

A FAMILY OF MEROMORPHICALLY MULTIVALENT FUNCTIONS WHICH ARE STARLIKE WITH RESPECT TO k -SYMMETRIC POINTS

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(Communicated by S. Hencł)

Abstract. In this paper, two new subclasses $\mathcal{P}_{p,k}(\lambda, A, B)$ and $\mathcal{T}_{p,k}(\lambda, A, B)$ of meromorphically multivalent functions starlike with respect to k -symmetric points are studied. Distortion bounds, inclusion relations and convolution properties for each of these classes are obtained.

1. Introduction, definitions and preliminaries

Throughout this paper, we assume that

$$\mathbb{N} = \{1, 2, 3, \dots\}, \quad k \in \mathbb{N} \setminus \{1\}, \quad -1 \leq B < 0, \quad B < A \leq -B \quad \text{and} \quad \lambda \geq 1. \quad (1.1)$$

For functions f and g analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

the function f is said to be subordinate to g , written $f(z) \prec g(z)$ ($z \in \mathbb{U}$), if there exists an analytic function w in \mathbb{U} , with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in \mathbb{N}), \quad (1.2)$$

which are analytic in the punctured open unit disk $\mathbb{U}_0 = \mathbb{U} \setminus \{0\}$.

A function $f \in \Sigma_p$ is said to be meromorphically starlike with respect to k -symmetric points, if it satisfies

$$\Re \left\{ -\frac{z f'(z)}{f_{p,k}(z)} \right\} > 0,$$

where

$$f_{p,k}(z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{jp} f(\varepsilon_k^j z) \quad \text{and} \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right).$$

Mathematics subject classification (2010): Primary 30C45.

Keywords and phrases: Meromorphically multivalent functions, Hadamard product (or convolution), distortion bounds, inclusion relations, convolution properties, subordination properties, symmetric points.

Let

$$f_j(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,j}z^n \in \Sigma_p \quad (j = 1, 2).$$

Then the Hadamard product (or convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,1}a_{n,2}z^n = (f_2 * f_1)(z).$$

The following lemma will be required in our investigation.

LEMMA. Let $f \in \Sigma_p$ defined by (1.2) satisfy

$$\sum_{n=p}^{\infty} [\lambda n(1 - B) + p(1 - A)\delta_{n,p,k}]|a_n| \leq p(A - B). \tag{1.3}$$

Then

$$\frac{p(1 - \lambda)z^{-p} - \lambda zf'(z)}{pf_{p,k}(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \tag{1.4}$$

where

$$f_{p,k}(z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{jp} f(\varepsilon_k^j z), \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right) \tag{1.5}$$

and

$$\delta_{n,p,k} = \begin{cases} 0 & (\frac{n+p}{k} \notin \mathbb{N}), \\ 1 & (\frac{n+p}{k} \in \mathbb{N}). \end{cases} \tag{1.6}$$

Proof. For $f \in \Sigma_p$ defined by (1.2), the function $f_{p,k}$ in (1.5) can be expressed as

$$f_{p,k}(z) = z^{-p} + \sum_{n=p}^{\infty} \delta_{n,p,k} a_n z^n \tag{1.7}$$

with

$$\delta_{n,p,k} = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{j(n+p)} = \begin{cases} 0 & (\frac{n+p}{k} \notin \mathbb{N}), \\ 1 & (\frac{n+p}{k} \in \mathbb{N}). \end{cases}$$

In view of (1.1) and (1.6), we see that

$$Ap\delta_{n,p,k} + B\lambda n \leq B(\lambda n - p\delta_{n,p,k}) \leq 0 \quad (n \geq p). \tag{1.8}$$

Let the inequality (1.3) hold true. Then from (1.7) and (1.8) we deduce that

$$\begin{aligned} \left| \frac{\frac{p(1-\lambda)z^{-p} - \lambda zf'(z)}{pf_{p,k}(z)} - 1}{A - B \frac{p(1-\lambda)z^{-p} - \lambda zf'(z)}{pf_{p,k}(z)}} \right| &= \left| \frac{\sum_{n=p}^{\infty} (\lambda n + p\delta_{n,p,k}) a_n z^{n+p}}{p(A - B) + \sum_{n=p}^{\infty} (Ap\delta_{n,p,k} + B\lambda n) a_n z^{n+p}} \right| \\ &\leq \frac{\sum_{n=p}^{\infty} (\lambda n + p\delta_{n,p,k}) |a_n|}{p(A - B) + \sum_{n=p}^{\infty} (Ap\delta_{n,p,k} + B\lambda n) |a_n|} \\ &\leq 1 \quad (|z| = 1). \end{aligned}$$

Hence, by the *Maximum Modulus Theorem*, we arrive at (1.4). \square

We now consider the following two subclasses of Σ_p .

DEFINITION 1. A function $f \in \Sigma_p$ defined by (1.2) is said to be in the class $\mathcal{R}_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (1.3).

It follows from the Lemma that, if $f \in \mathcal{R}_{p,k}(\lambda, A, B)$, then the subordination relation (1.4) holds true.

DEFINITION 2. A function $f \in \Sigma_p$ defined by (1.2) is said to be in the class $\mathcal{T}_{p,k}(\lambda, A, B)$ if and only if it satisfies

$$\sum_{n=p}^{\infty} n[\lambda n(1 - B) + p(1 - A)\delta_{n,p,k}]|a_n| \leq p^2(A - B). \tag{1.9}$$

For $f \in \Sigma_p$ defined by (1.2), we have

$$2z^{-p} + \frac{zf'(z)}{p} = z^{-p} + \sum_{n=p}^{\infty} \frac{n}{p} a_n z^n,$$

which implies that

$$f \in \mathcal{T}_{p,k}(\lambda, A, B) \quad \text{if and only if} \quad 2z^{-p} + \frac{zf'(z)}{p} \in \mathcal{R}_{p,k}(\lambda, A, B). \tag{1.10}$$

