdot w^q\big(\big\{x\in B:\big|[b,\mathscr{T}_\alpha](f_1)(x)\big|>\sigma/2\big\}\big)\right)^{1/q}\\
&+\left(\frac{1}{w^q(B)^{\kappa q}}\cdot w^q\big(\big\{x\in B:\big|[b,\mathscr{T}_\alpha](f_2)(x)\big|>\sigma/2\big\}\big)\right)^{1/q}\\
:=&I_1'+I_2'.
\end{aligned}$$

By using the assumption (1.7) and the inequality (2.1), we get

$$\begin{aligned}
I_1' &\leqslant C_0 \cdot \frac{1}{w^q(B)^\kappa} \int_B \Phi\left(\frac{|f_1(x)|}{\sigma}\right) \cdot w(x)\,dx \\
&= C_0 \cdot \frac{1}{w^q(B)^\kappa} \int_{2B} \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\,dx \\
&= C_0 \cdot \frac{w^q(2B)^\kappa}{w^q(B)^\kappa} \cdot \frac{1}{w^q(2B)^\kappa} \int_{2B} \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\,dx \\
&\leqslant C \cdot \sup_B \left\{ \frac{1}{w^q(B)^\kappa} \int_B \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\,dx \right\} \\
&\leqslant C \cdot \sup_B \left\{ \frac{1+\log^+\left(\frac{w(B)}{w^q(B)^\kappa}\right)}{w^q(B)^\kappa} \int_B \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\,dx \right\}.
\end{aligned}$$

For any $x \in B$, from the definition of (1.4), it follows that

$$\begin{aligned}
\big|[b,\mathscr{T}_\alpha](f_2)(x)\big| &\leqslant c_3 \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|\cdot|f_2(y)|}{|x-y|^{n-\alpha}}dy \\
&\leqslant c_3\big|b(x)-b_B\big| \cdot \int_{\mathbb{R}^n} \frac{|f_2(y)|}{|x-y|^{n-\alpha}}dy + c_3 \int_{\mathbb{R}^n} \frac{|b(y)-b_B|\cdot|f_2(y)|}{|x-y|^{n-\alpha}}dy \\
&:= \widetilde{\mu}(x) + \widetilde{\nu}(x).
\end{aligned}$$

So we can rewrite the term I_2' as follows:

$$\begin{aligned}
I_2' &\leqslant \left(\frac{1}{w^q(B)^{\kappa q}} \cdot w^q\big(\{x \in B : \widetilde{\mu}(x) > \sigma/4\}\big)\right)^{1/q} \\
&\quad + \left(\frac{1}{w^q(B)^{\kappa q}} \cdot w^q\big(\{x \in B : \widetilde{\nu}(x) > \sigma/4\}\big)\right)^{1/q} \\
&:= I_3' + I_4'.
\end{aligned}$$

For the term I_3', we can easily check that for given $0 < \alpha < n$ and every $x \in B$,

$$\int_{\mathbb{R}^n} \frac{|f_2(y)|}{|x-y|^{n-\alpha}}dy = \int_{(2B)^c} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy \leqslant C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{1-\alpha/n}} \int_{2^{j+1}B} |f(y)|\,dy. \tag{4.1}$$

Since w^q is in A_1, we know that there exists a number $r > 1$ such that $w^q \in RH_r$. Hence, by using the above pointwise estimate (4.1), Chebyshev's inequality together with Hölder's inequality and John–Nirenberg's inequality (see [12]), we deduce that

$$\begin{aligned}
I_3' &\leqslant \frac{1}{w^q(B)^\kappa} \cdot \frac{4}{\sigma} \left(\int_B |\widetilde{\mu}(x)|^q w^q(x)\,dx\right)^{1/q} \\
&\leqslant C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{1-\alpha/n}} \int_{2^{j+1}B} \frac{|f(y)|}{\sigma}\,dy \times \frac{1}{w^q(B)^\kappa} \cdot \left(\int_B \big|b(x)-b_B\big|^q w^q(x)\,dx\right)^{1/q}
\end{aligned}$$

$$\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy$$
$$\times\frac{1}{w^q(B)^{\kappa}}\cdot\left(\int_B\big|b(x)-b_B\big|^{qr'}dx\right)^{1/(qr')}\left(\int_B\big[w^q(x)\big]^{r}dx\right)^{1/(qr)}$$
$$\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\times w^q(B)^{1/q-\kappa}.$$

Moreover, by applying Hölder's inequality and then the reverse Hölder inequality in succession, we can show that $w\in A_1\cap RH_q$ if and only if $w^q\in A_1$ (see [13]). Thus, we are able to verify that for any $j\in\mathbb{Z}_+$,

$$w^q(2^{j+1}B)^{1/q}=\left(\int_{2^{j+1}B}w^q(x)\,dx\right)^{1/q}\leqslant C\cdot|2^{j+1}B|^{1/q-1}\cdot w(2^{j+1}B),$$

which is equivalent to

$$\frac{w^q(2^{j+1}B)^{1/q}}{|2^{j+1}B|^{1/q}}\leqslant C\cdot\frac{w(2^{j+1}B)}{|2^{j+1}B|}. \tag{4.2}$$

Therefore, by using the inequality (4.2) together with the facts that $1/q=1-\alpha/n$, $w\in A_1$ and $t\leqslant\Phi(t)$, we obtain

$$I_3'=C\sum_{j=1}^{\infty}\frac{w^q(2^{j+1}B)^{1/q}}{|2^{j+1}B|^{1-\alpha/n}}\cdot\frac{1}{w^q(2^{j+1}B)^{\kappa}}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\times\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}$$
$$\leqslant C\sum_{j=1}^{\infty}\frac{w(2^{j+1}B)}{|2^{j+1}B|}\cdot\frac{1}{w^q(2^{j+1}B)^{\kappa}}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\times\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}$$
$$\leqslant C\sum_{j=1}^{\infty}\frac{1}{w^q(2^{j+1}B)^{\kappa}}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\cdot w(y)\,dy\times\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}$$
$$\leqslant C\cdot\sup_B\left\{\frac{1}{w^q(B)^{\kappa}}\int_B\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy\right\}\times\sum_{j=1}^{\infty}\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}$$
$$\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\left(\frac{w(B)}{w^q(B)^{\kappa}}\right)}{w^q(B)^{\kappa}}\int_B\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy\right\}\times\sum_{j=1}^{\infty}\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}.$$

