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INEQUALITIES FROM GENERAL QUASI–LINEAR MEANS

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Abstract. The paper has a number of aims. The first is to demonstrate the use of the comparison theorem for quasi-linear means to see how mean inequalities, and other apparently unrelated inequalities, can be seen from the perspective of quasi-linear means. Second, we will be generalizing some means, such as the identric mean, by observing its representation as a quasi-linear mean. Finally, we will generalize the quasi-linear mean comparison theorem which provides an extension to the Jensen-Steffensen-Boas inequality for a strictly increasing concave function. This allows for new inequalities to be introduced.

1. Introduction

The quasi-linear mean ([3]) of distribution function F on $[a,b]$, written as $M(\phi,F)$, is given, for some strictly monotone (which we will always take to be increasing) and continuous function $\phi(\cdot)$, by

$$M(\phi,F)=\phi^{-1}\left(\int_a^b \phi(t)\,F(\mathrm{d}t)\right). \tag{1}$$

For a historical perspective on the quasi-linear mean and a collection of the key results, see [12]. Recent applications have been in measures of income inequality, see [10], where the form of (1) is shown to follow as a consequence of a number of axioms, or properties, a mean should possess.

There is also by now a vast literaure on mean inequalities. The most elementary of these being the geometric-arithmetic mean inequality; for positive and finite numbers a and b,

$$G(a,b)\leqslant A(a,b)\quad \text{where}\quad G(a,b)=\sqrt{ab}\quad\text{and}\quad A(a,b)=\tfrac{1}{2}(a+b).$$

There are by now many such inequalities relating to different means, such as the identric mean, the logarithmic mean, the harmonic mean, the Lehmer mean, the power mean, and so on. We provide a few references here, but there are many such articles; see [2], [16], and [7], for example.

There are also articles on inequalities which at first sight have no apparent connection to mean inequalities, yet can be demonstrated to be such; see [15]. The paper [15]

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makes use of quasi-linear means to look at a number of mean inequalities. The relevant result; i.e. the comparison theorem for quasi-linear means, is that

$$M(\phi_1,F) \leqslant M(\phi_2,F)\ \forall \text{ distributions } F \text{ on } [a,b] \iff \phi_1\circ\phi_2^{-1} \text{ is concave on } [a,b]. \tag{2}$$

So it is (2), specifically the sufficiency part, which provides many of the mean inequalities, and other ones beside. Once (ϕ_1,ϕ_2) have been set, the check is the concavity of $\phi_1\circ\phi_2^{-1}$, in order to obtain a mean inequality, which is in most cases an easier check that a direct study of the inequality. A proof of (2) is given in [9], and see also [14]. A novel proof using the Neyman-Pearson lemma is given later in the paper, and this new proof allows a generalization of (2).

Now [5] use (2) for establishing inequalities involving the digamma function and [15] use (2) for inequalities involving polygamma functions. As we have mentioed, the ease of proving proposed inequalities is reduced to checking the concavity of $\phi_1\circ\phi_2^{-1}$, which is equivalent to ϕ_1 being more concave than ϕ_2. On the other hand, when the same $\phi(t)$ is used, if $F_1 \leqslant F_2$ then

$$M(\phi,F_1) \geqslant M(\phi,F_2)$$

providing another source of mean inequalities.

There are a number of generalizations of (1), for example [18] describe the generalized weighted quasi-linear mean

$$M(\phi,F,g) = \phi^{-1}\left(\int_a^b \phi(g(t))\,F(\mathrm{d}t)\right),$$

where g is a real Lebesgue integrable function on $[a,b]$. On the other hand, [11] study a more general version

$$M(\phi,\mathbf{F},\mathbf{g}) = \phi^{-1}\left(\sum_{i=1}^{n}\int_a^b \phi(g_i(t))\,F_i(\mathrm{d}t)\right).$$

The extension considered in the present paper is to allow F in (1) to be more general than a distribution function. While F still takes values in $[0,1]$ and $F(b)=1$, there are no other restrictions. Then we can show that (2) still holds for such F and consequently this allows generalizations of inequalities in the literature. Moreover, the Jensen inequality which drives (2) and the ensuing inequalities is no longer available once we allow F to move away from a distribution function. This extension and new inequalities are the main contribution of the paper.