If we write

$$\alpha_n = \frac{\lambda n(1 - B) + p(1 - A)\delta_{n,p,k}}{p(A - B)} \quad \text{and} \quad \beta_n = \frac{n}{p} \alpha_n \quad (n \geq p), \tag{1.11}$$

then it is easy to verify that

$$\frac{\partial \beta_n}{\partial \lambda} = \frac{n}{p} \frac{\partial \alpha_n}{\partial \lambda} > 0, \quad \frac{\partial \beta_n}{\partial A} = \frac{n}{p} \frac{\partial \alpha_n}{\partial A} < 0 \quad \text{and} \quad \frac{\partial \beta_n}{\partial B} = \frac{n}{p} \frac{\partial \alpha_n}{\partial B} \geq 0.$$

Thus we have the following inclusion relations. If

$$1 \leq \lambda_1 \leq \lambda, \quad -1 \leq B_1 \leq B < 0, \quad B < A \leq -B \quad \text{and} \quad A \leq A_1 \leq -B_1,$$

then

$$\mathcal{T}_{p,k}(\lambda, A, B) \subset \mathcal{R}_{p,k}(\lambda, A, B) \subset \mathcal{R}_{p,k}(\lambda_1, A_1, B_1) \subset \mathcal{R}_{p,k}(1, 1, -1). \tag{1.12}$$

Therefore, by the Lemma, we see that each function in the classes $\mathcal{R}_{p,k}(\lambda, A, B)$ and $\mathcal{T}_{p,k}(\lambda, A, B)$ is meromorphically starlike with respect to k -symmetric points. Meromorphic (and analytic) functions which are starlike with respect to symmetric points and related functions have been extensively studied by several authors (see, e.g., [1, 2, 3, 6, 7, 8, 9] and [12] to [15]; see also the recent works [10] and [11]).

There are several papers which study the convolution properties of functions in different function classes, and sometimes these questions might turn out to be very difficult (see, e.g., [5] and the references therein). Also, many authors investigate the distortion bounds of functions in various function classes (see, e.g., [4] and the references therein). In the present paper, we obtain distortion bounds, inclusion relations and convolution properties for each of the above-defined classes $\mathcal{R}_{p,k}(\lambda, A, B)$ and $\mathcal{T}_{p,k}(\lambda, A, B)$.

2. Distortion bounds

THEOREM 1. Let $\frac{2p}{k} \in \mathbb{N}$ and suppose that either

(a) $1 - B \geq p(1 - A)$ and $\lambda \geq 1$ or

(b) $1 - B < p(1 - A)$ and $\lambda \geq \frac{p(1-A)}{1-B}$.

Then, if we denote

$$C_1 = \frac{A - B}{\lambda(1 - B) + 1 - A},$$

we have the following:

(i) If $f \in \mathcal{R}_{p,k}(\lambda, A, B)$, then for $z \in \mathbb{U}_0$,

$$|z|^{-p} - C_1|z|^p \leq |f(z)| \leq |z|^{-p} + C_1|z|^p. \quad (2.1)$$

(ii) If $f \in \mathcal{T}_{p,k}(\lambda, A, B)$, then for $z \in \mathbb{U}_0$,

$$p(|z|^{-p-1} - C_1|z|^{p-1}) \leq |f'(z)| \leq p(|z|^{-p-1} + C_1|z|^{p-1}). \quad (2.2)$$

The bounds in (2.1) and (2.2) are sharp.

Proof. Let $\frac{2p}{k} \in \mathbb{N}$. For $n \geq p$ and $\frac{n+p}{k} \in \mathbb{N}$, we have $n = p + k(m - 1)$ ($m \in \mathbb{N}$), $\delta_{n,p,k} = 1$, and so

$$\frac{\lambda n(1 - B) + p(1 - A)\delta_{n,p,k}}{p(A - B)} \geq \frac{\lambda(1 - B) + 1 - A}{A - B}. \quad (2.3)$$

For $n \geq p$ and $\frac{n+p}{k} \notin \mathbb{N}$, we have $\delta_{n,p,k} = \delta_{p+1,p,k} = 0$ and

$$\frac{\lambda n(1 - B) + p(1 - A)\delta_{n,p,k}}{p(A - B)} \geq \frac{\lambda(p + 1)(1 - B)}{p(A - B)}. \quad (2.4)$$

If either (a) or (b) is satisfied, then

$$\frac{\lambda(p + 1)(1 - B)}{p(A - B)} \geq \frac{\lambda(1 - B) + 1 - A}{A - B}. \quad (2.5)$$

(i) If

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in \mathcal{R}_{p,k}(\lambda, A, B),$$

then it follows from (1.3) and (2.3) to (2.5) that

$$\frac{\lambda(1 - B) + 1 - A}{A - B} \sum_{n=p}^{\infty} |a_n| \leq 1.$$

Hence we have

$$|f(z)| \leq |z|^{-p} + |z|^p \sum_{n=p}^{\infty} |a_n| \leq |z|^{-p} + \frac{A - B}{\lambda(1 - B) + 1 - A} |z|^p$$

and

$$|f(z)| \geq |z|^{-p} - |z|^p \sum_{n=p}^{\infty} |a_n| \geq |z|^{-p} - \frac{A-B}{\lambda(1-B)+1-A} |z|^p > 0$$

for $z \in \mathbb{U}_0$.

(ii) If

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in \mathcal{T}_{p,k}(\lambda, A, B),$$

then it follows from (1.9) and (2.3) to (2.5) that

$$\frac{\lambda(1-B)+1-A}{p(A-B)} \sum_{n=p}^{\infty} n|a_n| \leq 1.$$

This leads to (2.2).