Applying the inequality (2.2) and the property $w^q\in A_1\subset A_\infty$, we can get

$$\sum_{j=1}^{\infty}\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\leqslant C\sum_{j=1}^{\infty}\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta^*(1/q-\kappa)}\leqslant C\sum_{j=1}^{\infty}\left(\frac{1}{2^{(j+1)n}}\right)^{\delta^*(1/q-\kappa)}\leqslant C, \tag{4.3}$$

where in the last inequality we have used the facts that $\delta^*>0$ and $0<\kappa<1/q$. Substituting the above inequality (4.3) into the term I_3', we thus obtain

$$I_3'\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\left(\frac{w(B)}{w^q(B)^{\kappa}}\right)}{w^q(B)^{\kappa}}\int_B\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy\right\}.$$

For the term I_4', similar to the proof of (4.1), for all $0<\alpha<n$ and all $x\in B$, we can show the following pointwise estimate as well.

$$\big|\widetilde{\nu}(x)\big| \leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\big|b(y)-b_B\big|\cdot\big|f(y)\big|\,dy. \tag{4.4}$$

Following the same arguments as in the proof of Theorem 1.1 and using the pointwise inequality (4.4) and Chebyshev's inequality, we have

$$\begin{aligned}
I_4' &\leqslant \frac{1}{w^q(B)^{\kappa}}\cdot\frac{4}{\sigma}\left(\int_B|\widetilde{\nu}(x)|^q w^q(x)\,dx\right)^{1/q}\\
&\leqslant \frac{w^q(B)^{1/q}}{w^q(B)^{\kappa}}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\big|b(y)-b_B\big|\cdot\big|f(y)\big|\,dy\\
&\leqslant \frac{w^q(B)^{1/q}}{w^q(B)^{\kappa}}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|\cdot\big|f(y)\big|\,dy\\
&\quad+\frac{w^q(B)^{1/q}}{w^q(B)^{\kappa}}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\big|b_{2^{j+1}B}-b_B\big|\cdot\big|f(y)\big|\,dy\\
&:= I_5'+I_6'.
\end{aligned}$$

To deal with the term I_5', it then follows from the inequality (4.2) and the facts $1/q=1-\alpha/n$ and $w\in A_1$ that

$$\begin{aligned}
I_5' &= \frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\cdot\frac{1}{w^q(2^{j+1}B)^{\kappa}}\\
&\quad\times\frac{w^q(2^{j+1}B)^{1/q}}{|2^{j+1}B|^{1-\alpha/n}}\cdot\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|\cdot\big|f(y)\big|\,dy\\
&\leqslant \frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\cdot\frac{1}{w^q(2^{j+1}B)^{\kappa}}\\
&\quad\times\frac{w(2^{j+1}B)}{|2^{j+1}B|}\cdot\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|\cdot\big|f(y)\big|\,dy\\
&\leqslant \frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\cdot\frac{1}{w^q(2^{j+1}B)^{\kappa}}\times\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|\cdot\big|f(y)\big|w(y)\,dy.
\end{aligned}$$

Furthermore, by using the generalized Hölder's inequality with weight (2.7) and (2.9) together with (2.5), we can conclude that

$$\begin{aligned}
I_5' &\leqslant \frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\cdot\frac{w(2^{j+1}B)}{w^q(2^{j+1}B)^{\kappa}}\big\|b-b_{2^{j+1}B}\big\|_{\exp L(w),2^{j+1}B}\big\|f\big\|_{L\log L(w),2^{j+1}B}\\
&\leqslant \frac{C\|b\|_*}{\sigma}\cdot\sum_{j=1}^{\infty}\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\cdot\frac{w(2^{j+1}B)}{w^q(2^{j+1}B)^{\kappa}}\\
&\quad\times\inf_{\eta>0}\left\{\eta+\frac{\eta}{w(2^{j+1}B)}\int_{2^{j+1}B}\Phi\left(\frac{|f(z)|}{\eta}\right)\cdot w(z)\,dz\right\}.
\end{aligned}$$

For $j=1,2,\dots$, we may choose $\eta=\dfrac{\sigma\cdot w^q(2^{j+1}B)^\kappa}{w(2^{j+1}B)}$. Then by using (4.3) and the fact that $\Phi(t)$ is submultiplicative ($\Phi(a\cdot b)\leqslant\Phi(a)\cdot\Phi(b)$ for any $a,b>0$), we deduce

$$\begin{aligned}
I_5' &\leqslant \frac{C\|b\|_*}{\sigma}\cdot\sum_{j=1}^\infty \frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\cdot\frac{w(2^{j+1}B)}{w^q(2^{j+1}B)^\kappa}\\
&\quad\times\left\{\frac{\sigma\cdot w^q(2^{j+1}B)^\kappa}{w(2^{j+1}B)}+\frac{\sigma}{w(2^{j+1}B)}\cdot\frac{w^q(2^{j+1}B)^\kappa}{w(2^{j+1}B)}\cdot\Phi\left(\frac{w(2^{j+1}B)}{w^q(2^{j+1}B)^\kappa}\right)\right.\\
&\quad\left.\times\int_{2^{j+1}B}\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}\\
&\leqslant C\|b\|_*\cdot\left[1+\sup_B\left\{\frac{1+\log^+\left(\frac{w(B)}{w^q(B)^\kappa}\right)}{w^q(B)^\kappa}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}\right]\\
&\quad\times\sum_{j=1}^\infty\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\\
&\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\left(\frac{w(B)}{w^q(B)^\kappa}\right)}{w^q(B)^\kappa}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}.
\end{aligned}$$

For the last term I_6' we proceed as follows. Since $b\in BMO(\mathbb{R}^n)$, as before, a straightforward computation shows that

$$\left|b_{2^{j+1}B}-b_B\right|\leqslant C\cdot(j+1)\|b\|_*. \tag{4.5}$$

Thus, by (4.5), (4.2), the A_1 condition and the fact that $t\leqslant\Phi(t)$, we obtain

$$\begin{aligned}
I_6' &\leqslant C\cdot w^q(B)^{1/q-\kappa}\sum_{j=1}^\infty(j+1)\|b\|_*\cdot\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\\
&= C\cdot\|b\|_*\sum_{j=1}^\infty(j+1)\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\cdot\frac{1}{w^q(2^{j+1}B)^\kappa}\cdot\frac{w^q(2^{j+1}B)^{1/q}}{|2^{j+1}B|^{1/q}}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\\
&\leqslant C\cdot\|b\|_*\sum_{j=1}^\infty(j+1)\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\cdot\frac{1}{w^q(2^{j+1}B)^\kappa}\cdot\frac{w(2^{j+1}B)}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\\
&\leqslant C\cdot\|b\|_*\sum_{j=1}^\infty(j+1)\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}\cdot\frac{1}{w^q(2^{j+1}B)^\kappa}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\cdot w(y)\,dy\\
&\leqslant C\cdot\sup_B\left\{\frac{1+\log^+\left(\frac{w(B)}{w^q(B)^\kappa}\right)}{w^q(B)^\kappa}\int_B\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy\right\}\times\sum_{j=1}^\infty(j+1)\cdot\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}}.
\end{aligned}$$