The layout of the paper is as follows; in Section 2 we characterize some well known means in terms of quasi-linear means. Some known and some new representations. The sub-section 2.1 uses such characterizations to introduce new generalized means to the literature and sub-section 2.2 highlights the full extent of the use of the quasi-linear mean comparison theorem by presenting inequalities realted to the polygamma functions. Section 3 then sets about generalizing the comparison theorem, using the Neyman-Pearson lemma. In particular, a new generalization of the Jensen inequality is given for a strictly increasing concave function and illustrations are presented in Section 4.

2. Basic means and inequalities

First we will list some means in the form of (ϕ, F) for pairs (a,b), so F is a probability distribution on the interval $[a,b]$. We will also use f to denote the density function corresponding to F. In the following $U(a,b)$ denotes the uniform distribution on $[a,b]$.

1. $\phi(t) = s$ and $F = U(a,b)$ gives the arithmetic mean $A(a,b) = \frac{1}{2}(a+b)$. This can also follow from $\phi(t) = t$ and $f(t) = \frac{1}{2}$ at $t = a$ and $f(t) = \frac{1}{2}$ at $t = b$.

2. $\phi(t) = \log t$ and $F = U(a,b)$ gives the identric mean
$$I(a,b) = e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}.$$

3. $\phi(t) = \log t$ and $f(t) = \frac{1}{2}$ at $t = a$ and $f(t) = \frac{1}{2}$ at $t = b$ yields the geometric mean $G(a,b) = \sqrt{ab}$.

4. $\phi(t) = -1/t$ and $F = U(a,b)$ yields the logarithmic mean
$$L(a,b) = \frac{b-a}{\log b - \log a}.$$

5. $\phi(t) = t^r$ and $f(t) = \frac{1}{2}$ at $t = a$ and $f(t) = \frac{1}{2}$ at $t = b$ yields the power mean
$$P_r(a,b) = \left(\tfrac{1}{2}(a^r + b^r)\right)^{1/r}.$$

6. $\phi(t) = -1/t$ and $f(t) = \frac{1}{2}$ at $t = a$ and $f(t) = \frac{1}{2}$ at $t = b$ yields the harmonic mean
$$H(a,b) = (1/a + 1/b)^{-1}.$$

7. $\phi(t) = t^r$ and $F = U(a,b)$ yields the general logarithmic mean
$$L_r(a,b) = \left(\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right)^{1/r}.$$

8. We conclude such examples with the Lehmer mean,
$$L_h(a,b) = \frac{a^{r+1} + b^{r+1}}{a^r + b^r},$$
which becomes the arithmetic mean with $r = 0$. While $\phi(t) = t$, the density function $f(t)$ is more exotic; an extended beta density on $[a,b]$, given by
$$f(t) = c(\alpha, \beta)\,(t-a)^{\alpha-1}(b-t)^{\beta-1}$$

where $c(\alpha,\beta)$ is the normalizing constant. Then

$$\int_a^b t\,f(t)\,\mathrm{d}t = a+(b-a)\frac{\alpha}{\alpha+\beta} = a\frac{\beta}{\alpha+\beta}+b\frac{\alpha}{\alpha+\beta}$$

and hence we recover the Lehmer mean with $\beta = a^r$ and $\alpha = b^r$.