Furthermore, the bounds in (2.1) and (2.2) are sharp for the function

$$f(z) = z^{-p} + \frac{A-B}{\lambda(1-B)+1-A} z^p \in \mathcal{T}_{p,k}(\lambda, A, B) \subset \mathcal{R}_{p,k}(\lambda, A, B). \quad \square \quad (2.6)$$

THEOREM 2. Let $\frac{2p}{k} \in \mathbb{N}$ and suppose that

$$(1-B) < p(1-A) \quad \text{and} \quad 1 \leq \lambda < \frac{p(1-A)}{1-B},$$

and let

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n.$$

Then, if we denote

$$C_2 = \frac{p(A-B) - p(\lambda(1-B)+1-A)|a_p|}{\lambda(p+1)(1-B)},$$

we have the following:

(i) If $f \in \mathcal{R}_{p,k}(\lambda, A, B)$, then for $z \in \mathbb{U}_0$,

$$|z|^{-p} - |a_p||z|^p - C_2|z|^{p+1} \leq |f(z)| \leq |z|^{-p} + |a_p||z|^p + C_2|z|^{p+1}. \quad (2.7)$$

(ii) If $f \in \mathcal{T}_{p,k}(\lambda, A, B)$, then for $z \in \mathbb{U}_0$,

$$p(|z|^{-p-1} - |a_p||z|^{p-1} - C_2|z|^p) \leq |f'(z)| \leq p(|z|^{-p-1} + |a_p||z|^{p-1} + C_2|z|^p). \quad (2.8)$$

The bounds in (2.7) and (2.8) are sharp.

Proof. Note that $1 \leq \lambda < \frac{p(1-A)}{1-B}$ implies that

$$\frac{\lambda(1-B)+1-A}{A-B} \geq \frac{\lambda(p+1)(1-B)}{p(A-B)}. \quad (2.9)$$

(i) For $f(z) = z^{-p} + a_p z^p + \dots \in \mathcal{R}_{p,k}(\lambda, A, B)$, it follows from (2.3), (2.4) (used in the proof of Theorem 1) and (2.9) that

$$\frac{\lambda(1-B) + 1 - A}{A - B} |a_p| + \frac{\lambda(p+1)(1-B)}{p(A-B)} \sum_{n=p+1}^{\infty} |a_n| \leq 1.$$

From this we easily have (2.7).

The bounds in (2.7) are sharp for the function

$$f(z) = z^{-p} + \frac{p(A-B)}{\lambda(p+1)(1-B)} z^{p+1} \in \mathcal{R}_{p,k}(\lambda, A, B). \tag{2.10}$$

(ii) For $f(z) = z^{-p} + a_p z^p + \dots \in \mathcal{T}_{p,k}(\lambda, A, B)$, from (2.3), (2.4) and (2.9) we deduce that

$$\frac{\lambda(1-B) + 1 - A}{A - B} |a_p| + \frac{\lambda(p+1)(1-B)}{p^2(A-B)} \sum_{n=p+1}^{\infty} n |a_n| \leq 1.$$

Hence we have (2.8).

The bounds in (2.8) are sharp for the function

$$f(z) = z^{-p} + \frac{p^2(A-B)}{\lambda(p+1)^2(1-B)} z^{p+1} \in \mathcal{T}_{p,k}(\lambda, A, B). \quad \square \tag{2.11}$$

THEOREM 3. Let $\frac{2p}{k} \notin \mathbb{N}$. Then, if we denote

$$C_3 = \frac{A - B}{\lambda(1 - B)},$$

we have the following:

(i) If $f \in \mathcal{R}_{p,k}(\lambda, A, B)$, then for $z \in \mathbb{U}_0$,

$$|z|^{-p} - C_3 |z|^p \leq |f(z)| \leq |z|^{-p} + C_3 |z|^p. \tag{2.12}$$

(ii) If $f \in \mathcal{T}_{p,k}(\lambda, A, B)$, then for $z \in \mathbb{U}_0$,

$$p (|z|^{-p-1} - C_3 |z|^{p-1}) \leq |f'(z)| \leq p (|z|^{-p-1} + C_3 |z|^{p-1}). \tag{2.13}$$

The bounds in (2.12) and (2.13) are sharp.

Proof. Let $\frac{2p}{k} \notin \mathbb{N}$. For $n \geq p$ and $\frac{n+p}{k} \notin \mathbb{N}$, we have $\delta_{n,p,k} = \delta_{p,p,k} = 0$ and

$$\frac{\lambda n(1-B) + p(1-A)\delta_{n,p,k}}{p(A-B)} \geq \frac{\lambda(1-B)}{A-B}. \tag{2.14}$$

For $n \geq p$ and $\frac{n+p}{k} \in \mathbb{N}$, we have

$$\delta_{n,p,k} = 1, \quad n = k \left(\left[\frac{2p}{k} \right] + m \right) - p > p \quad (m \in \mathbb{N}),$$

and

$$\frac{\lambda n(1-B) + p(1-A)\delta_{n,p,k}}{p(A-B)} > \frac{\lambda(1-B) + 1-A}{A-B} \geq \frac{\lambda(1-B)}{A-B}, \tag{2.15}$$

where $[a]$ denotes the integer part of a given real number a .

(i) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in \mathcal{R}_{p,k}(\lambda, A, B)$, then it follows from (2.14) and (2.15) that

$$\frac{\lambda(1-B)}{A-B} \sum_{n=p}^{\infty} |a_n| \leq 1,$$

which leads to (2.12).

(ii) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in \mathcal{T}_{p,k}(\lambda, A, B)$, then (2.14) and (2.15) give

$$\frac{\lambda(1-B)}{p(A-B)} \sum_{n=p}^{\infty} n|a_n| \leq 1,$$

which yields (2.13).

Furthermore, the function f defined by

$$f(z) = z^{-p} + \frac{A-B}{\lambda(1-B)} z^p \in \mathcal{T}_{p,k}(\lambda, A, B) \subset \mathcal{R}_{p,k}(\lambda, A, B) \tag{2.16}$$

shows that the bounds in (2.12) and (2.13) are best possible. \square

3. Inclusion relations

In this section, we generalize the above-mentioned inclusion relation (1.12)

$$\mathcal{T}_{p,k}(\lambda, A, B) \subset \mathcal{R}_{p,k}(\lambda, A, B) \tag{3.1}$$

as follows.

THEOREM 4. *If $-1 \leq D \leq B$, then*

$$\mathcal{T}_{p,k}(\lambda, A, B) \subset \mathcal{R}_{p,k}(\lambda, C(D), D), \tag{3.2}$$

where

$$C(D) = D + \frac{(1-D)(A-B)}{1-B}. \tag{3.3}$$

The number $C(D)$ cannot be decreased for each D .