Moreover, since $w^q \in A_1 \subset A_\infty$, by using the inequality (2.2) again, we have

$$\sum_{j=1}^{\infty}(j+1)\cdot\frac{w^q(B)^{1/q-\kappa}}{w^q(2^{j+1}B)^{1/q-\kappa}} \leqslant C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta^*(1/q-\kappa)}$$
$$\leqslant C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{1}{2^{(j+1)n}}\right)^{\delta^*(1/q-\kappa)} \leqslant C, \tag{4.6}$$

which in turn gives that

$$I_6' \leqslant C\cdot\sup_B\left\{\frac{1+\log^+\left(\frac{w(B)}{w^q(B)^\kappa}\right)}{w^q(B)^\kappa}\int_B\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy\right\}.$$

Combining all the above estimates, we are done. □

Proof of Theorem 1.4. For any ball $B = B(x_0,r) \subseteq \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r > 0$, we set $f = f\cdot\chi_{2B} + f\cdot\chi_{(2B)^c} := f_1 + f_2$. Then for each fixed $\sigma > 0$, we have

$$\begin{aligned}
&\left(\frac{1}{\Theta^q(r)}\cdot\left|\left\{x\in B : \left|[b,\mathscr{T}_\alpha](f)(x)\right| > \sigma\right\}\right|\right)^{1/q}\\
\leqslant&\left(\frac{1}{\Theta^q(r)}\cdot\left|\left\{x\in B : \left|[b,\mathscr{T}_\alpha](f_1)(x)\right| > \sigma/2\right\}\right|\right)^{1/q}\\
&+\left(\frac{1}{\Theta^q(r)}\cdot\left|\left\{x\in B : \left|[b,\mathscr{T}_\alpha](f_2)(x)\right| > \sigma/2\right\}\right|\right)^{1/q}\\
:=&J_1' + J_2'.
\end{aligned}$$

We consider the term J_1' first. The assumption (1.8) and the inequality (2.3) yield that

$$\begin{aligned}
J_1' &\leqslant C_0\cdot\frac{1}{\Theta(r)}\int_B\Phi\left(\frac{|f_1(x)|}{\sigma}\right)dx\\
&= C_0\cdot\frac{1}{\Theta(r)}\int_{2B}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\\
&= C_0\cdot\frac{\Theta(2r)}{\Theta(r)}\cdot\frac{1}{\Theta(2r)}\int_{B(x_0,2r)}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\\
&\leqslant C\cdot\sup_{r>0}\left\{\frac{1}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\right\}\\
&\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\right\}.
\end{aligned}$$

We now turn our attention to the estimate of J_2'. Recall that the following estimate holds for given $0 < \alpha < n$ and any $x \in B$,

$$\left|[b,\mathscr{T}_\alpha](f_2)(x)\right| \leqslant \widetilde{\mu}(x) + \widetilde{\nu}(x),$$

where

$$\widetilde{\mu}(x)=c_3\big|b(x)-b_B\big|\cdot\int_{\mathbb{R}^n}\frac{|f_2(y)|}{|x-y|^{n-\alpha}}dy,$$

and

$$\widetilde{\nu}(x)=c_3\int_{\mathbb{R}^n}\frac{|b(y)-b_B|\cdot|f_2(y)|}{|x-y|^{n-\alpha}}dy.$$

Thus, we have

$$\begin{aligned}J_2'&\leqslant\left(\frac{1}{\Theta^q(r)}\cdot\big|\{x\in B:\widetilde{\mu}(x)>\sigma/4\}\big|\right)^{1/q}+\left(\frac{1}{\Theta^q(r)}\cdot\big|\{x\in B:\widetilde{\nu}(x)>\sigma/4\}\big|\right)^{1/q}\\&:=J_3'+J_4'.\end{aligned}$$

Using the previous pointwise estimate (4.1), Chebyshev's inequality, John–Nirenberg's inequality and the fact that $1/q=1-\alpha/n$, we deduce that

$$\begin{aligned}J_3'&\leqslant\frac{1}{\Theta(r)}\cdot\frac{4}{\sigma}\left(\int_B\widetilde{\mu}(x)^q\,dx\right)^{1/q}\\&\leqslant\frac{C}{\Theta(r)}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}dy\times\left\{|B|\cdot\frac{1}{|B|}\int_B\big|b(x)-b_B\big|^q dx\right\}^{1/q}\\&\leqslant C\|b\|_*\sum_{j=1}^{\infty}\frac{|B|^{1/q}}{|2^{j+1}B|^{1/q}}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\cdot\frac{1}{\Theta(2^{j+1}r)}\int_{B(x_0,2^{j+1}r)}\frac{|f(y)|}{\sigma}dy.\end{aligned}$$

Observe that $t\leqslant t\cdot(1+\log^+t)=\Phi(t)$, we get

$$\begin{aligned}J_3'&\leqslant C\|b\|_*\sum_{j=1}^{\infty}\frac{|B|^{1/q}}{|2^{j+1}B|^{1/q}}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\times\frac{1}{\Theta(2^{j+1}r)}\int_{B(x_0,2^{j+1}r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\\&\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\right\}\times\sum_{j=1}^{\infty}\frac{|B|^{1/q}}{|2^{j+1}B|^{1/q}}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}.\end{aligned}$$

Note that $1\leqslant D(\Theta)<2^{n/q}$, then by using the doubling condition (2.3) of Θ, we are able to verify that

$$\sum_{j=1}^{\infty}\frac{|B|^{1/q}}{|2^{j+1}B|^{1/q}}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\leqslant\sum_{j=1}^{\infty}\left(\frac{D(\Theta)}{2^{n/q}}\right)^{j+1}\leqslant C. \tag{4.7}$$

Hence

$$J_3'\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\right\}.$$

Applying the previous pointwise estimate (4.4) and Chebyshev's inequality, we have

$$\begin{aligned}
J_4' &\leqslant \frac{1}{\Theta(r)}\cdot\frac{4}{\sigma}\left(\int_B \widetilde{\nu}(x)^q\,dx\right)^{1/q}\\
&\leqslant \frac{|B|^{1/q}}{\Theta(r)}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\big|b(y)-b_B\big|\cdot\big|f(y)\big|\,dy\\
&\leqslant \frac{|B|^{1/q}}{\Theta(r)}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\big|b(y)-b_{2^{j+1}B}\big|\cdot\big|f(y)\big|\,dy\\
&\quad+\frac{|B|^{1/q}}{\Theta(r)}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\big|b_{2^{j+1}B}-b_B\big|\cdot\big|f(y)\big|\,dy\\
&:= J_5'+J_6'.
\end{aligned}$$