Let us first consider the example 5 and the well known power means inequality, which is that

$$P_{r_1}(a,b) \leqslant P_{r_2}(a,b) \quad \text{for} \quad r_1 \leqslant r_2.$$

Now if $\phi_1(t) = t^{r_1}$ and $\phi_2(t) = t^{r_2}$ then $\phi_1 \circ \phi_2^{-1}(t) = t^{r_1/r_2}$ which is concave for $r_1/r_2 \leqslant 1$. Hence, the power means ordering follows from (2). The geometric mean-arithmetic mean inequality follows from the fact that $\log t$ is concave.

Other inequalities which have been published include

$$L(a,b) \leqslant I(a,b) \leqslant A(a,b),$$

the first inequality following from the fact that $-e^{-t}$ is concave and the second inequality follows from the fact that $\log t$ is concave. Also known is that

$$H(a,b) \leqslant G(a,b)$$

which again follows from $-e^{-t}$ being concave. Also published, see for example [7], is the inequality

$$G(a,b) \leqslant L(a,b). \tag{3}$$

To prove this using quasi-linear means we need to find an alternative version for the logarithmic mean.

THEOREM 1. *It is that, with $b > a$,*

$$L(a,b) = \exp\left(\int_a^b \log t\, F_L(dt)\right)$$

where

$$f_L(t) = w\,\delta_a(t) + (1-w)\,\delta_b(t)$$

with

$$w = 1 - \frac{\log\frac{\lambda-1}{\log\lambda}}{\log\lambda},$$

and $\lambda = b/a$ and $0 < w < \frac{1}{2}$.

Proof. This is quite straightforward to show by equating

$$w\log a + (1-w)\log b = \log\frac{b-a}{\log b - \log a} = \log a + \log\frac{\lambda-1}{\log\lambda},$$

from which w follows. That $0 < w < \frac{1}{2}$ follows from the inequality

$$\tfrac{1}{2} < \frac{\log \frac{\lambda-1}{\log\lambda}}{\log\lambda} < 1$$

which follows from the two inequalities, for $\lambda > 1$, $\sqrt{\lambda}\log\lambda < \lambda - 1$ and $\lambda - 1 < \lambda\log\lambda$. $\square$

Theorem 1 could be proven directly, but it is insightful to see it working via the quasi-linear means. Now

$$G(a,b) = \exp\left(\int_a^b \log t\, F_G(\mathrm{d}t)\right)$$

where $f_G(t) = \frac{1}{2}\delta_a(t) + \frac{1}{2}\delta_b(t)$. So $F_L \leqslant F_G$ which implies, see for example [10], that $M(\phi, F_G) \leqslant M(\phi, F_L)$, proving (3).

So it is possible to run through all the combinations checking for the concavity of $\phi_1 \circ \phi_2^{-1}$, or a common ϕ with $F_1 \leqslant F_2$, to establish orderings of means. Recall $\phi_1 \circ \phi_2^{-1}$ is concave means that ϕ_1 is more concave than ϕ_2 and if $\succ$ denotes "more concave than", then

$$-t^{-1-r} \succ -t^{-r} \succ \log t \succ t^r \succ t^{r+1}$$

with $r > 0$.

2.1. New means

Most of the means in common use, or the ones which are well known, are derived from either $F = U(a,b)$ or $f(t) = \frac{1}{2}$ at $t = a$ and $f(t) = \frac{1}{2}$ at $t = b$, though the latter could be replaced by $f(t) = w$ at $t = a$ and $f(t) = 1 - w$ at $t = b$ for any $0 < w < 1$. An F which provides new means to the literature has density function on $[a,b]$ given by

$$f_r(t) = \frac{(1+r)t^r}{b^{r+1} - a^{r+1}} \tag{4}$$

for $r \geqslant 0$. For example, we can generalize the identric mean to

$$I_r(a,b) = \frac{(1+r)\int_a^b \log t\, t^r\,\mathrm{d}t}{b^{r+1} - a^{r+1}},$$

and, since

$$\int_a^b \log t\, t^r\,\mathrm{d}t = \frac{1}{r+1}\left[b^{r+1}\log b - a^{r+1}\log a - (b^{r+1} - a^{r+1})/(1+r)\right],$$

the generalized identric mean is given by

$$I_r(a,b) = \exp\{-1/(1+r)\}\left(\frac{b^{b^{1+r}}}{a^{a^{1+r}}}\right)^{1/(b^{r+1}-a^{r+1})}.$$