Proof. Since $B < A \leq -B$ and $-1 \leq D \leq B < 0$, we see that

$$D < C(D) \leq D - \frac{2B(1-D)}{1-B} \leq -D.$$

Let $f \in \mathcal{T}_{p,k}(\lambda, A, B)$. In order to prove that $f \in \mathcal{R}_{p,k}(\lambda, C(D), D)$, we only need to find the smallest C ($D < C \leq -D$) and show that it equals to $C(D)$ such that

$$\frac{\lambda n(1-D) + p(1-C)\delta_{n,p,k}}{p(C-D)} \leq \frac{n[\lambda n(1-B) + p(1-A)\delta_{n,p,k}]}{p^2(A-B)} \tag{3.4}$$

for all $n \geq p$, that is, that

$$\frac{(\lambda n + p\delta_{n,p,k})(1-D)}{p(C-D)} - \delta_{n,p,k} \leq \frac{n}{p} \left\{ \frac{(\lambda n + p\delta_{n,p,k})(1-B)}{p(A-B)} - \delta_{n,p,k} \right\} \quad (n \geq p). \quad (3.5)$$

For $n \geq p$ and $\frac{n+p}{k} \in \mathbb{N}$, (3.5) is equivalent to

$$C \geq D + \frac{1-D}{\frac{n(1-B)}{p(A-B)} - \frac{n-p}{\lambda n+p}} = \varphi(n) \quad (\text{say}). \quad (3.6)$$

Noting that (1.1), a simple calculation shows that $\varphi(n)$ ($n \geq p, \lambda \geq 1$) is decreasing in n . Therefore

$$\varphi(n) \leq \begin{cases} \varphi(p) & \left(\frac{2p}{k} \in \mathbb{N}\right), \\ \varphi\left(k\left(\left[\frac{2p}{k}\right] + 1\right) - p\right) & \left(\frac{2p}{k} \notin \mathbb{N}\right), \end{cases} \quad (3.7)$$

where $[a]$ in (3.7) denotes the integer part of a given real number a .

For $n \geq p$ and $\frac{n+p}{k} \notin \mathbb{N}$, (3.5) becomes

$$C \geq D + \frac{1-D}{\frac{n(1-B)}{p(A-B)}} = \psi(n) \quad (\text{say}) \quad (3.8)$$

and

$$\psi(n) \leq \begin{cases} \psi(p+1) & \left(\frac{2p}{k} \in \mathbb{N}\right), \\ \psi(p) & \left(\frac{2p}{k} \notin \mathbb{N}\right). \end{cases} \quad (3.9)$$

Consequently, by taking

$$C = \varphi(p) = \psi(p) = D + \frac{(1-D)(A-B)}{1-B} = C(D), \quad (3.10)$$

it follows from (3.4) to (3.10) that $f \in \mathcal{R}_{p,k}(\lambda, C(D), D)$.

Furthermore, for $\frac{2p}{k} \in \mathbb{N}$ and $D < C_0 < C(D)$, we have

$$\frac{\lambda(1-D) + 1 - C_0}{C_0 - D} \cdot \frac{A-B}{\lambda(1-B) + 1 - A} > \frac{\lambda(1-D) + 1 - C(D)}{C(D) - D} \cdot \frac{A-B}{\lambda(1-B) + 1 - A} = 1,$$

which implies that the function $f \in \mathcal{T}_{p,k}(\lambda, A, B)$ defined by (2.6) is not in the class $\mathcal{R}_{p,k}(\lambda, C_0, D)$. Also, for $\frac{2p}{k} \notin \mathbb{N}$ and $D < C_0 < C(D)$, we have

$$\frac{\lambda(1-D)}{C_0 - D} \cdot \frac{A-B}{\lambda(1-B)} > \frac{\lambda(1-D)}{C(D) - D} \cdot \frac{A-B}{\lambda(1-B)} = 1,$$

which implies that the function $f \in \mathcal{T}_{p,k}(\lambda, A, B)$ defined by (2.16) is not in the class $\mathcal{R}_{p,k}(\lambda, C_0, D)$. The proof of Theorem 4 is thus completed. \square

4. Convolution properties

In this section, we assume that

$$-1 \leq B_j < 0 \quad \text{and} \quad B_j < A_j \leq -B_j \quad (j = 1, 2). \tag{4.1}$$

Furthermore, we denote by λ_1 the root in $(1, +\infty)$ of the equation:

$$h(\lambda) = a\lambda^2 + b\lambda + c = 0,$$

where

$$\begin{cases} a = -(1 - B_1)(1 - B_2), \\ b = (p - 1)(1 - B_1)(1 - B_2) - p[(1 - B_1)(A_2 - B_2) + (1 - B_2)(A_1 - B_1)], \\ c = p[(1 - A_1)(1 - A_2) + (A_1 - B_1)(A_2 - B_2)]. \end{cases} \tag{4.2}$$

We also denote

$$\tilde{A}(B) = B + \frac{1 - B}{(\lambda + 1) \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \sum_{j=1}^2 \frac{1 - B_j}{A_j - B_j} + \frac{2}{\lambda + 1}} \tag{4.3}$$

and

$$A(B) = B + \frac{p(1 - B)}{\lambda(p + 1)} \prod_{j=1}^2 \frac{A_j - B_j}{1 - B_j}. \tag{4.4}$$

THEOREM 5. *Let*

$$f_j \in \mathcal{R}_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2)$$

with

$$\frac{2p}{k} \in \mathbb{N} \quad \text{and} \quad -1 \leq B \leq \max\{B_1, B_2\}.$$

Then we have the following:

(i) *If $p(1 - A_1)(1 - A_2) \leq (1 - B_1)(1 - B_2)$ and $\lambda \geq 1$, then*

$$f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, \tilde{A}(B), B).$$

(ii) *If $p(1 - A_1)(1 - A_2) > (1 - B_1)(1 - B_2)$ and $\lambda \geq \lambda_1$, then*

$$f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, \tilde{A}(B), B).$$

(iii) *If $p(1 - A_1)(1 - A_2) > (1 - B_1)(1 - B_2)$ and $1 \leq \lambda < \lambda_1$, then*

$$f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, A(B), B).$$

In all cases (i)–(iii) the numbers $A(B)$ and $\tilde{A}(B)$ are optimal in the sense that they cannot be decreased for each B .