For the term J_5', notice that the inequality $\Phi(a\cdot b)\leqslant\Phi(a)\cdot\Phi(b)$ holds for any $a,b>0$, when $\Phi(t)=t\cdot(1+\log^+ t)$. We then use the generalized Hölder's inequality (2.6), (2.8) and (2.4) together with (4.7) to obtain

$$\begin{aligned}
J_5' &\leqslant \frac{|B|^{1/q}}{\Theta(r)}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\big|2^{j+1}B\big|^{\alpha/n}\cdot\big\|b-b_{2^{j+1}B}\big\|_{\exp L,2^{j+1}B}\big\|f\big\|_{L\log L,2^{j+1}B}\\
&\leqslant \frac{C\|b\|_*}{\sigma}\cdot\frac{|B|^{1/q}}{\Theta(r)}\sum_{j=1}^{\infty}\big|2^{j+1}B\big|^{\alpha/n}\times\inf_{\eta>0}\left\{\eta+\frac{\eta}{|2^{j+1}B|}\int_{2^{j+1}B}\Phi\left(\frac{|f(z)|}{\eta}\right)dz\right\}\\
&\leqslant \frac{C\|b\|_*}{\sigma}\cdot\frac{|B|^{1/q}}{\Theta(r)}\sum_{j=1}^{\infty}\big|2^{j+1}B\big|^{\alpha/n}\\
&\quad\times\left\{\frac{\sigma\cdot\Theta(2^{j+1}r)}{|2^{j+1}B|}+\frac{\sigma}{|2^{j+1}B|}\cdot\frac{\Theta(2^{j+1}r)}{|2^{j+1}B|}\cdot\Phi\left(\frac{|2^{j+1}B|}{\Theta(2^{j+1}r)}\right)\int_{2^{j+1}B}\Phi\left(\frac{|f(z)|}{\sigma}\right)dz\right\}\\
&\leqslant C\|b\|_*\cdot\left[1+\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(z)|}{\sigma}\right)dz\right\}\right]\\
&\quad\times\sum_{j=1}^{\infty}\frac{|B|^{1/q}}{|2^{j+1}B|^{1/q}}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\\
&\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(z)|}{\sigma}\right)dz\right\}.
\end{aligned}$$

For the last term J_6', in view of the inequality (4.5) and the fact that $t\leqslant\Phi(t)$, we get

$$\begin{aligned}
J_6' &\leqslant C\cdot\frac{|B|^{1/q}}{\Theta(r)}\sum_{j=1}^{\infty}(j+1)\|b\|_*\cdot\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}\frac{|f(y)|}{\sigma}\,dy\\
&\leqslant C\cdot\frac{|B|^{1/q}}{\Theta(r)}\sum_{j=1}^{\infty}(j+1)\|b\|_*\cdot\frac{\Theta(2^{j+1}r)}{|2^{j+1}B|^{1-\alpha/n}}\cdot\frac{1}{\Theta(2^{j+1}r)}\int_{B(x_0,2^{j+1}r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy
\end{aligned}$$

$$\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\right\}$$
$$\times\sum_{j=1}^{\infty}(j+1)\cdot\frac{|B|^{1/q}}{|2^{j+1}B|^{1/q}}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}.$$

Moreover, by using the doubling condition (2.3) of Θ again and the fact that $1\leqslant D(\Theta)<2^{n/q}$, we find that

$$\sum_{j=1}^{\infty}(j+1)\cdot\frac{|B|^{1/q}}{|2^{j+1}B|^{1/q}}\cdot\frac{\Theta(2^{j+1}r)}{\Theta(r)}\leqslant C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{D(\Theta)}{2^{n/q}}\right)^{j+1}\leqslant C. \tag{4.8}$$

Substituting the above inequality (4.8) into the term J_6', we finally obtain

$$J_6'\leqslant C\cdot\sup_{r>0}\left\{\frac{1+\log^+\left(\frac{|B(x_0,r)|}{\Theta(r)}\right)}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(y)|}{\sigma}\right)dy\right\}.$$

Summing up all the above estimates, we finish the proof of the main theorem. □

5. Some applications

In this section, we will give some applications of our main theorems to several integral operators such as θ-type Calderón–Zygmund operators, Marcinkiewicz integral operators, Littlewood–Paley operators, Bochner–Riesz means, fractional maximal functions and fractional integrals.

5.1. θ-type Calderón–Zygmund operators

Calderón–Zygmund singular integral operators and their generalizations on the Euclidean space $\mathbb{R}^n$ have been extensively studied (see [7, 8, 31, 33] for instance). In particular, Yabuta [33] introduced certain θ-type Calderón–Zygmund operators to facilitate his study of certain classes of pseudo-differential operators. Let θ be a non-negative, non-decreasing function on $(0,+\infty)$ with

$$\int_0^1\frac{\theta(t)\cdot|\log t|}{t}dt<\infty.$$

A measurable function K on $\mathbb{R}^n\times\mathbb{R}^n\backslash\{(x,x):x\in\mathbb{R}^n\}$ is said to be a θ-type kernel if it satisfies

(i) $|K(x,y)|\leqslant C\cdot|x-y|^{-n}$, for any $x\neq y$;

(ii) $|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)|\leqslant C\cdot\theta(|x-z|/|x-y|)|x-y|^{-n}$, for $|x-z|<|x-y|/2$.

Let T_θ be a linear operator from $\mathscr{S}(\mathbb{R}^n)$ into its dual $\mathscr{S}'(\mathbb{R}^n)$. We say that T_θ is a θ-type Calderón–Zygmund operator if

(1) T_θ can be extended to be a bounded operator on $L^2(\mathbb{R}^n)$;

(2) There is a θ-type kernel K such that $T_\theta f(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy$ for all $f \in C_0^\infty(\mathbb{R}^n)$ and for all $x \notin \operatorname{supp} f$, where $C_0^\infty(\mathbb{R}^n)$ is the space consisting of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports. If $b \in BMO(\mathbb{R}^n)$, we define the commutator $[b,T_\theta]$ to be the operator

$$[b,T_\theta]f(x) = b(x)\cdot T_\theta f(x) - T_\theta(bf)(x) = \int_{\mathbb{R}^n}[b(x)-b(y)]K(x,y)f(y)\,dy.$$

The following endpoint estimates for commutator of the θ-type Calderón–Zygmund operator were established in [16] and [36].