A general arithmetic mean is given by

$$A_r(a,b) = \frac{r+1}{r+2}\frac{b^{r+2}-a^{r+2}}{b^{r+1}-a^{r+1}}.$$

We could also derive the same generalized arithmetic mean by applying $f_r(t)$ with $\phi(t) = -1/t$.

Finally, we can apply $f_r(t)$ with $\phi(t) = t^q$ yielding the mean

$$L_{r,q}(a,b) = \left(\frac{r+1}{r+q+1}\frac{b^{r+q+1}-a^{r+q+1}}{b^{r+1}-a^{r+1}}\right)^{1/q}.$$

Now it is easy to check that $F_r(t)$ is decreasing in r for all t and so the means just given; i.e. $I_r(a,b)$, $A_r(a,b)$ and $L_{r,q}(a,b)$, are all increasing with r.

2.2. Inequalities involving the gamma and modified Bessel functions

In this sub-section we first look at inequalities involving the digamma function, specifically using $\Psi(t)$, since this is known to be a strictly monotone increasing function. The idea here then is that for any strictly increasing continuous function ϕ, we have

$$\frac{1}{b-a}\int_a^b \Psi(t)\,\mathrm{d}t = \frac{1}{b-a}\log\big(\Gamma(b)/\Gamma(a)\big) \leqslant \Psi\left(\phi^{-1}\left(\frac{1}{b-a}\int_a^b \phi(t)\,\mathrm{d}t\right)\right) \tag{5}$$

whenever $\Psi\circ\phi^{-1}$ is concave.

Now [15] use $\phi(t) = \log t$ since $\Psi(e^t)$ is concave and the corresponding inequality becomes

$$\exp\left(\Psi(I(a,b))\right) \geqslant \left(\frac{\Gamma(b)}{\Gamma(a)}\right)^{1/(b-a)}.$$

It is also known that

$$\left(\frac{\Gamma(b)}{\Gamma(a)}\right)^{1/(b-a)} \geqslant \exp\left(\Psi(L(a,b))\right),$$

which follows from the fact that $-1/t$ is more concave than $\Psi(t)$.

Here we see the extent of the use of the quasi-linear mean comparison theorem for results which involve the di-gamma and tri-gamma functions. Now, the tri-gamma function, $\Psi^{(1)}(t)$, is convex, non-negative and strictly decreasing for $t > 0$, and so $-\Psi^{(1)}(t)$ is concave and strictly increasing for $t > 0$. Hence

$$\widetilde{\Psi^{(1)}}\left(\frac{1}{b-a}\int_a^b -\Psi^{(1)}(t)\,\mathrm{d}t\right) \leqslant A(a,b),$$

where $\widetilde{\Psi^{(1)}}$ denotes the inverse of the negative of the tri-gamma function. Thus

$$\Psi^{(1)}(A(a,b)) \leqslant \frac{\Psi(b)-\Psi(a)}{b-a}. \tag{6}$$

This, and more (improved) results like it, are documented in [6] and [15], the former extending results in [8] and the latter developing work in [4].