Proof. Suppose that $-1 \leq B \leq \max\{B_1, B_2\} = B_j$ ($j = 1$ or 2). It follows from (4.1) and (4.4) that

$$\frac{1 - B}{A(B) - B} = \frac{\lambda(p + 1)}{p} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} \geq \frac{1 - B_j}{A_j - B_j} \geq -\frac{1 - B_j}{2B_j} \geq -\frac{1 - B}{2B} > 0,$$

which implies that $B < A(B) \leq -B$. Also, (4.1) and (4.3) give that

$$\begin{aligned} \frac{1-B}{\widetilde{A}(B)-B} &= (\lambda+1) \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} - \sum_{j=1}^2 \frac{1-B_j}{A_j-B_j} + \frac{2}{\lambda+1} \\ &= (\lambda+1) \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} - \left(\prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} - \prod_{j=1}^2 \frac{1-A_j}{A_j-B_j} + 1 \right) + \frac{2}{\lambda+1} \\ &= \lambda \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} + \prod_{j=1}^2 \frac{1-A_j}{A_j-B_j} - \frac{\lambda-1}{\lambda+1} \\ &\geq \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} \geq -\frac{1-B}{2B} > 0, \end{aligned}$$

which implies that $B < \widetilde{A}(B) \leq -B$.

Let $\frac{2p}{k} \in \mathbb{N}$ and

$$f_j(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,j} z^n \in \mathcal{R}_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2).$$

Then

$$\begin{aligned} &\sum_{n=p}^{\infty} \left\{ \prod_{j=1}^2 \frac{\lambda n(1-B_j) + p(1-A_j)\delta_{n,p,k}}{p(A_j-B_j)} \right\} |a_{n,1}a_{n,2}| \\ &\leq \prod_{j=1}^2 \left\{ \sum_{n=p}^{\infty} \frac{\lambda n(1-B_j) + p(1-A_j)\delta_{n,p,k}}{p(A_j-B_j)} |a_{n,j}| \right\} \leq 1. \end{aligned} \tag{4.5}$$

Also, $f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, A, B)$ if and only if

$$\sum_{n=p}^{\infty} \frac{\lambda n(1-B) + p(1-A)\delta_{n,p,k}}{p(A-B)} |a_{n,1}a_{n,2}| \leq 1. \tag{4.6}$$

In order to prove Theorem 5, it follows from (4.5) and (4.6) that we need only to find the smallest A such that

$$\frac{\lambda n(1-B) + p(1-A)\delta_{n,p,k}}{p(A-B)} \leq \prod_{j=1}^2 \frac{\lambda n(1-B_j) + p(1-A_j)\delta_{n,p,k}}{p(A_j-B_j)} \quad (n \geq p). \tag{4.7}$$

For $n \geq p$ and $\frac{n+p}{k} \in \mathbb{N}$, (4.7) is equivalent to

$$A \geq B + \frac{1-B}{\frac{(\lambda n+p)}{p} \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} - \sum_{j=1}^2 \frac{1-B_j}{A_j-B_j} + \frac{2p}{\lambda n+p}} = \varphi_1(n) \text{ (say)}. \tag{4.8}$$

It can be verified that $\varphi_1(n)$ ($n \geq p, \lambda \geq 1$) is decreasing in n and so, in view of $\frac{2p}{k} \in \mathbb{N}$,

$$\varphi_1(n) \leq \varphi_1(p) = B + \frac{1-B}{(\lambda+1) \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} - \sum_{j=1}^2 \frac{1-B_j}{A_j-B_j} + \frac{2}{\lambda+1}}. \tag{4.9}$$

For $n \geq p$ and $\frac{n+p}{k} \notin \mathbb{N}$, (4.7) becomes

$$A \geq B + \frac{1 - B}{\frac{\lambda n}{p} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j}} = \psi_1(n) \text{ (say)} \tag{4.10}$$

and we have

$$\psi_1(n) \leq \psi_1(p + 1) = B + \frac{1 - B}{\frac{\lambda(p+1)}{p} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j}}. \tag{4.11}$$

Now

$$\begin{aligned} & (\lambda + 1) \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \sum_{j=1}^2 \frac{1 - B_j}{A_j - B_j} + \frac{2}{\lambda + 1} - \frac{\lambda(p + 1)}{p} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} \\ &= \frac{h(\lambda)}{p(\lambda + 1)(A_1 - B_1)(A_2 - B_2)}, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} h(\lambda) &= (p - \lambda)(\lambda + 1)(1 - B_1)(1 - B_2) - p(\lambda + 1)[(1 - B_1)(A_2 - B_2) \\ &\quad + (1 - B_2)(A_1 - B_1)] + 2p(A_1 - B_1)(A_2 - B_2) \\ &= a\lambda^2 + b\lambda + c, \end{aligned} \tag{4.13}$$

$$a = -(1 - B_1)(1 - B_2),$$

$$b = (p - 1)(1 - B_1)(1 - B_2) - p[(1 - B_1)(A_2 - B_2) + (1 - B_2)(A_1 - B_1)],$$

$$\begin{aligned} c &= p(1 - B_1)(1 - B_2) + 2p(A_1 - B_1)(A_2 - B_2) - p[(1 - B_1)(A_2 - B_2) + (1 - B_2)(A_1 - B_1)] \\ &= p[(1 - A_1)(1 - A_2) + (A_1 - B_1)(A_2 - B_2)]. \end{aligned}$$

Note that $a < 0$, $h(0) = c > 0$ and

$$\begin{aligned} h(1) &= 2(p - 1)(1 - B_1)(1 - B_2) - 2p[(1 - B_1)(A_2 - B_2) + (1 - B_2)(A_1 - B_1)] \\ &\quad + 2p(A_1 - B_1)(A_2 - B_2) \\ &= 2[p(1 - A_1)(1 - A_2) - (1 - B_1)(1 - B_2)]. \end{aligned} \tag{4.14}$$

Therefore, if (i) or (ii) is satisfied, then it follows from (4.7) to (4.14) that $h(\lambda) \leq 0$ for $\lambda \geq \lambda_1$, $\psi_1(p + 1) \leq \varphi_1(p) = \tilde{A}(B)$, and $f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, \tilde{A}(B), B)$.