THEOREM 5.1. ([36]) *Let $w \in A_1$ and $b \in BMO(\mathbb{R}^n)$. Then for all $\sigma > 0$, there is a constant $C_0 > 0$ independent of f and σ such that*

$$w\big(\big\{x \in \mathbb{R}^n : \big|[b,T_\theta](f)(x)\big| > \sigma\big\}\big) \leqslant C_0 \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx,$$

where $\Phi(t) = t(1+\log^+ t)$.

THEOREM 5.2. ([16]) *Let $b \in BMO(\mathbb{R}^n)$. Then for all $\sigma > 0$, there is a constant $C_0 > 0$ independent of f and σ such that*

$$\big|\big\{x \in \mathbb{R}^n : \big|[b,T_\theta](f)(x)\big| > \sigma\big\}\big| \leqslant C_0 \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right)dx,$$

where $\Phi(t) = t(1+\log^+ t)$.

Then, from Theorem 1.1 and Theorem 1.2, we immediately get the following:

COROLLARY 5.3. *Let $0 < \kappa < 1$, $w \in A_1$ and $b \in BMO(\mathbb{R}^n)$. Then for any given $\sigma > 0$ and any ball B, there exists a constant $C > 0$ independent of f, B and σ such that*

$$\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x \in B : \big|[b,T_\theta](f)(x)\big| > \sigma\big\}\big) \leqslant C\cdot\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\kappa}_{L\log L}(w)},$$

where $\Phi(t) = t(1+\log^+ t)$.

COROLLARY 5.4. *Let $b \in BMO(\mathbb{R}^n)$. Suppose that Θ satisfies (2.3) and $1 \leqslant D(\Theta) < 2^n$, then for any given $\sigma > 0$ and any ball $B(x_0,r)$, there exists a constant $C > 0$ independent of f, $B(x_0,r)$ and σ such that*

$$\frac{1}{\Theta(r)}\cdot\big|\big\{x \in B(x_0,r) : \big|[b,T_\theta](f)(x)\big| > \sigma\big\}\big| \leqslant C\cdot\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\Theta}_{L\log L}},$$

where $\Phi(t) = t(1+\log^+ t)$.

5.2. Marcinkiewicz integral operators

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \geqslant 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(S^{n-1})$ and $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$, where $x' = x/|x|$ for any $x \neq 0$. Then the Marcinkiewicz integral of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \big|F_{\Omega,t}(x)\big|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy.$$

For $b \in BMO(\mathbb{R}^n)$, the commutator operator $[b,\mu_\Omega]$ is defined by (see [6])

$$[b,\mu_\Omega](f)(x) = \left(\int_0^\infty \big|F^b_{\Omega,t}(x)\big|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F^b_{\Omega,t}(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)] f(y)\,dy.$$

For $0 < \alpha \leqslant 1$, we say that $\Omega \in Lip_\alpha(S^{n-1})$, if there exists a constant $L > 0$ such that

$$\big|\Omega(x') - \Omega(y')\big| \leqslant L|x'-y'|^\alpha, \quad \text{for any } x', y' \in S^{n-1}.$$

Let $\mathbb{H}$ be the Banach space

$$\mathbb{H} = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t^3}\right)^{1/2} < \infty \right\}.$$

Then, it is clear that $[b,\mu_\Omega](f)(x) = \|F^b_{\Omega,t}(x)\|$. By Minkowski's inequality and the condition on Ω, we can get

$$\begin{aligned} \big|[b,\mu_\Omega](f)(x)\big| &\leqslant \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} \big|b(x)-b(y)\big| \cdot \big|f(y)\big| \left(\int_{|x-y|}^\infty \frac{dt}{t^3}\right)^{1/2} dy \\ &\leqslant c_2 \int_{\mathbb{R}^n} \frac{|b(x)-b(y)| \cdot |f(y)|}{|x-y|^n} dy, \end{aligned}$$

where c_2 is an absolute constant independent of f and $x \in \mathbb{R}^n$. Thus, $[b,\mu_\Omega]$ satisfies the condition (1.3). Moreover, in [6], Ding et al. considered the weighted weak $L\log L$-type estimate for the commutator $[b,\mu_\Omega]$ and proved:

THEOREM 5.5. ([6]) *Let $0 < \alpha \leqslant 1$, $\Omega \in Lip_\alpha(S^{n-1})$, $w \in A_1$ and $b \in BMO(\mathbb{R}^n)$. Then for all $\sigma > 0$, there is a constant $C_0 > 0$ independent of f and σ such that*

$$w\big(\{x \in \mathbb{R}^n : \big|[b,\mu_\Omega](f)(x)\big| > \sigma\}\big) \leqslant C_0 \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\,dx,$$

where $\Phi(t) = t(1+\log^+ t)$.

In particular, we have the following estimate if w is taken to be a constant function.

THEOREM 5.6. ([6]) *Let $0<\alpha\leqslant 1$, $\Omega\in Lip_\alpha(S^{n-1})$ and $b\in BMO(\mathbb{R}^n)$. Then for all $\sigma>0$, there is a constant $C_0>0$ independent of f and σ such that*

$$\left|\left\{x\in\mathbb{R}^n:\left|[b,\mu_\Omega](f)(x)\right|>\sigma\right\}\right|\leqslant C_0\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx,$$

where $\Phi(t)=t(1+\log^+ t)$.

As a consequence of Theorem 1.1 and Theorem 1.2, we obtain the following results:

COROLLARY 5.7. *Let $0<\alpha\leqslant 1$, $\Omega\in Lip_\alpha(S^{n-1})$, $0<\kappa<1$, $w\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x\in B:\left|[b,\mu_\Omega](f)(x)\right|>\sigma\big\}\big)\leqslant C\cdot\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\kappa}_{L\log L}(w)},$$

where $\Phi(t)=t(1+\log^+ t)$.

COROLLARY 5.8. *Let $0<\alpha\leqslant 1$, $\Omega\in Lip_\alpha(S^{n-1})$ and $b\in BMO(\mathbb{R}^n)$. Suppose that Θ satisfies (2.3) and $1\leqslant D(\Theta)<2^n$, then for any given $\sigma>0$ and any ball $B(x_0,r)$, there exists a constant $C>0$ independent of f, $B(x_0,r)$ and σ such that*

$$\frac{1}{\Theta(r)}\cdot\left|\left\{x\in B(x_0,r):\left|[b,\mu_\Omega](f)(x)\right|>\sigma\right\}\right|\leqslant C\cdot\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\Theta}_{L\log L}},$$

where $\Phi(t)=t(1+\log^+ t)$.