Using the comparison theorem we can work some results for the modified Bessel functions and we start with the modified Bessel function of the first kind $I_\alpha(t)$ and specifically with $I_0(t)$. In the following, the functions can be found in [17], for example. Let

$$L_\nu(t) = \left(\tfrac{1}{2}t\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{\left(\tfrac{1}{2}t\right)^{2k}}{\Gamma(k+3/2)\Gamma(k+\nu+3/2)}$$

be the modified Struve function and

$$\Phi(t) = \tfrac{1}{2}\pi t\,[I_0(t)L_1(t) - I_1(t)L_0(t)],$$

which is listed in [17]. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}[tI_0(t) + \Phi(t)] = I_0(t)$$

and so, since $-I_0(t)$ is concave and strictly increasing for $t > 0$, we have, in a similar manner to (6), that

$$I_0\big(A(a,b)\big) \leqslant \frac{bI_0(b) - aI_0(a) + \Phi(b) - \Phi(a)}{b-a}.$$

The interesting aspect to this type of function is that there are known forms for

$$\int_a^b I_0(t)\,t^r\,\mathrm{d}t$$

and hence we can obtain mean inequalities using $f_r(t)$ given in (4). If we define $D_r(x)$ by $\mathrm{d}D_r(x)/\mathrm{d}x = x^r I_0(x)$, then

$$\widetilde{I_0}\left(\int_a^b -I_0(t)\,f_r(t)\,\mathrm{d}t\right) \leqslant \int_a^b t\,f_r(t)\,\mathrm{d}t$$

where $\widetilde{I_0}$ is the inverse of $-I_0$. The resulting inequality is

$$I_0\left(A_r(a,b)\right) \leqslant (1+r)\,\frac{D_r(b) - D_r(a)}{b^{r+1} - a^{r+1}}.$$

There would appear to be unlimited access to inequalities by using suitable special functions and suitable density functions on $[a,b]$. Moreover, inequalities which would appear to be unrelated to quasi-linear means, such as the connection between identric mean and polygamma functions, can indeed be understood from this perspective.

3. Generalizing the comparison theorem

The aim for the remainder of the paper is first to prove (2) using the Neyman-Pearson lemma ([13]). The Neyman-Pearson lemma is a piece of theory related to most powerful tests of statistical hypotheses. The structure which allows us to do this then permits a generalization of (2) in which F can be a non-monotone function on $[a,b]$. That is, we have

$$\mathbb{D} = \{F \ : \quad F(b) = 1 \quad \text{and} \quad F(t) \in [0,1]\}$$

so we aim to prove that

$$M(\phi_1, F) \leqslant M(\phi_2, F) \quad \forall\, F \in \mathbb{D} \iff \phi_1 \circ \phi_2^{-1} \text{ is concave on } [a,b]. \tag{7}$$

This then allows for the introduction of new inequalities of the type already set out in the paper.

We first take $\phi(t)$ to be a distribution function on $[a,b]$ and then relax this assumption by connecting the distribution $\phi(t)\,(=\phi_D(t))$ by its more general counterpart $\phi(t)$ using the transform

$$\phi_D(t) = \frac{\phi(t) - \phi(a)}{\phi(b) - \phi(a)}.$$

We denote by ψ the derivative function corresponding to ϕ.

3.1. The Neyman-Pearson lemma

This is a fundamental result relating to most powerful tests. To set the scene, suppose there is a single (observable) random variable $T \in [a,b]$, with distribution function $\phi(t)$, and the aim is to test

$$H_0 : \phi = \phi_1 \quad \text{vs} \quad H_1 : \phi = \phi_2.$$

The test proceeds via a test function $\delta : [a,b] \to \{0,1\}$ with

$$\delta(t) = \begin{cases} 1 & \text{if} \quad t \in C \\ 0 & \text{if} \quad t \notin C, \end{cases} \tag{8}$$

i.e., $\delta(t) = \mathbf{1}(t \in C)$, for some critical set C, and H_0 is rejected if $T \in C$. The type I error of the test is given by

$$\alpha = \int_a^b \delta(t)\,\psi_1(t)\,\mathrm{d}t = \phi_1(C)$$

and the power of the test, the probability of rejecting H_0 when it is not true, is given by

$$\beta = \int_a^b \delta(t)\,\psi_2(t)\,\mathrm{d}t = \phi_2(C).$$

A most powerful test δ^* is a test function for which; if

$$\int_a^b (\phi - \phi^*)(t)\,\psi_1(t)\,\mathrm{d}t \leqslant 0$$

then

$$\int_a^b (\delta - \delta^*)(t)\,\psi_2(t)\,\mathrm{d}t \leqslant 0$$

for any δ of the form (8).