Furthermore, for $B < A_0 < \tilde{A}(B)$, we have

$$\frac{\lambda(1 - B) + 1 - A}{A_0 - B} \prod_{j=1}^2 \frac{A_j - B_j}{\lambda(1 - B_j) + 1 - A_j} > \frac{\lambda(1 - B) + 1 - \tilde{A}(B)}{\tilde{A}(B) - B} \prod_{j=1}^2 \frac{A_j - B_j}{\lambda(1 - B_j) + 1 - A_j} = 1.$$

Hence the functions f_j defined by

$$f_j(z) = z^{-p} + \frac{A_j - B_j}{\lambda(1 - B_j) + 1 - A_j} z^p \in \mathcal{R}_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2)$$

show that $f_1 * f_2 \notin \mathcal{R}_{p,k}(\lambda, A_0, B)$.

(iii) If $p(1 - A_1)(1 - A_2) > (1 - B_1)(1 - B_2)$ and $1 \leq \lambda < \lambda_1$, then we have $h(\lambda) > 0$, $\varphi_1(p) < \psi_1(p + 1) = A(B)$, and $f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, A(B), B)$. Furthermore, the number $A(B)$ cannot be decreased as can be seen from the functions $f_j(z)$ defined by

$$f_j(z) = z^{-p} + \frac{p(A_j - B_j)}{\lambda(p + 1)(1 - B_j)} z^{p+1} \in \mathcal{R}_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2). \quad \square$$

THEOREM 6. *Let*

$$f_1 \in \mathcal{R}_{p,k}(\lambda, A_1, B_1) \quad \text{and} \quad f_2 \in \mathcal{T}_{p,k}(\lambda, A_2, B_2)$$

with

$$\frac{2p}{k} \in \mathbb{N} \quad \text{and} \quad -1 \leq B \leq \max\{B_1, B_2\}.$$

Also let $A(B)$, $\tilde{A}(B)$ and λ_1 be given as in Theorem 5. Then we have the following:

(i) If $p(1 - A_1)(1 - A_2) \leq (1 - B_1)(1 - B_2)$ and $\lambda \geq 1$, then

$$f_1 * f_2 \in \mathcal{T}_{p,k}(\lambda, \tilde{A}(B), B).$$

(ii) If $p(1 - A_1)(1 - A_2) > (1 - B_1)(1 - B_2)$ and $\lambda \geq \lambda_1$, then

$$f_1 * f_2 \in \mathcal{T}_{p,k}(\lambda, \tilde{A}(B), B).$$

(iii) If $p(1 - A_1)(1 - A_2) > (1 - B_1)(1 - B_2)$ and $1 \leq \lambda < \lambda_1$, then

$$f_1 * f_2 \in \mathcal{T}_{p,k}(\lambda, A(B), B).$$

In all cases (i)–(iii) the numbers $A(B)$ and $\tilde{A}(B)$ are optimal in the sense that they cannot be decreased for each B .

Proof. Since [see Eq. (1.10)]

$$f_1 \in \mathcal{R}_{p,k}(\lambda, A_1, B_1), \quad 2z^{-p} + \frac{zf_2'(z)}{p} \in \mathcal{R}_{p,k}(\lambda, A_2, B_2)$$

and

$$f_1(z) * \left(2z^{-p} + \frac{zf_2'(z)}{p} \right) = 2z^{-p} + \frac{z(f_1 * f_2)'}{p} \quad (z \in \mathbb{U}_0),$$

an application of Theorem 5 yields the theorem. \square

Next, we denote by λ_2 the root in $(1, +\infty)$ of the equation:

$$h_1(\lambda) = a_1\lambda^2 + b_1\lambda + c_1 = 0,$$

where

$$\begin{cases} a_1 = -(2p + 1)(1 - B_1)(1 - B_2), \\ b_1 = (p^2 - 2p - 1)(1 - B_1)(1 - B_2) - p^2[(1 - B_1)(A_2 - B_2) + (1 - B_2)(A_1 - B_1)], \\ c_1 = p^2(1 - B_1)(1 - B_2) - p^2[(1 - B_1)(A_2 - B_2) + (1 - B_2)(A_1 - B_1)] \\ \quad + 2p^2(A_1 - B_1)(A_2 - B_2) \\ \quad = p^2[(1 - A_1)(1 - A_2) + (A_1 - B_1)(A_2 - B_2)]. \end{cases} \tag{4.15}$$

We also denote

$$\widetilde{A}_1(B) = B + \frac{p^2(1 - B)}{\lambda(p + 1)^2} \prod_{j=1}^2 \frac{A_j - B_j}{1 - B_j}. \tag{4.16}$$

THEOREM 7. *Let*

$$f_1 \in \mathcal{R}_{p,k}(\lambda, A_1, B_1) \quad \text{and} \quad f_2 \in \mathcal{F}_{p,k}(\lambda, A_2, B_2)$$

with

$$\frac{2p}{k} \in \mathbb{N} \quad \text{and} \quad -1 \leq B \leq \max\{B_1, B_2\}.$$

(i) *If* $p^2(1 - A_1)(1 - A_2) \leq (2p + 1)(1 - B_1)(1 - B_2)$ *and* $\lambda \geq 1$, *then*

$$f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, \widetilde{A}(B), B).$$

(ii) *If* $p^2(1 - A_1)(1 - A_2) > (2p + 1)(1 - B_1)(1 - B_2)$ *and* $\lambda \geq \lambda_2$, *then*

$$f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, \widetilde{A}(B), B).$$

(iii) *If* $p^2(1 - A_1)(1 - A_2) > (2p + 1)(1 - B_1)(1 - B_2)$ *and* $1 \leq \lambda < \lambda_2$, *then*

$$f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, \widetilde{A}_1(B), B).$$

In all cases (i)–(iii) the numbers $\widetilde{A}(B)$ *and* $\widetilde{A}_1(B)$ *are optimal in the sense that they cannot be decreased for each* B .