5.3. Littlewood–Paley operators

Let $\varepsilon>0$ and ψ be a fixed function which satisfies the following properties:
(1) $\psi\in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}\psi(x)\,dx=0$;
(2) $\psi(x)\leqslant C\cdot(1+|x|)^{-(n+1)}$;
(3) $\left|\psi(x+y)-\psi(x)\right|\leqslant C\cdot|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y|<|x|$.

We set $\psi_t(x)=t^{-n}\psi(x/t)$ and $\Gamma(x)=\big\{(y,t)\in\mathbb{R}^{n+1}_+:|x-y|<t\big\}$. The Littlewood–Paley g-function, Lusin area integral and the g^*_λ-function will be defined respectively by (see [32])

$$g_\psi(f)(x)=\left(\int_0^\infty\left|\psi_t*f(x)\right|^2\frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x)=\left(\iint_{\Gamma(x)}\left|\psi_t*f(y)\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

and

$$g^*_{\lambda,\psi}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \psi_t * f(y) \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad \lambda > 1.$$

For $b \in BMO(\mathbb{R}^n)$, we will consider the commutators generated by b and Littlewood–Paley operators, which are defined respectively by the following expressions (see [34]):

$$[b, g_\psi](f)(x) = \left(\int_0^\infty \left| \int_{\mathbb{R}^n} [b(x) - b(y)] \psi_t(x-y) f(y)\, dy \right|^2 \frac{dt}{t} \right)^{1/2},$$

$$[b, S_\psi](f)(x) = \left(\iint_{\Gamma(x)} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \psi_t(y-z) f(z)\, dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

and

$$[b, g^*_{\lambda,\psi}](f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \psi_t(y-z) f(z)\, dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \lambda > 1.$$

Let $\mathbb{H}$ be the Banach space

$$\mathbb{H} = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty \right\}$$

or

$$\mathbb{H} = \left\{ h : \|h\| = \left(\iint_{\mathbb{R}^{n+1}_+} |h(y,t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} < \infty \right\}.$$

If we set

$$F^b_{\psi,t}(x) = \int_{\mathbb{R}^n} [b(x) - b(y)] \psi_t(x-y) f(y)\, dy,$$

$$F^b_{\psi,t}(x,y) = \int_{\mathbb{R}^n} [b(x) - b(z)] \psi_t(y-z) f(z)\, dz,$$

and denote the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$, then, for each fixed $x \in \mathbb{R}^n$, it is easy to see that

$$[b, g_\psi](f)(x) = \left\| F^b_{\psi,t}(x) \right\|, \quad [b, S_\psi](f)(x) = \left\| \chi_{\Gamma(x)} \cdot F^b_{\psi,t}(x,y) \right\|,$$

and

$$[b, g^*_{\lambda,\psi}](f)(x) = \left\| \left(\frac{t}{t+|x-y|} \right)^{\lambda n/2} \cdot F^b_{\psi,t}(x,y) \right\|.$$

By using Minkowski's inequality and the condition on ψ, we can get

$$\begin{aligned}\big|[b,g_{\psi}](f)(x)\big| &\leqslant c_2\int_{\mathbb{R}^n}|b(x)-b(y)|\cdot|f(y)|\left(\int_0^{\infty}\left(\frac{1}{t^n}\cdot\frac{1}{[1+t^{-1}|x-y|]^{n+1}}\right)^2\frac{dt}{t}\right)^{1/2}dy\\ &\leqslant c_2\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|\cdot|f(y)|}{|x-y|^n}dy.\end{aligned}$$

Similarly, we can also prove

$$\big|[b,S_{\psi}](f)(x)\big|\leqslant c_2\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|\cdot|f(y)|}{|x-y|^n}dy,$$

and

$$\big|[b,g^*_{\lambda,\psi}](f)(x)\big|\leqslant c_2\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|\cdot|f(y)|}{|x-y|^n}dy,$$

where c_2 is an absolute constant independent of f and $x\in\mathbb{R}^n$. Thus, $[b,g_{\psi}]$, $[b,S_{\psi}]$ and $[b,g^*_{\lambda,\psi}]$ all satisfy the condition (1.3). The following endpoint estimates for these commutator operators $[b,g_{\psi}]$, $[b,S_{\psi}]$ and $[b,g^*_{\lambda,\psi}]$ were proved by Xue and Ding in [34], when $b\in BMO(\mathbb{R}^n)$ and $w\in A_1$.

THEOREM 5.9. ([34]) *Let $\lambda>3$, $w\in A_1$, $b\in BMO(\mathbb{R}^n)$ and ψ be a function on $\mathbb{R}^n$ satisfying (1)–(3) mentioned above. Then for all $\sigma>0$, there is a constant $C_0>0$ independent of f and σ such that*

$$w\big(\big\{x\in\mathbb{R}^n:\big|[b,T_{\psi}](f)(x)\big|>\sigma\big\}\big)\leqslant C_0\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx,$$

where $\Phi(t)=t(1+\log^+t)$ and T_{ψ} is g_{ψ} or S_{ψ} or $g^*_{\lambda,\psi}$.

THEOREM 5.10. ([34]) *Let $\lambda>3$, $b\in BMO(\mathbb{R}^n)$ and ψ be a function on $\mathbb{R}^n$ satisfying (1)–(3) mentioned above. Then for all $\sigma>0$, there is a constant $C_0>0$ independent of f and σ such that*

$$\big|\big\{x\in\mathbb{R}^n:\big|[b,T_{\psi}](f)(x)\big|>\sigma\big\}\big|\leqslant C_0\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx,$$

where $\Phi(t)=t(1+\log^+t)$ and T_{ψ} is g_{ψ} or S_{ψ} or $g^*_{\lambda,\psi}$.

Then, from Theorem 1.1 and Theorem 1.2, we can show that:

COROLLARY 5.11. *Let $\lambda>3$, $0<\kappa<1$, $w\in A_1$, $b\in BMO(\mathbb{R}^n)$ and ψ be a function on $\mathbb{R}^n$ satisfying (1)–(3) mentioned above. Then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{w(B)^{\kappa}}\cdot w\big(\big\{x\in B:\big|[b,T_{\psi}](f)(x)\big|>\sigma\big\}\big)\leqslant C\cdot\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\kappa}_{L\log L}(w)},$$

where $\Phi(t)=t(1+\log^+t)$ and T_{ψ} is g_{ψ} or S_{ψ} or $g^*_{\lambda,\psi}$.