A well known result relating to most powerful tests is that if $\delta^*(t) = \mathbf{1}(t > t^*)$; i.e. the critical region is $C = (t^*, b]$ for some t^*, then $(\psi_1/\psi_2)(t)$ is decreasing. The critical region is equivalently given by $(\psi_1/\psi_2)(T) < \lambda$ for some $\lambda > 0$. This forms a part of the Neyman-Pearson lemma. Now we state and prove a minor modification of the standard presentation of the Neyman-Pearson lemma which starts with $(\psi_1/\psi_2)(t)$ decreasing.

LEMMA 1. *Let $H(t)$ be a function defined on $\mathbb{D}$. If $(\psi_1/\psi_2)(t)$ is decreasing on $[a,b]$, then:*

$$\phi_1(t^*) \leqslant \int_a^b \bar{H}(t)\,\psi_1(t)\,dt$$

implies

$$\phi_2(t^*) \leqslant \int_a^b \bar{H}(t)\,\psi_2(t)\,dt,$$

where $\bar{H}(t) = 1 - H(t)$.

Proof. Since ψ_1/ψ_2 is decreasing we know there exists a $0 < k < \infty$ for which

$$\psi_2(t) > k\,\psi_1(t) \quad \text{if and only if} \quad t > t^*.$$

Now consider

$$\xi(t) = [\mathbf{1}(t > t^*) - H(t)]\,[\psi_2(t) - k\,\psi_1(t)].$$

It is easy to show that $\xi(t) \geqslant 0$. Hence, $\int_a^b \xi(t)\,\mathrm{d}t \geqslant 0$ and so

$$1 - \phi_2(t^*) - \int_a^b H(t)\,\psi_2(t)\,\mathrm{d}t \geqslant k\left[1 - \phi_1(t^*) - \int_a^b H(t)\,\psi_1(t)\,\mathrm{d}t\right],$$

and thus the lemma is proved. □

The lemma is usually proven with $H(t) = \delta(t)$. The fact that it works for any $H \in \mathbb{D}$, which does not help for testing purposes, is, however, what allows us to make the general comparisons of quasi-linear means

The Neyman-Pearson lemma works around ψ_1/ψ_2 being decreasing and the order of the quasi-linear means works around $\phi_1 \circ \phi_2^{-1}$ being concave. Ultimately, the connection between the ordering of quasi-linear means and the Neyman-Pearson lemma relies on the fact that

$$\frac{\psi_1}{\psi_2} \quad \text{decreasing} \quad \Longleftrightarrow \quad \phi_1 \circ \phi_2^{-1} \quad \text{concave}. \tag{9}$$

That (9) holds true is a simple consequence of the fact that

$$\frac{d}{dz}\phi_1 \circ \phi_2^{-1}(z) = \frac{\psi_1}{\psi_2}(\phi_2^{-1}(z))$$

and that $\phi_2^{-1}(z)$ is increasing in z.

We now use the above theory for most powerful tests to obtain a general result relating to ordered quasi-linear means.

3.2. Main results

The main results of the paper are the following theorem and corollaries; and these generalize (2).

THEOREM 2. *For distribution functions ϕ_1 and ϕ_2, it is that*

$$\phi_1^{-1}\left(\int_a^b \phi_1(t)F(dt)\right) \leqslant \phi_2^{-1}\left(\int_a^b \phi_2(t)F(dt)\right),$$

for all $F \in \mathbb{D}$ if, and only if, $\phi_1 \circ \phi_2^{-1}$ is concave.