Proof. It can be verified that

$$\frac{1 - B}{\widetilde{A}_1(B) - B} = \frac{\lambda(p + 1)^2}{p^2} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} > \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} \geq -\frac{1 - B}{2B} > 0$$

and so $B < \widetilde{A}_1(B) < -B$.

In order to prove Theorem 7, we need only to find the smallest A such that

$$\frac{\lambda n(1 - B) + p(1 - A)\delta_{n,p,k}}{p(A - B)} \leq \frac{n}{p} \prod_{j=1}^2 \frac{\lambda n(1 - B_j) + p(1 - A_j)\delta_{n,p,k}}{p(A_j - B_j)} \tag{4.17}$$

for all $n \geq p$.

For $n \geq p$ and $\frac{n+p}{k} \in \mathbb{N}$, (4.17) is equivalent to

$$A \geq B + \frac{1 - B}{\frac{n(\lambda n+p)}{p^2} \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} - \frac{n}{p} \sum_{j=1}^2 \frac{1-B_j}{A_j-B_j} + \frac{n+p}{\lambda n+p}} = \varphi_2(n) \quad (\text{say}). \tag{4.18}$$

Defining the function $g(\lambda, x)$ by

$$g(\lambda, x) = \frac{x(\lambda x + p)}{p^2} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \frac{x}{p} \sum_{j=1}^2 \frac{1 - B_j}{A_j - B_j} + \frac{x + p}{\lambda x + p} \quad (x \geq p; \lambda \geq 1),$$

then

$$\begin{aligned} \frac{\partial g(\lambda, x)}{\partial x} &= \frac{2\lambda x + p}{p^2} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \frac{1}{p} \sum_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \frac{p(\lambda - 1)}{(\lambda x + p)^2} \\ &\geq \frac{2\lambda + 1}{p} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \frac{1}{p} \left(\prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \prod_{j=1}^2 \frac{1 - A_j}{A_j - B_j} + 1 \right) - \frac{\lambda - 1}{p(\lambda + 1)^2} \\ &= \frac{2\lambda}{p} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} + \frac{1}{p} \left(\prod_{j=1}^2 \frac{1 - A_j}{A_j - B_j} - 1 \right) - \frac{\lambda - 1}{p(\lambda + 1)^2} \\ &\geq \frac{2\lambda}{p} - \frac{1}{p} - \frac{\lambda - 1}{p(\lambda + 1)^2} > 0 \quad (x \geq p; \lambda \geq 1), \end{aligned}$$

which implies that $\varphi_2(n)$ defined by (4.18) is decreasing in n ($n \geq p$). Hence, in view of $\frac{2p}{k} \in \mathbb{N}$, we have

$$\varphi_2(n) \leq \varphi_2(p) = B + \frac{1 - B}{(\lambda + 1) \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \sum_{j=1}^2 \frac{1 - B_j}{A_j - B_j} + \frac{2}{\lambda + 1}}.$$

For $n \geq p$ and $\frac{n+p}{k} \notin \mathbb{N}$, (4.17) reduces to

$$A \geq B + \frac{1 - B}{\frac{\lambda n^2}{p^2} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j}} = \psi_2(n) \quad (\text{say})$$

and, in view of $\frac{2p}{k} \in \mathbb{N}$, we have

$$\psi_2(n) \leq \psi_2(p + 1) = B + \frac{1 - B}{\frac{\lambda(p+1)^2}{p^2} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j}}.$$

Now

$$\begin{aligned} &(\lambda + 1) \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \sum_{j=1}^2 \frac{1 - B_j}{A_j - B_j} + \frac{2}{\lambda + 1} - \frac{\lambda(p + 1)^2}{p^2} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} \\ &= \frac{h_1(\lambda)}{p^2(\lambda + 1)(A_1 - B_1)(A_2 - B_2)}, \end{aligned} \tag{4.19}$$

where $h_1(\lambda) = a_1\lambda^2 + b_1\lambda + c_1$ and a_1, b_1, c_1 are given by (4.15). Note that $a_1 < 0, h_1(0) = c_1 > 0$ and

$$\begin{aligned} h_1(1) &= [4p^2 - 2(p+1)^2](1-B_1)(1-B_2) - 2p^2[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)] \\ &\quad + 2p^2(A_1-B_1)(A_2-B_2) \\ &= 2[p^2(1-A_1)(1-A_2) - (2p+1)(1-B_1)(1-B_2)]. \end{aligned}$$

The remaining part of the proof of Theorem 7 is much akin to Theorem 5 and hence we omit it. The proof of the theorem is completed. \square

By applying Theorem 7, we can derive the following theorem immediately.

THEOREM 8. *Let*

$$f_j \in \mathcal{T}_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2)$$

with

$$\frac{2p}{k} \in \mathbb{N} \quad \text{and} \quad -1 \leq B \leq \max\{B_1, B_2\}.$$

Also let $\tilde{A}(B), \tilde{A}_1(B)$ and λ_2 be given as in Theorem 7.

(i) *If $p^2(1-A_1)(1-A_2) \leq (2p+1)(1-B_1)(1-B_2)$ and $\lambda \geq 1$, then*

$$f_1 * f_2 \in \mathcal{T}_{p,k}(\lambda, \tilde{A}(B), B).$$

(ii) *If $p^2(1-A_1)(1-A_2) > (2p+1)(1-B_1)(1-B_2)$ and $\lambda \geq \lambda_2$, then*

$$f_1 * f_2 \in \mathcal{T}_{p,k}(\lambda, \tilde{A}(B), B).$$

(iii) *If $p^2(1-A_1)(1-A_2) > (2p+1)(1-B_1)(1-B_2)$ and $1 \leq \lambda < \lambda_2$, then*

$$f_1 * f_2 \in \mathcal{T}_{p,k}(\lambda, \tilde{A}_1(B), B).$$

In all cases (i)–(iii) the numbers $\tilde{A}(B)$ and $\tilde{A}_1(B)$ are optimal in the sense that they cannot be decreased for each B .