COROLLARY 5.12. *Let $\lambda > 3$, $b \in BMO(\mathbb{R}^n)$ and ψ be a function on $\mathbb{R}^n$ satisfying* (1)–(3) *mentioned above. Suppose that Θ satisfies* (2.3) *and $1 \leqslant D(\Theta) < 2^n$, then for any given $\sigma > 0$ and any ball $B(x_0,r)$, there exists a constant $C > 0$ independent of f, $B(x_0,r)$ and σ such that*

$$\frac{1}{\Theta(r)} \cdot \left|\left\{x \in B(x_0,r) : \left|[b,T_\psi](f)(x)\right| > \sigma\right\}\right| \leqslant C \cdot \left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathcal{M}^{1,\Theta}_{L\log L}},$$

where $\Phi(t) = t(1+\log^+ t)$ and T_ψ is g_ψ or S_ψ or $g^*_{\lambda,\psi}$.

5.4. Bochner–Riesz means

The Bochner–Riesz means of order $\delta > 0$ in $\mathbb{R}^n$ are defined initially for Schwartz functions in terms of Fourier transforms by

$$\big(\widehat{T^\delta_R f}\big)(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta_+ \widehat{f}(\xi), \quad 0 < R < \infty,$$

where $\widehat{f}$ denotes the Fourier transform of f. We recall that the Bochner–Riesz means can be expressed as convolution operators (see [17, 30])

$$T^\delta_R f(x) = (\phi_{1/R} * f)(x),$$

where $\phi(x) = [(1-|\cdot|^2)^\delta_+]^\wedge(x)$ and $\phi_{1/R}(x) = R^n \cdot \phi(Rx)$. It is well known that the kernel ϕ can be represented as (see [17, 30])

$$\phi(x) = \pi^{-\delta}\Gamma(\delta+1)|x|^{-(\frac{n}{2}+\delta)}J_{\frac{n}{2}+\delta}(2\pi|x|),$$

where $J_\mu(t)$ is the Bessel function

$$J_\mu(t) = \frac{(\frac{t}{2})^\mu}{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{its}(1-s^2)^{\mu-\frac{1}{2}}\,ds.$$

Let $b \in BMO(\mathbb{R}^n)$ and $0 < R < \infty$. Consider the commutator $\big[b,T^\delta_R\big]$ defined by

$$\big[b,T^\delta_R\big](f)(x) = b(x)\cdot T^\delta_R f(x) - T^\delta_R(bf)(x) = \int_{\mathbb{R}^n}[b(x)-b(y)]\phi_{1/R}(x-y)f(y)\,dy.$$

The maximal operator $\big[b,T^\delta_*\big]$ associated with the commutator is defined by

$$\big[b,T^\delta_*\big](f)(x) = \sup_{R>0}\Big|\big[b,T^\delta_R\big](f)(x)\Big|.$$

Let $\mathbb{H}$ be the space

$$\mathbb{H} = \left\{h : \|h\| = \sup_{R>0}\big|h(R)\big| < \infty\right\}.$$

Then, it is clear that $\big[b,T_*^{\delta}\big](f)(x) = \|b(x)\cdot T_R^{\delta} f(x) - T_R^{\delta}(bf)(x)\|$. If $\delta \geqslant (n-1)/2$, by the kernel estimates of T_R^{δ}, we have

$$\begin{aligned}\big|\big[b,T_*^{\delta}\big](f)(x)\big| &\leqslant c_2\cdot\sup_{R>0}\int_{\mathbb{R}^n}\big|b(x)-b(y)\big|\frac{R^n}{(1+R|x-y|)^{\delta+\frac{n+1}{2}}}\cdot\big|f(y)\big|\,dy\\ &\leqslant c_2\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|\cdot|f(y)|}{|x-y|^n}\,dy,\end{aligned}$$

where c_2 is an absolute constant independent of f and $x\in\mathbb{R}^n$. Thus, $\big[b,T_*^{\delta}\big]$ satisfies the condition (1.3). Furthermore, in [15], Liu and Lu established weighted endpoint estimates of $L\log L$-type for maximal commutators of the Bochner–Riesz means.

THEOREM 5.13. ([15]) *Let $\delta > (n-1)/2$, $w\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Then for all $\sigma>0$, there is a constant $C_0>0$ independent of f and σ such that*

$$w\big(\big\{x\in\mathbb{R}^n : \big|\big[b,T_*^{\delta}\big](f)(x)\big| > \sigma\big\}\big) \leqslant C_0\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx,$$

where $\Phi(t) = t(1+\log^+ t)$.

THEOREM 5.14. ([15]) *Let $\delta > (n-1)/2$ and $b\in BMO(\mathbb{R}^n)$. Then for all $\sigma>0$, there is a constant $C_0>0$ independent of f and σ such that*

$$\big|\big\{x\in\mathbb{R}^n : \big|\big[b,T_*^{\delta}\big](f)(x)\big| > \sigma\big\}\big| \leqslant C_0\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx,$$

where $\Phi(t) = t(1+\log^+ t)$.

As a consequence of Theorem 1.1 and Theorem 1.2, we can prove the following results:

COROLLARY 5.15. *Let $\delta > (n-1)/2$, $0<\kappa<1$, $w\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{w(B)^{\kappa}}\cdot w\big(\big\{x\in B : \big|\big[b,T_*^{\delta}\big](f)(x)\big| > \sigma\big\}\big) \leqslant C\cdot\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathcal{M}^{1,\kappa}_{L\log L}(w)},$$

where $\Phi(t) = t(1+\log^+ t)$.

COROLLARY 5.16. *Let $\delta > (n-1)/2$ and $b\in BMO(\mathbb{R}^n)$. Suppose that Θ satisfies (2.3) and $1\leqslant D(\Theta)<2^n$, then for any given $\sigma>0$ and any ball $B(x_0,r)$, there exists a constant $C>0$ independent of f, $B(x_0,r)$ and σ such that*

$$\frac{1}{\Theta(r)}\cdot\big|\big\{x\in B(x_0,r) : \big|\big[b,T_*^{\delta}\big](f)(x)\big| > \sigma\big\}\big| \leqslant C\cdot\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathcal{M}^{1,\Theta}_{L\log L}},$$

where $\Phi(t) = t(1+\log^+ t)$.