Proof. Given that $\phi(a) = 0$ and $\phi(b) = 1$ and $\bar{F}(b) = 0$, where $\bar{F} = 1 - F$, we can show that

$$\int_a^b \bar{F}(t)\,\psi(t)\,\mathrm{d}t = \int_a^b \phi(t)\,F(\mathrm{d}t). \tag{10}$$

Let us start by assuming that

$$M(\phi_1, F) \leqslant M(\phi_2, F) \tag{11}$$

for all $F \in \mathbb{D}$, which includes all distribution functions F on $[a,b]$ and thus $\phi_1 \circ \phi_2^{-1}$ is concave.

Now assume that $\phi_1 \circ \phi_2^{-1}$ is concave. Thus usual appeal to the Jensen inequality is no longer valid since F is not necessarily a distribution function. The concavity of $\phi_1 \circ \phi_2^{-1}$ implies ψ_1/ψ_2 is decreasing which, in turn, implies, using Lemma 1, that

$$\phi_1(t^*) = \int_a^b \bar{F}(t)\,\psi_1(t)\,\mathrm{d}t \Rightarrow \phi_2(t^*) \leqslant \int_a^b \bar{F}(t)\,\psi_2(t)\,\mathrm{d}t$$

for all $F \in \mathbb{D}$. We can rewrite this as

$$\phi_1(t^*) = \int_a^b \phi_1(t)\,F(\mathrm{d}t) \Rightarrow \phi_2(t^*) \leqslant \int_a^b \phi_2(t)\,F(\mathrm{d}t),$$

which is equivalent to

$$t^* = \phi_1^{-1}\left(\int_a^b \phi_1(t)\,F(\mathrm{d}t)\right) \leqslant \phi_2^{-1}\left(\int_a^b \phi_2(t)\,F(\mathrm{d}t)\right).$$

Thus, (11) holds for all $F \in \mathbb{D}$. $\square$

This theorem gives us a comparison theorem when ϕ is a distribution function and $F \in \mathbb{D}$. To see how the result looks when we take a general increasing ϕ we switch

$$\phi(t) \to \frac{\phi(t)-\phi(a)}{\phi(b)-\phi(a)}.$$

COROLLARY 1. *For continuous and strictly increasing functions ϕ_1 and ϕ_2 on $[a,b]$, it is that*

$$\phi_1^{-1}\left(\int_a^b \phi_1(t)F(dt)+\phi_1(a)F(a)\right) \leqslant \phi_2^{-1}\left(\int_a^b \phi_2(t)F(dt)+\phi_2(a)F(a)\right),$$

for all $F \in \mathbb{D}$ if, and only if, $\phi_1 \circ \phi_2^{-1}$ is concave.

COROLLARY 2. *For a continuous concave function ϕ on $[\xi(a),\xi(b)]$, where $\xi(t)$ is a strictly increasing continuous function on $[a,b]$, it is that*

$$\int_a^b \phi\circ\xi(t)F(dt)+\phi\circ\xi(a)F(a) \leqslant \phi\left(\int_a^b \xi(t)F(dt)+\xi(a)F(a)\right),$$

for all $F \in \mathbb{D}$.

This follows by taking $\phi(t)=\phi_1\circ\phi_2^{-1}(t)$ and $\xi(t)=\phi_2(t)$ in Corollary 1. This is a stronger result than the Jensen-Steffensen-Boas inequality (see for example [1]) which requires $F(a)\leqslant F(t)$ for all $t\in[a,b]$. That is, F a distribution function on $[a,b]$, i.e. $F(a)=0$, gives the Jensen inequality, whereas $F \in \mathbb{D}$ with $F(a)\leqslant F(t)$ for all $t\in[a,b]$ gives the Jensen-Steffensen-Boas inequality. See in particular Theorem 1 in [1], where the $\lambda(t)$ in [1], which needs $\lambda(a)\leqslant\lambda(t)$ for all $t\in[a,b]$, is related to the $F(\mathrm{d}t)$ here by

$$F(\mathrm{d}t)=\frac{\mathrm{d}\lambda(t)}{\int_a^b \mathrm{d}\lambda(s)}$$

for $t\in[a,b]$.