Finally, we denote by λ_3 the root in $(1, +\infty)$ of the equation:

$$h_2(\lambda) = a_2\lambda^2 + b_2\lambda + c_2 = 0,$$

where

$$\begin{cases} a_2 = -(3p^2 + 3p + 1)(1-B_1)(1-B_2), \\ b_2 = (p^3 - 3p^2 - 3p - 1)(1-B_1)(1-B_2) - p^3[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)], \\ c_2 = p^3[(1-A_1)(1-A_2) + (A_1-B_1)(A_2-B_2)]. \end{cases} \tag{4.20}$$

We also denote

$$\tilde{A}_2(B) = B + \frac{p^3(1-B)}{\lambda(p+1)^3} \prod_{j=1}^2 \frac{A_j - B_j}{1 - B_j}. \tag{4.21}$$

THEOREM 9. *Let*

$$f_j \in \mathcal{T}_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2)$$

with

$$\frac{2p}{k} \in \mathbb{N} \quad \text{and} \quad -1 \leq B \leq \max\{B_1, B_2\}.$$

(i) *If* $p^3(1 - A_1)(1 - A_2) \leq (3p^2 + 3p + 1)(1 - B_1)(1 - B_2)$ *and* $\lambda \geq 1$, *then*

$$f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, \tilde{A}(B), B).$$

(ii) *If* $p^3(1 - A_1)(1 - A_2) > (3p^2 + 3p + 1)(1 - B_1)(1 - B_2)$ *and* $\lambda \geq \lambda_3$, *then*

$$f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, \tilde{A}(B), B).$$

(iii) *If* $p^3(1 - A_1)(1 - A_2) > (3p^2 + 3p + 1)(1 - B_1)(1 - B_2)$ *and* $1 \leq \lambda < \lambda_3$, *then*

$$f_1 * f_2 \in \mathcal{R}_{p,k}(\lambda, \tilde{A}_2(B), B).$$

In all cases (i)–(iii) the numbers $\tilde{A}(B)$ *and* $\tilde{A}_2(B)$ *are optimal in the sense that they cannot be decreased for each* B .

Proof. It can be seen that $B < \tilde{A}_2(B) < -B$. In order to prove Theorem 9, we need only to find the smallest A such that

$$\frac{\lambda n(1 - B) + p(1 - A)\delta_{n,p,k}}{p(A - B)} \leq \left(\frac{n}{p}\right)^2 \prod_{j=1}^2 \frac{\lambda n(1 - B_j) + p(1 - A_j)\delta_{n,p,k}}{p(A_j - B_j)} \quad (4.22)$$

for all $n \geq p$.

For $n \geq p$ and $\frac{n+p}{k} \in \mathbb{N}$, (4.22) can be written as

$$A \geq B + \frac{1 - B}{\frac{n^2(\lambda n + p)}{p^3} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \frac{n^2}{p^2} \sum_{j=1}^2 \frac{1 - B_j}{A_j - B_j} + \frac{n^2 + p^2}{p(\lambda n + p)}} = \varphi_3(n) \quad (\text{say}). \quad (4.23)$$

Since $\varphi_3(n)$ ($n \geq p, \lambda \geq 1$) is decreasing in n and so

$$\begin{aligned} \varphi_3(n) &\leq \varphi_3(p) = B + \frac{1 - B}{(\lambda + 1) \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j} - \sum_{j=1}^2 \frac{1 - B_j}{A_j - B_j} + \frac{2}{\lambda + 1}} \\ &= \tilde{A}(B). \end{aligned}$$

For $n \geq p$ and $\frac{n+p}{k} \notin \mathbb{N}$, (4.22) becomes

$$A \geq B + \frac{1 - B}{\frac{\lambda n^3}{p^3} \prod_{j=1}^2 \frac{1 - B_j}{A_j - B_j}} = \psi_3(n) \quad (\text{say})$$

and we have

$$\psi_3(n) \leq \psi_3(p+1) = B + \frac{1-B}{\frac{\lambda(p+1)^3}{p^3} \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j}}.$$

Now

$$\begin{aligned} & (\lambda+1) \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} - \sum_{j=1}^2 \frac{1-B_j}{A_j-B_j} + \frac{2}{\lambda+1} - \frac{\lambda(p+1)^3}{p^3} \prod_{j=1}^2 \frac{1-B_j}{A_j-B_j} \\ &= \frac{h_2(\lambda)}{p^3(\lambda+1)(A_1-B_1)(A_2-B_2)}, \end{aligned}$$

where $h_2(\lambda) = a_2\lambda^2 + b_2\lambda + c_2$ and a_2, b_2, c_2 are given by (4.20).

We note that $a_2 < 0, h_2(0) = c_2 > 0$ and

$$h_2(1) = 2[p^3(1-A_1)(1-A_2) - (3p^2 + 3p + 1)(1-B_1)(1-B_2)].$$

The remaining part of the proof is similar to that of Theorem 5 and thus we omit it. \square

5. Concluding remarks and observations

In our present investigation, we have introduced and studied several properties of the two new subclasses $\mathcal{R}_{p,k}(\lambda, A, B)$ and $\mathcal{T}_{p,k}(\lambda, A, B)$ of meromorphically multivalent functions which are starlike with respect to k -symmetric points. Among the various properties derived in this paper for each of these classes are obtained, we include distortion bounds, inclusion relations and convolution properties. Our results are motivated by a number of recent works (see, for example, [1] to [15]).

Acknowledgement. The authors would like to express sincere thanks to the referee for careful reading and suggestions which helped us to improve the paper. This work is supported by National Natural Science Foundation of China(Grant No. 11571299) and Natural Science Foundation of Jiangsu Province(Grant No. BK20151304).

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(Received April 10, 2016)

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