5.5. Fractional integrals

For given α, $0<\alpha<n$, the fractional integral operator (or the Riesz potential) I_α is defined by (see [29])

$$I_\alpha f(x)=\frac{1}{\gamma(\alpha)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy,\quad \gamma(\alpha)=\frac{2^\alpha\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

We also define the associated fractional maximal function with order α by

$$M_\alpha(f)(x)=\sup_{x\in B}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B|f(y)|\,dy,$$

where the supremum is taken over all balls containing x. When $b\in BMO(\mathbb{R}^n)$, the commutators $[b,I_\alpha]$ and $[b,M_\alpha]$ are defined as

$$[b,I_\alpha]f(x)=b(x)\cdot I_\alpha f(x)-I_\alpha(bf)(x)=\int_{\mathbb{R}^n}[b(x)-b(y)]\cdot\frac{f(y)}{|x-y|^{n-\alpha}}\,dy,$$

$$[b,M_\alpha](f)(x)=\sup_{x\in B}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B\big|b(x)-b(y)\big|\cdot|f(y)|\,dy.$$

In [3, 4], Cruz-Uribe and Fiorenza discussed the unweighted and weighted endpoint inequalities for commutators of fractional integrals and proved the following:

THEOREM 5.17. *Let $0<\alpha<n$, $q=n/(n-\alpha)$, $w^q\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Then for any given $\sigma>0$ and any bounded domain $\Omega\subset\mathbb{R}^n$, there is a constant $C_0>0$ which does not depend on f, Ω and σ such that*

$$\Big[w^q\big(\big\{x\in\Omega:\big|[b,I_\alpha](f)(x)\big|>\sigma\big\}\big)\Big]^{1/q}\leqslant C_0\int_\Omega\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx,$$

where $\Phi(t)=t(1+\log^+t)$.

THEOREM 5.18. *Let $0<\alpha<n$, $q=n/(n-\alpha)$ and $b\in BMO(\mathbb{R}^n)$. Then for any given $\sigma>0$ and any bounded domain $\Omega\subset\mathbb{R}^n$, there is a constant $C_0>0$ which does not depend on f, Ω and σ such that*

$$\big|\big\{x\in\Omega:\big|[b,I_\alpha](f)(x)\big|>\sigma\big\}\big|^{1/q}\leqslant C_0\int_\Omega\Phi\left(\frac{|f(x)|}{\sigma}\right)dx,$$

where $\Phi(t)=t(1+\log^+t)$.

Then, from Theorem 1.3 and Theorem 1.4, we immediately get the following:

COROLLARY 5.19. *Let $0<\alpha<n$, $q=n/(n-\alpha)$, $0<\kappa<1/q$, $w^q\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Then for any given $\sigma>0$ and any ball $B\subset\mathbb{R}^n$, there exists a constant $C>0$ independent of f, B and σ such that*

$$\left(\frac{1}{w^q(B)^{\kappa q}}\cdot w^q\big(\big\{x\in B:\big|[b,I_\alpha](f)(x)\big|>\sigma\big\}\big)\right)^{1/q}\leqslant C\cdot\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathcal{M}^{1,\kappa}_{L\log L}(w,w^q)},$$

where $\Phi(t)=t(1+\log^+t)$.

COROLLARY 5.20. *Let* $0 < \alpha < n$, $q = n/(n-\alpha)$ *and* $b \in BMO(\mathbb{R}^n)$. *Suppose that* Θ *satisfies* (2.3) *and* $1 \leqslant D(\Theta) < 2^{n/q}$, *then for any given* $\sigma > 0$ *and any ball* $B(x_0,r) \subset \mathbb{R}^n$, *there exists a constant* $C > 0$ *independent of* f, $B(x_0,r)$ *and* σ *such that*

$$\left(\frac{1}{\Theta^q(r)} \cdot \left|\left\{x \in B(x_0,r) : \left|[b,I_\alpha](f)(x)\right| > \sigma\right\}\right|\right)^{1/q} \leqslant C \cdot \left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\Theta}_{L\log L}},$$

where $\Phi(t) = t(1+\log^+ t)$.

If we define the commutator $[b,I_\alpha]^+$ by

$$[b,I_\alpha]^+(f)(x) = \int_{\mathbb{R}^n} \left|b(x)-b(y)\right| \cdot \frac{f(y)}{|x-y|^{n-\alpha}}\,dy,$$

then from the proof of Theorem 1.3 and Theorem 1.4, we know that the conclusions of Corollaries 5.19 and 5.20 still hold if one has $[b,I_\alpha]^+$ instead of $[b,I_\alpha]$. It should be pointed out that $[b,M_\alpha](f)$ can be controlled pointwise by $[b,I_\alpha]^+(|f|)$ for any $f(x)$ (see [5]). In fact, for any $0 < \alpha < n$, $x \in \mathbb{R}^n$ and $r > 0$, we have

$$[b,I_\alpha]^+(|f|)(x) \geqslant \int_{|y-x|\leqslant r} \frac{|b(x)-b(y)| \cdot |f(y)|}{|x-y|^{n-\alpha}}\,dy \geqslant \frac{1}{r^{n-\alpha}} \int_{|y-x|\leqslant r} |b(x)-b(y)| \cdot |f(y)|\,dy.$$

Taking the supremum for all $r > 0$ on both sides of the above inequality, we get

$$[b,M_\alpha](f)(x) \leqslant [b,I_\alpha]^+(|f|)(x), \quad \text{for all } x \in \mathbb{R}^n.$$

Hence, as a direct consequence of the above results, we finally obtain

COROLLARY 5.21. *Let* $0 < \alpha < n$, $q = n/(n-\alpha)$, $0 < \kappa < 1/q$, $w^q \in A_1$ *and* $b \in BMO(\mathbb{R}^n)$. *Then for any given* $\sigma > 0$ *and any ball* $B \subset \mathbb{R}^n$, *there exists a constant* $C > 0$ *independent of* f, B *and* σ *such that*

$$\left(\frac{1}{w^q(B)^{\kappa q}} \cdot w^q\left(\left\{x \in B : \left|[b,M_\alpha](f)(x)\right| > \sigma\right\}\right)\right)^{1/q} \leqslant C \cdot \left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\kappa}_{L\log L}(w,w^q)},$$

where $\Phi(t) = t(1+\log^+ t)$.

COROLLARY 5.22. *Let* $0 < \alpha < n$, $q = n/(n-\alpha)$ *and* $b \in BMO(\mathbb{R}^n)$. *Suppose that* Θ *satisfies* (2.3) *and* $1 \leqslant D(\Theta) < 2^{n/q}$, *then for any given* $\sigma > 0$ *and any ball* $B(x_0,r) \subset \mathbb{R}^n$, *there exists a constant* $C > 0$ *independent of* f, $B(x_0,r)$ *and* σ *such that*

$$\left(\frac{1}{\Theta^q(r)} \cdot \left|\left\{x \in B(x_0,r) : \left|[b,M_\alpha](f)(x)\right| > \sigma\right\}\right|\right)^{1/q} \leqslant C \cdot \left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathscr{M}^{1,\Theta}_{L\log L}},$$

where $\Phi(t) = t(1+\log^+ t)$.

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