4. Illustrations and new inequalities

In this section we illustrate the result in Section 3.

EXAMPLE 1. Here we use $\phi_1(t)=\log t$ and $\phi_2(t)=t$. Consequently, $\psi_1(t)=1/t$ and $\psi_2(t)=1$. Then we take the $F\in\mathbb{D}$, which is not a distribution function on $[a,b]$, and neither has $F(a)\leqslant F(t)$ for all $t\in[a,b]$, to be

$$F(t)=1-\frac{(b-t)(t-a)}{(b-a)^2}.$$

It is easy to verify in both cases that

$$\int_a^b \bar{F}(t)\,\psi_j(t)\,\mathrm{d}t=\int_a^b \phi_j(t)\,f(t)\,\mathrm{d}t$$

where $f(t)=F'(t)$. The general comparison theorem then yields

$$\phi_1^{-1}\left(\int_a^b \bar F(t)\,\psi_1(t)\,\mathrm{d}t+\phi_1(a)\right)\leqslant \phi_2^{-1}\left(\int_a^b \bar F(t)\,\psi_2(t)\,\mathrm{d}t+\phi_2(a)\right);$$

which after some algebra gives the inequality,

$$\exp\left\{\tfrac{1}{2}\,\frac{a+b}{b-a}\right\}\left(\frac{b}{a}\right)^{-ab/(b-a)^2}\leqslant \frac{5+b/a}{6}.$$

Many other such new inequalities can be derived through different choices of ϕ_1 and ϕ_2 and F.

EXAMPLE 2. In our second example we retain the same $F(t)$ as in Example 1 but now consider $\phi(t)=t^r$. Then based on the concavity orderings, we have, after some algebra, that

$$\left[\frac{1}{(b-a)^2}\frac{1}{(r+1)(r+2)}\left\{r(b^{r+2}-a^{r+2})-(r+2)ab(b^r-a^r)\right\}+a^r\right]^{1/r}$$

is increasing in r.

EXAMPLE 3. We retain $\phi(t)=t^r$ but now take

$$F(t)=\begin{cases}\frac{c-t}{c-a} & \text{for}\quad a\leqslant t\leqslant c\\ \frac{t-c}{b-c} & \text{for}\quad c\leqslant t\leqslant b\end{cases}$$

with $c=\frac{1}{2}(a+b)$. So $F(a)$ is not smaller than $F(t)$ for all $t\in[a,b]$. Again, after some algebra, we have that

$$\left[\frac{2}{b-a}\frac{1}{r+1}\left\{b^{r+1}+a^{r+1}-2c^{r+1}\right\}+a^r\right]^{1/r}$$

is increasing in r.

EXAMPLE 4. We retain the same $F(t)$ as in Example 3 but now take $\phi(t)=\Psi_2(t)$, so $\psi(t)=\Psi_1(t)$, are polygamma functions. Using

$$\int_\alpha^\beta t\,\Psi_1(t)\,\mathrm{d}t=\beta\Psi(\beta)-\alpha\Psi(\alpha)-\log(\Gamma(\beta)/\Gamma(\alpha))\quad\text{and}\quad\int_\alpha^\beta \Psi_1(t)\,\mathrm{d}t=\Psi(\beta)-\Psi(\alpha)$$

and the fact that $\Psi_2(t)$ is concave for $t>0$, we have the inequality

$$\frac{2}{b-a}\left[2c\Psi(c)-a\Psi(a)-b\Psi(b)+\log\{\Gamma(a)\Gamma(b)/\Gamma(c)^2\}\right]+\Psi_2(a)$$
$$\leqslant \Psi_2\left(\frac{1}{b-a}\left(b^2+a^2-2c^2\right)+a\right)$$

for all $0<a<b<\infty$.

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